AN ANALYSIS OF A STOKES FLOW IN AN ANNULAR REGION

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ABSTRACT. A study is made of the steady two-dimensional Stokes flow stirred by an infinitesimal rotating cylinder (a line rotlet) in the annular region between two fixed concentric cylindrical walls. It is shown that this simple flow exhibits a rich diversity of possible flow structures. Generally, one component of the flow is associated with a net flux through the annular region in either a clockwise or anticlockwise direction depending on whether the radial distance between the rotlet and the center axis of the bounding cylinders is greater or less than a unique value at which this flow component vanishes. The remaining components of the flow are either eddies attached to the boundary or free eddies in the interior of the flow. Owing to some of the dynamical features of this flow, the results obtained here may be applicable in engineering settings where mixing by the chaotic advection of fluid particles is desirable.

1. Introduction. Singularities play an important role in the study of Stokes flows (see Pozrikidis [14] for useful background and references in this regard). A number of studies (e.g., [2], [4], [6–9], [15], [16], [18], [19]) have used one or more rotlet singularities in the presence of various no-slip boundaries to generate interesting flow structures. (A line rotlet or point rotlet may be regarded as a rotating cylinder or rotating sphere, respectively, of infinitesimal radius). The use of a rotlet is very advantageous in the sense that if a small rotating cylinder or sphere in a flow is replaced by a line or point rotlet, respectively, the resulting flow generally has the same topological structure but is far more tractable mathematically. This is illustrated, for example, by the Stokes flow inside a fixed or rotating cylindrical wall induced by a line rotlet (Ranger [16]) or a rotating cylinder (Ballal and Rivlin [1]).

The present paper investigates a Stokes flow, due to a line rotlet, with a rich diversity of possible flow structures, including most of those observed in the aforementioned studies as well as some that have...
not been observed therein. An important characteristic of the flow domain is that it is both doubly-connected and bounded; the simplest two-dimensional domain with these characteristics, the annular region between two concentric cylindrical walls, is the flow domain here. One interesting aspect of the flow is the net flux through the annular region: it is clockwise, anticlockwise or vanishes depending on the position of the rotlet. A closely related study by Hajjam [11] deals with the Stokes flow due to a rotating cylinder between fixed concentric cylindrical walls. Although [11] deals with a genuine rotating cylinder (as opposed to a line rotlet), the numerical method used in [11] seems unable to treat cases in which the flux between the cylindrical walls vanishes or nearly vanishes, and these cases are the most remarkable with respect to the number and qualities of the eddies in the flow. The use of a line rotlet here allows an easy derivation of an analytical expression for the stream function in terms of an infinite series which converges quickly and from which the stream function can be computed to any required accuracy. It is expected that the flow considered here has the same topological features as the Stokes flow induced by a rotating cylinder of small radius between fixed concentric cylindrical walls.

It is significant that the flow examined in this paper has a saddle stagnation point for some rotlet positions as well as at least one matching pair of separation and reattachment points on its boundary for all rotlet positions. It is well known (see Guckenheimer and Holmes [5]) that a time-periodic perturbation of a steady flow possessing a saddle stagnation point can cause a homoclinic streamline, joining such a point to itself in the steady flow, to become a homoclinic tangle in the perturbed flow. Moreover, it has recently been shown (Yuster and Hackborn [20]) that a time-periodic perturbation of a steady Stokes flow possessing a matching pair of separation and reattachment points on its boundary can cause the heteroclinic streamline, joining these points in the steady flow, to become a heteroclinic tangle in the perturbed flow. Tangles of these kinds, corresponding to chaotic mixing regions, have been created in several Stokes flows (see, e.g., [10], [13]) by periodic motion of one or more walls. For the flow investigated here, chaotic mixing regions could be created by, for instance, a slow rotational oscillation of one of the bounding cylinders. Although turbulence is the most effective mechanism for mixing at small length scales, there are cases where chaotic mixing in a laminar
flow may be preferable. It has been noted, for example, by Smith [17] that the large-scale culture of mammalian cells is constrained by the high susceptibility of such cells to stress damage in the highly agitated environments common in turbulent fermentation reactors. It is possible that the results of this paper can assist in the design of efficient reactors that make optimal use of chaotic mixing.

2. Statement and solution of the problem. Let \((x, y)\) be Cartesian coordinates and \((r, \theta)\) the corresponding polar coordinates. The flow region lies between two fixed concentric circular cylinders, the center axis of which is perpendicular to the \((x, y)\) plane and passes through its origin; \(x\) and \(y\) are assumed to be dimensionless relative to the radius of the inner cylinder, and so the cylinders coincide with \(r = 1\) and \(r = a\), where \(a > 1\). The flow is driven by a line rotlet coinciding with \(x = c, y = 0\), with \(1 < c < a\); see Figure 1. A line rotlet is a singularity, mathematically identical to a line vortex of potential flow theory, that exerts a torque per unit length of magnitude \(2\Gamma \mu\), where \(\Gamma\) is the circulation around the rotlet and \(\mu\) the viscosity of the fluid.
The flow is represented by a stream function $\psi$, dimensionless relative to $\Gamma/2\pi$, in terms of which the components of the dimensionless fluid velocity in the directions of increasing $r$ and $\theta$ are $-(1/r)\partial\psi/\partial\theta$ and $\partial\psi/\partial r$, respectively. It is well known that the stream function for a Stokes flow satisfies $\nabla^4 \psi = 0$, the biharmonic equation. The presence of the line rotlet is expressed by

\[ (1) \quad \psi = \log R + \tilde{\psi}, \quad R = [(x - c)^2 + y^2]^{1/2}, \]

where $\tilde{\psi}$ is a function of class $C^4$ in the flow region $1 < r < a$. The condition that the flow velocity vanish on the cylindrical walls is satisfied if

\[ (2) \quad \psi = 0 \quad \text{at} \quad r = 1, \quad \psi = Q \quad \text{at} \quad r = a, \quad \frac{\partial \psi}{\partial r} = 0 \quad \text{at} \quad r = 1, a, \]

where $Q$, a constant to be determined, is the volume flux per unit depth in the direction of increasing $\theta$ across a surface on which $\theta$ is constant and $1 < r < a$. Note that the flow velocity is antisymmetric about the plane $\theta = 0$ and periodic (with period $2\pi$) in $\theta$; it follows that $\tilde{\psi}$ is even and periodic in $\theta$ and biharmonic. A suitable series representation for $\tilde{\psi}$ is

\[ (3) \quad \tilde{\psi} = A_0 + B_0 r^2 + C_0 \log r \\
+ (A_1 r + B_1 r^3 + C_1 r^{-1} + D_1 \log r) \cos \theta \\
+ \sum_{n=2}^{\infty} (A_n r^n + B_n r^{n+2} + C_n r^{-n} + D_n r^{2-n}) \cos n\theta, \]

where $A_n, B_n, C_n, D_n, n = 0, 1, 2, \ldots$, and $D_n, n = 1, 2, \ldots$, are constants to be determined. (An additional term, $D_0 r^2 \log r$, could have been included in (3), but it gives rise to a fluid pressure which is not periodic in $\theta$ unless $D_0 = 0$). Now, use of the Maclaurin series for $\log(1 - z)$ and the fact that $R^2 = (re^{i\theta} - c)(re^{-i\theta} - c)$ yields

\[ (4) \quad \log R = \begin{cases} 
\log c - \sum_{n=1}^{\infty} \frac{r^n}{nc^n} \cos n\theta & \text{if} \ 1 \leq r < c, \\
\log r - \sum_{n=1}^{\infty} \frac{c^n}{nr^n} \cos n\theta & \text{if} \ c < r \leq a. 
\end{cases} \]
On substituting (3) and (4) into (1) and applying the boundary conditions in (2), linear equations for the unknown constants can be obtained and solved, giving

\[ A_0 = \frac{1}{2} (a^2 - 1)^{-1} - \log c, \quad B_0 = -\frac{1}{2} (a^2 - 1)^{-1}, \]

\[ C_0 = (a^2 - 1)^{-1}, \]

\[ A_1 = \left[ (a^2 - 1)(a^2 - 1 + 2c^2) + 2(1 - a^4 - 2c^2) \log a \right] / \Delta_1, \]

\[ B_1 = \left[ 1 - a^2 + 2c^2 \log a \right] / \Delta_1, \]

\[ C_1 = \left[ (a^2 - 1)(a^2 - 2e^2) + 2c^2 \log a \right] / \Delta_1, \]

\[ D_1 = 2(a^2 - 1)(a^2 + 1 - 2c^2) / \Delta_1, \]

\[ A_n = \left[ (n+1)(1-a^{-2n})c^n + (n^2 - n^2a^2 + a^{-2n} - 1)c^{-n} \right] / n \Delta_n, \]

\[ B_n = \left[ (a^{-2n} - a^{-2})c^n + (n-1)(1-a^{-2})c^{-n} \right] / \Delta_n, \]

\[ C_n = \left[ (n^2 - n^2a^{-2} + a^{-2n} - 1)c^n + (n-1)(1-a^{2n})c^{-n} \right] / n \Delta_n, \]

\[ D_n = \left[ (n + 1)(a^{-2} - 1)c^n + (a^{2n} - a^{-2})c^{-n} \right] / \Delta_n, \]

where

\[ \Delta_1 = 2c(a^2 - 1)[a^2 - 1 - (a^2 + 1) \log a], \]

\[ \Delta_n = a^{2n} - n^2a^{-2} - 2 - n^2a^{-2} + a^{-2n}, \]

for \( n = 2, 3, \ldots \), and the net volume flux per unit depth between the cylindrical walls in the anticlockwise direction is

\[ Q = a^2(a^2 - 1)^{-1} \log a - \frac{1}{2} - \log c. \]

The solution for the stream function \( \psi \) is given by (1), (3) and (5)–(15). It can be shown that the general term of the series in (3) is bounded absolutely by \( K\lambda^n \), where \( K \) is a sufficiently large constant.
and \( \lambda = \max(cr/a^2, 1/cr) \). This indicates that the series in (3) generally converges quickly, leads to an error bound on its partial sums, and proves (via the Weierstrass test) that it converges uniformly in the region \( 1 \leq r \leq a \). Since similar reasoning proves that termwise partial derivatives of the series in (3) up to any order converge uniformly in the same region, it follows that the series can be differentiated termwise up to any order in this region and \( \tilde{\psi} \) is of class \( C^\infty \) there.

3. Flow description. Since the flow region is bounded (in a two-dimensional sense), all streamlines (except separatrices attached to either a wall or a saddle stagnation point) must be simple closed curves. A simple closed streamline that does not enclose the inner cylinder \( r = 1 \) must lie in a cellular separation region (also termed an eddy unless the velocity fails to exist at some point, such as a rotlet, inside the region). The simple closed streamlines, if any, that enclose the inner cylinder form a flow component which circulates around the inner cylinder with flux \( Q \) given by (16) above. This flow component, which is unique to all studies (known to the author) of Stokes flows driven by rotlets, exists if and only if \( Q \neq 0 \). It is clear from (16) that there is a unique value of \( c \) at which \( Q \) vanishes; denoting this value by \( c^*(a) \), (16) yields

\[
(17) \quad c^*(a) = \exp \left[ a^2(a^2 - 1)^{-1} \log a - \frac{1}{2} \right].
\]

It is easily shown that \( 1 < c^*(a) < a \) when \( a > 1 \). Moreover, \( c^*(a) = (1 + a)/2 \) within an error of \( O[(a - 1)^2] \) as \( a \to 1^+ \); hence, when the annular flow region is narrow, the rotlet position at which \( Q \) vanishes is roughly halfway between the cylindrical walls. Finally, note that \( Q > 0 \) (anticlockwise flow) when \( 1 < c < c^*(a) \) and \( Q < 0 \) (clockwise flow) when \( c^*(a) < c < a \).

The flow for the (typical) case \( a = 2 \) will now be examined in detail. The stream function \( \psi \) and the scalar vorticity \( \nabla^2 \psi \) are the principal tools used to investigate the flow. The level curves of the former are the streamlines of the flow; the latter indicates the flow direction near the boundary, and flow separation or reattachment occurs at points on
the boundary at which the vorticity vanishes. From (1) and (3),

\[\nabla^2 \psi = 4B_0 + (8B_1 r + 2D_1 r^{-1}) \cos \theta + 4 \sum_{n=2}^{\infty} [(n + 1)B_n r^n + (1 - n)D_n r^{-n}] \cos n\theta.\]

Figure 2 depicts significant streamlines of the flow for several values of \(c\) (and \(a = 2\)). For all \(c\) values in the interval (1, 2), the flow has a separation region containing the rotlet and centered at \(\theta = 0\). This separation region is attached to the inner wall \(r = 1\) when \(c \in (1, c^*(2))\), the outer wall \(r = 2\) when \(c \in (c^*(2), 2)\), and both walls when \(c = c^*(2) \approx 1.528361491\). The sequence of flow structures associated with increasing values of \(c\) in the interval \([1, c^*(2)]\) and that associated with decreasing values of \(c\) in the interval \([c^*(2), 2)\) are qualitatively similar but opposite in the sense of the wall, if any, to which corresponding separation regions adhere. For example, if \(c\) is slightly greater than 1, the separation region containing the rotlet is attached to the inner wall and an eddy is attached to the outer wall (see Figure 2(a)), but if \(c\) is slightly less than 2, the separation region containing the rotlet is attached to the outer wall and the eddy is attached to the inner wall. As \(c\) approaches \(c^*(2)\), starting at a value close to 1 or 2, the separation region containing the rotlet grows and the distance at \(\theta = 0\) between the separatrix and the wall bounding the eddy shrinks and eventually vanishes, splitting the eddy into two. A saddle stragnation point appears inside the eddy prior to its splitting (see Figure 2(b)); the region inside either of the two homoclinic separatrices attached to this point is termed a free eddy. As \(c\) continues to approach \(c^*(2)\), the two attached eddies resulting from the split move apart (see Figure 2(c)), and eventually two more pairs of attached eddies develop; the eddies in each newly developed pair are situated symmetrically with respect to the \(x\)-axis and, relative to the previously developed pair, are farther from the rotlet and attached to the opposite wall. The flow component associated with the flux \(Q\) narrows as \(c\) approaches \(c^*(2)\), following a winding route between the attached separation regions reminiscent of the flow through the gap between a cylinder and a plane (Davis and O’Neill [3]) as the gap narrows, and vanishes when \(c = c^*(2)\) (see Figure 2(d)).

Subtle changes in the flow structure for values of \(c\) very close to \(c^*(2)\) as \(c \to c^*(2)^-\) are illustrated (schematically, to emphasize small-
FIGURE 2. Significant streamlines (separatrices and one streamline, if any, enclosing the inner cylindrical wall) for $a = 2$ and several values of $c$. The dashed circle centered at the rotlet (indicated by $+$) represents the small, nearly circular streamlines that enclose the rotlet.

(a) $c = 1.1$.

(b) $c = 1.75$. 
(c) \( c = 1.5 \).

(d) \( c = c^*(2) \approx 1.528361491 \).
scale features) in Figure 3; the case in which \( c \to c^*(2)^+ \) is similar but opposite as mentioned above. Essentially, parts (a)–(c) depict the joining of the two attached eddies farthest from the rotlet in part (a) to form a single attached eddy containing two free eddies in part (c); part (b) shows the flow at the unique value of \( c \approx 1.528361476 \) at which three attached eddies meet at a single point. Part (d) shows a tiny eddy that subsequently emerges on the inner wall, and no further changes in flow structure occur prior to the eventual vanishing of the narrow channel conveying the flux \( Q \) when \( c = c^*(2) \) (see Figure 2(d); the tiny eddy on the inner wall is just visible). The bifurcation portrayed in the sequence (a)–(c) of Figure 3, involving a single free eddy in part (a) and a small attached eddy in parts (b) and (c), has rarely been observed in studies of Stokes flow separation (it was also observed in Hackborn [8]). This bifurcation can be seen as either a joining of two eddies or, when viewed in reverse order, a splitting of one eddy, but it differs qualitatively from the splitting of the eddy discussed earlier. The method used by Hajjam [11] to study the flow due to a rotating cylinder between fixed concentric cylindrical walls seems unable to capture subtle flow features like those shown in Figure 3. In fact, the ratio of the value of \( \psi \) on the outer wall to that on the inner wall was found in all cases considered in [11] to be a constant \( \neq 1 \), thus precluding the vanishing of the flux \( Q \); this suggests a flaw in the numerical method used in [11].

For \( a \neq 2 \), the flow structures that occur are generally similar to those for \( a = 2 \). However, the number of attached eddies in the flow for values of \( c \) close to \( c^*(a) \) tends to decrease as \( a \) increases. For example, there are seven attached eddies when \( a = 2 \) and \( c = c^*(2) \), but there are only three such eddies when \( a = 3 \) and \( c = c^*(3) \). For values of \( a \) slightly greater than 1 and \( c = c^*(a) \), the flow near the rotlet resembles the flow induced by a line rotlet (Hackborn [6]) or a rotating cylinder (Hellou and Coutanceau [12]) midway between fixed parallel planar walls, and the number of attached eddies in the flow increases without bound as \( a \to 1^+ \) with \( c = c^*(a) \). It is also noteworthy that, when \( a \) is slightly greater than 1 and \( c \neq c^*(a) \), the flow resembles the superposition of the flow due to a line rotlet and a Couette flow between fixed parallel planar walls.
FIGURE 3. Schematic sketch of significant streamlines for $a = 2$ with values of $c$ slightly less than $c^*(2)$ and increasing through parts (a) to (d). Parts (a) to (c) depict the joining of the two attached eddies in part (a) to form a single attached eddy containing two free eddies in part (c); in part (b), $c \approx 1.528361476$. Part (d) is similar to part (c) except for the tiny eddy attached to the inner wall.
REFERENCES


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