SOLVABILITY OF A NONLINEAR CONJUGATE EIGENVALUE PROBLEM

JOHN M. DAVIS, JOHNNY HENDERSON, K. RAJENDRA PRASAD AND WILLIAM K.C. YIN

ABSTRACT. We consider the nonlinear conjugate eigenvalue problem

\[ (-1)^{n-k} y^{(n)}(t) = \lambda a(t)f(y), \quad 0 \leq t \leq 1, \]
\[ y^{(i)}(0) = 0, \quad 0 \leq i \leq k - 1, \]
\[ y^{(j)}(1) = 0, \quad 0 \leq j \leq n - k - 1. \]

Values of the parameter \( \lambda \) are determined for which the problem above has a positive solution. The methods used here extend recent works by allowing for a broader class of functions for \( a(t) \). Optimal eigenvalue intervals are given for some relevant examples.

1. Introduction. We consider the nonlinear conjugate eigenvalue problem

\[ (-1)^{n-k} y^{(n)}(t) = \lambda a(t)f(y), \quad 0 \leq t \leq 1, \]
\[ y^{(i)}(0) = 0, \quad 0 \leq i \leq k - 1, \]
\[ y^{(j)}(1) = 0, \quad 0 \leq j \leq n - k - 1, \]

where \( k \) and \( n \) are fixed with \( 1 \leq k \leq n - 1 \). Our aim is to state intervals of eigenvalues, \( \lambda \), for which there exists at least one positive solution of (1), (2).

Boundary value problems (BVPs) of this form (when \( n = 2 \)) arise in the modeling of nonlinear diffusions via nonlinear sources, thermal ignition of gases, and in chemical concentrations in biological problems; see, for example, [5], [17], [19]. In these applied settings, only positive solutions are meaningful. Second order problems of this type were also

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FIGURE 1. The “tent” function $h(t)$ for $n = 5$ and $k = 1, 2, 3, 4$. 
dealt with by Fink, Gatica, and Hernandez [11] in modeling the one-dimensional case of the Dirichlet problem when \( a(t) \) satisfies certain integrability conditions.

However, higher order problems have received attention as well. Two-point problems for the fourth order equation

\[
u^{(4)}(t) - p(t)q(u) = 0
\]

subject to various boundary conditions arise in describing the deformations of an elastic beam. We refer the reader to [12], [14], [20] and the references therein for details on beam equations and [18], [20] for other higher order problems.

Eigenvalue problems similar to (1), (2) have often been tackled using methods arising from nonlinear elliptic problems in annular regions [3], [4], [5], [12]. Along these lines, Erbe and Wang [10] first dealt with the existence of a positive solution assuming \( f \) is either superlinear or sublinear.

The existence of positive solutions for eigenvalue problems has been studied by many authors. An excellent resource for a comprehensive treatment of these problems is the very recent book by Agarwal, O’Regan, and Wong [2] and its extensive bibliography. Although several authors [7], [8], [9], [10], [18] have presented eigenvalue interval theorems, each requires that \( a(t) \) is nonvanishing on \([0, 1] \). The main contribution of this work is that we can develop optimal eigenvalue intervals even if \( a(t) \) vanishes on some subinterval(s) of \([0, 1] \). Also, unlike [1], [7], [10], [15], we do not restrict ourselves to the superlinear or sublinear cases (although our theorems handle those as well).

Jiang and Liu [15] consider (1), (2) with \( \lambda = 1 \) and use a novel method of bounding the Green’s function above and below by “tent” functions. (See Figure 1.) Eloe and Henderson’s often-used bounds [6] originated from tent functions but concentrate mainly on providing lower bounds on positive concave functions as a function of their maximum rather than in terms of the tent functions themselves. As it turns out—at least in this application—using the actual tent functions for bounds provides sharper results. Graef and Yang [13] also implement the tent function bounding technique for second order conjugate and right focal problems.

We make the following assumptions throughout:
(A1) $a(t)$ is a nonnegative, measurable function defined on $[0, 1]$ and satisfies

$$0 < \int_0^1 h(s)a(s) \, ds < \infty$$

where $h(t) = \min\{t^{n-k}, (1-t)^k\}$.

(A2) $f : [0, \infty) \to [0, \infty)$ is continuous and

$$f_0 = \lim_{x \to 0^+} \frac{f(x)}{x} \quad \text{and} \quad f_\infty = \lim_{x \to \infty} \frac{f(x)}{x}$$

both exist.

We note that (A1) allows for $a(t) \equiv 0$ on some subinterval(s) of $[0, 1]$ and allows $a(t)$ to have a singularity at $t = 0$ and/or $t = 1$. Jiang and Liu [15] provide

$$a(t) = t^{-\alpha}(1-t)^{-\beta}(|\cos 2\pi t| + \cos 2\pi t)$$

as such an example. Note that (3) is satisfied if $\alpha \in (0, n+1-k)$ and $\beta \in (0, k+1)$. Assumption (A1) is important because it admits a much larger class of functions than those allowed in [7], [8], [9], [10], [18].

This is the inequality due to Jiang and Liu [15].

**Theorem 1.** Let $G(t, s)$ be the Green’s function for the homogeneous problem

$$(-1)^{n-k}y^{(n)}(t) = 0$$

satisfying the boundary conditions (2). Define

$$g(t) = \min\{t^k, (1-t)^{n-k}\}.$$

Then there exist numbers $B > 0$ and $0 < \gamma < 1$ such that

$$\gamma B g(t) h(s) \leq G(t, s) \leq B h(s), \quad (t, s) \in [0, 1] \times [0, 1].$$

**Theorem 2** (Krasnosel’kii) [16]. Let $E$ be a Banach space, $K \subseteq E$ be a cone, and suppose that $\Omega_1$, $\Omega_2$ are open subsets of $E$ with $0 \in \Omega_1$.
and \( \Omega_1 \subset \Omega_2 \). Suppose further that \( A : K \cap (\Omega_2 \setminus \Omega_1) \to K \) is a completely continuous operator such that either

(i) \( \|Au\| \leq \|u\|, \ u \in K \cap \partial \Omega_1 \) and \( \|Au\| \geq \|u\|, \ u \in K \cap \partial \Omega_2 \), or

(ii) \( \|Au\| \geq \|u\|, \ u \in K \cap \partial \Omega_1 \) and \( \|Au\| \leq \|u\|, \ u \in K \cap \partial \Omega_2 \)

holds. Then \( A \) has a fixed point in \( K \cap (\Omega_2 \setminus \Omega_1) \).

2. Main results. Let \( B \) denote the Banach space \( C[0,1] \) with the norm \( \|x\| = \sup_{t \in [0,1]} |x(t)| \). Define the cone \( P \subset B \) by

\[
P = \{ x \in B : x(t) \geq \gamma g(t) \|x\|, \ t \in [0,1] \}.
\]

Let \( \delta \in (0, 1/2) \) be chosen such that

\[
(5) \quad \int_{\delta}^{1-\delta} a(s)h(s) \, ds > 0,
\]

and for \( \tau \in [\delta, 1-\delta] \), define

\[
g(\tau) = \max_{t \in [\delta,1-\delta]} g(t).
\]

We are now ready to state our first theorem which establishes an open interval of values of \( \lambda \) for which the BVP (1), (2) has a positive solution.

**Theorem 3.** Suppose (A1) and (A2) hold, and let \( m = \max\{k, n-k\} \). Then for each \( \lambda \) satisfying

\[
(6) \quad \frac{1}{(B\gamma^2 g(\tau)\delta^m \int_{\delta}^{1-\delta} h(s)a(s) \, ds) f_{\infty}} < \lambda < \frac{1}{(B \int_{0}^{1} h(s)a(s) \, ds) f_{0}},
\]

there exists at least one solution of (1), (2) in \( P \).

**Proof.** Let \( \lambda \) be given as in (6) and let \( \varepsilon > 0 \) be such that

\[
(7) \quad \frac{1}{(B\gamma^2 g(\tau)\delta^m \int_{\delta}^{1-\delta} h(s)a(s) \, ds)(f_{\infty}-\varepsilon)} \leq \lambda \leq \frac{1}{(B \int_{0}^{1} h(s)a(s) \, ds)(f_{0}+\varepsilon)}.
\]
We note that \( y(t) \) is a solution of (1), (2) if and only if
\[
y(t) = \lambda \int_0^1 G(t,s)a(s)f(y(s)) \, ds, \quad 0 \leq t \leq 1.
\]
Motivated by this, we define the operator \( T : \mathcal{P} \to \mathcal{B} \) by
\[
(Ty)(t) = \lambda \int_0^1 G(t,s)a(s)f(y(s)) \, ds, \quad y \in \mathcal{P}.
\]
We seek a fixed point of \( T \) in \( \mathcal{P} \). We prove this by showing the conditions in Theorem 2 hold.

First, if \( y \in \mathcal{P} \), then
\[
(Ty)(t) = \lambda \int_0^1 G(t,s)a(s)f(y(s)) \, ds \leq \lambda B \int_0^1 h(s)a(s)f(y(s)) \, ds
\]
so that \( ||Ty|| \leq \lambda B \int_0^1 h(s)a(s)f(y(s)) \, ds \). Next, if \( y \in \mathcal{P} \), then by (4) and the inequality above we see
\[
(Ty)(t) = \lambda \int_0^1 G(t,s)a(s)f(y(s)) \, ds \geq \lambda B g(t) \gamma \int_0^1 h(s)a(s)f(y(s)) \, ds \geq g(t) \gamma ||Ty||.
\]
Hence, \( T : \mathcal{P} \to \mathcal{P} \). Standard arguments involving the Arzela-Ascoli theorem show that \( T \) is completely continuous.

We begin with \( f_0 \). By the definition of \( f_0 \), there exists an \( H_1 > 0 \) such that \( f(x) \leq (f_0 + \varepsilon)x \) for \( 0 < x \leq H_1 \). Choose \( y \in \mathcal{P} \) with \( ||y|| = H_1 \).
Using (4) we have
\[
(Ty)(t) = \lambda \int_0^1 G(t, s)a(s)f(y(s)) \, ds \\
\leq \lambda B \int_1^0 h(s)a(s)f(y(s)) \, ds \\
\leq \lambda B \int_1^0 h(s)a(s)(f_0 + \varepsilon)y(s) \, ds \\
\leq \lambda B \int_1^0 h(s)a(s)(f_0 + \varepsilon)||y|| \, ds \\
\leq ||y||.
\]

The last inequality follows from the right side of (7). Therefore \(||Ty|| \leq ||y||\). So, if we set
\[
\Omega_1 = \{x \in B : ||x|| < H_1\},
\]
then
\[
||Ty|| \leq ||y||, \quad y \in \mathcal{P} \cap \partial \Omega_1.
\]

Now we turn our attention to \(f_\infty\). By the definition of \(f_\infty\), there exists an \(H_2 > H_1\) such that \(f(x) \geq (f_\infty - \varepsilon)x\) for \(x \geq H_2\).

If \(y \in \mathcal{P}\) with \(||y|| = H_2\), then for \(t \in [\delta, 1 - \delta]\), we have
\[
y(t) \geq \gamma g(t)||y|| = \gamma g(t)H_2 \geq \gamma H_2 \delta^m.
\]

Using (9), we get
\[
(Ty)(\tau) = \lambda \int_0^1 G(\tau, s)a(s)f(y(s)) \, ds \\
\geq \lambda \int_0^{1-\delta} G(\tau, s)a(s)f(y(s)) \, ds \\
\geq \gamma g(\tau)\lambda B \int_0^{1-\delta} h(s)a(s)(f_\infty - \varepsilon)y(s) \, ds
\]

\[ \geq \gamma g(\tau) \lambda B \int_{\delta}^{1-\delta} h(s) a(s) (f_\infty - \varepsilon) \gamma H_2 \delta^m ds \]
\[ = \gamma^2 g(\tau) \lambda B \delta^m (f_\infty - \varepsilon) H_2 \int_{\delta}^{1-\delta} h(s) a(s) ds \]
\[ \geq H_2 \]
\[ = ||y||. \]

(The next to last inequality follows from the left side of (7)). If we define
\[ \Omega_2 = \{ x \in B : ||x|| < H_2 \}, \]
then we have shown that
\[ ||Ty|| \geq ||y||, \quad y \in \mathcal{P} \cap \partial \Omega_2. \]

An application of Theorem 2 yields the conclusion of our theorem.

\[ \square \]

Likewise, we can use the latter part of Theorem 2 to obtain the following result.

**Theorem 4.** Suppose (A1) and (A2) hold, and let \( m = \max\{k, n - k\} \). Then for each \( \lambda \) satisfying

\[ \frac{1}{(B \gamma^2 g(\tau) \delta^m \int_{\delta}^{1-\delta} h(s) a(s) ds f_0 - \eta)} < \lambda < \frac{1}{(B \int_{0}^1 h(s) a(s) ds f_\infty + \eta)}, \]

there exists at least one solution of (1), (2) in \( \mathcal{P} \).

**Proof.** Let \( \lambda \) be given as in (10), and let \( \eta > 0 \) be such that

\[ \frac{1}{(B \gamma^2 g(\tau) \delta^m \int_{\delta}^{1-\delta} h(s) a(s) ds (f_0 - \eta)} \leq \lambda \leq \frac{1}{(B \int_{0}^1 h(s) a(s) ds (f_\infty + \eta)}. \]

Let \( T \) be the cone preserving, completely continuous operator defined in (8).
Beginning with $f_0$, there exists an $H_1 > 0$ such that $f(x) \geq (f_0 - \eta)x$ for $0 < x \leq H_1$. Choose $y \in \mathcal{P}$ with $|y| = H_1$. Note that for $t \in [\delta, 1 - \delta]$,

$$y(t) \geq \gamma g(t)|y| = \gamma g(t)H_1 \geq \gamma H_1 \delta^m. \tag{12}$$

Using (12) and the left side of (11), we obtain

$$(Ty)(\tau) = \lambda \int_0^1 G(\tau, s)a(s)f(y(s)) \, ds$$

$$\geq \lambda \int_{\delta}^{1-\delta} G(\tau, s)a(s)f(y(s)) \, ds$$

$$\geq \lambda \int_{\delta}^{1-\delta} G(\tau, s)a(s)(f_0 - \eta)y(s) \, ds$$

$$\geq \gamma g(\tau)\lambda B \int_{\delta}^{1-\delta} h(s)a(s)(f_0 - \eta)\gamma H_1 \delta^m \, ds$$

$$= \gamma^2 g(\tau)\lambda B \delta^m (f_0 - \eta)H_1 \int_{\delta}^{1-\delta} h(s)a(s) \, ds$$

$$\geq H_1$$

$$= |y|.$$

Thus $||Ty|| \geq ||y||$. So, if we define

$$\Omega_1 = \{x \in \mathcal{B} : |x| < H_1\},$$

then we have shown

$$||Ty|| \geq ||y||, \quad y \in \mathcal{P} \cap \partial \Omega_1.$$

Now we turn our attention to $f_\infty$. By the definition of $f_\infty$, there exists an $\overline{H}_2 > H_1$ such that $f(x) \leq (f_\infty + \eta)x$ for $x \geq \overline{H}_2$. We have two cases: when $f$ is bounded and when $f$ is unbounded.

First, suppose $f$ is bounded. Then there exists $N > 0$ such that $f(x) \leq N$ for all $0 < x < \infty$. Let

$$H_2 = \max \left\{ 2\overline{H}_2, NB\lambda \int_0^1 h(s)a(s) \, ds \right\}.$$
If \( y \in \mathcal{P} \) with \( ||y|| = H_2 \), then we have

\[
(Ty)(t) = \lambda \int_0^1 G(t, s)a(s)f(y(s)) \, ds
\]

\[
\leq N\lambda \int_0^1 G(t, s)a(s) \, ds
\]

\[
\leq N\lambda B \int_0^1 h(s)a(s) \, ds
\]

\[
\leq H_2
\]

\[
= ||y||.
\]

Thus \( ||Ty|| \leq ||y|| \). So, if we define

\[
\Omega_2 = \{ x \in \mathcal{B} : ||x|| < H_2 \},
\]

then we have shown

\[
||Ty|| \leq ||y||, \quad y \in \mathcal{P} \cap \partial \Omega_2.
\]

Now suppose \( f \) is unbounded. Let \( H_2 = \max\{2H_1, \overline{H_2}\} \) be such that \( f(x) \leq f(H_2) \) for \( 0 < x \leq H_2 \). If \( y \in \mathcal{P} \) with \( ||y|| = H_2 \), then we have (using the right side of (11)),

\[
(Ty)(t) = \lambda \int_0^1 G(t, s)a(s)f(y(s)) \, ds
\]

\[
\leq \lambda \int_0^1 Bh(s)a(s)f(y(s)) \, ds
\]

\[
\leq \lambda B \int_0^1 h(s)a(s) f(H_2) \, ds
\]

\[
\leq \lambda B \int_0^1 h(s)a(s)(f_{\infty} + \eta)H_2 \, ds
\]

\[
\leq H_2
\]

\[
= ||y||.
\]

Thus \( ||Ty|| \leq ||y|| \). So, if we define

\[
\Omega_2 = \{ x \in \mathcal{B} : ||x|| < H_2 \},
\]
then we have shown
\[ ||Ty|| \leq ||y||, \quad y \in \mathcal{P} \cap \partial \Omega_2. \]

An application of Theorem 2 yields the conclusion of our theorem. \( \square \)

3. Examples.

**Example 1.** Consider the second order problem
\[
-\gamma''(t) = \lambda ty(2000e^{-\gamma} + 1), \quad 0 \leq t \leq 1,
\]
\[ y(0) = y(1) = 0. \]

We are taking \( a(t) = t \) and \( f(y) = y(2000e^{-\gamma} + 1) \) so \( f_0 = 2001 \) and \( f_\infty = 1. \) Recalling (4), it is easy to see that \( B = 1 \) is optimal (for the second order case) and we can select any \( 0 < \gamma < 1 \) with this choice of \( B. \) Also, for the second order problem, \( g(\tau) = g(1/2) = (1/2) \) and \( m = 1. \) If we pick \( \gamma = 0.5 \) and \( \delta = 0.1, \) Theorem 4 says that for any
\[ 0.333167 < \lambda < 8, \]

the BVP (13) has at least one positive solution. (Theorem 3 is vacuous in this case so we appeal to Theorem 4.) Since \( B \) is optimal and the right endpoint of this interval of eigenvalues depends only on \( B, \) \( 8 \) is the least upper bound of \( \lambda \) for which we can conclude this problem is solvable.

Based on Example 1, it is natural to ask how one might choose \( B, \) \( \gamma, \) and \( \delta \) in general. First, \( B \) depends only on the Green’s function for the homogeneous problem; once \( n \) is specified, it is easy to choose an optimal \( B. \) As we said, for any second order problem, \( B = 1 \) is best. Clearly, if \( \gamma \) is close to 1, then the left endpoint of the eigenvalue interval will be smaller. We will denote this optimal parameter value by \( \gamma^* \approx 1. \)

But how do we choose \( \delta? \) Of course, we must keep (3) in mind. In Example 1, \( a(t) = t \) and so any \( 0 < \delta < (1/2) \) can be selected. To choose an optimal \( \delta, \) which we will denote by \( \delta^*, \) we just need to
minimize the left side of (10) as a function of \( \delta \). To this end, define

\[
L(\delta) = \frac{1}{(B\gamma^2 g(\tau) \delta^m \int_0^{1-\delta} h(s) a(s) \, ds) f_0}
\]

\[
= \frac{1}{B\gamma^2 g(\tau) \delta^m f_0 (\int_0^{1/2} s \cdot s \, ds + \int_{1/2}^{1-\delta} (1-s) \, ds)}
\]

\[
= \frac{1}{B\gamma^2 g(\tau) \delta^m f_0} \cdot \frac{1}{\delta - 4\delta^3}
\]

\[
= \frac{0.00815835}{\delta - 4\delta^3}.
\]

Minimizing \( L(\delta) \) for \( 0 < \delta < \frac{1}{2} \), we see that \( \delta^* = 0.288675 \). Using \( \gamma^* = 0.99 \) and this \( \delta^* \), Theorem 4 gives us the optimal eigenvalue interval

\[
0.042392 < \lambda < 8
\]

for which (13) has a positive solution.

**Example 2.** Consider

\[
-\frac{y''(t)}{t} = \lambda \frac{1}{t} y(2000e^{-y} + 1), \quad 0 < t < 1,
\]

\[
y(0) = y(1) = 0.
\]

As previously stated, we take \( B = 1 \) and \( \gamma^* \approx 1 \). To find \( \delta^* \), we want to minimize \( L(\delta) \) defined in (14) for \( 0 < \delta < (1/2) \). We find

\[
L(\delta) = \frac{1}{(B\gamma^2 g(\tau) \delta^m \int_0^{1-\delta} h(s) a(s) \, ds) f_0} \cdot \frac{1}{\delta - 4\delta^3}
\]

\[
= \frac{0.00101979}{\log(2 - 2\delta)}.
\]

Hence \( \delta^* = 0.272633 \) and Theorem 4 yields the optimal eigenvalue interval

\[
0.0345612 < \lambda < 1.4427
\]

for which (15) has a positive solution.
In order to emphasize the flexibility that assumption (A1) allows, we want to look at an example posed by Jiang and Liu [15] where $a(t)$ vanishes on a subinterval of $[0, 1]$.

**Example 3.** Consider

\begin{equation}
- y''(t) = \lambda a(t) y(2000e^{-y} + 1), \quad 0 \leq t \leq 1,
\end{equation}

\begin{equation*}
 y(0) = y(1) = 0,
\end{equation*}

where

\begin{equation*}
a(t) = t(1 - t)(|\cos 2\pi t| + \cos 2\pi t).
\end{equation*}

We have already discussed choosing $B = 1$ and $\gamma^* = 0.99$. From (5) and Figure 2, this time $0 < \delta \leq 1/4$. Again, we want to minimize $L(\delta)$ defined in (14). We generated the graph of $L(\delta)$ and $L'(\delta)$ for $0 < \delta \leq 1/4$ with *Mathematica*. See Figure 3. Using the numerical *FindRoot* command on $L'(\delta)$, we found $\delta^* = 0.1216320932$. Employing Theorem 4 we get the optimal eigenvalue interval

\begin{equation*}
1.83695 < \lambda < 156.747.
\end{equation*}

for which (16) has a positive solution.
FIGURE 3. Minimizing $L(\delta)$ on $(0, 1/4]$ with Mathematica.
Remark. For a given \( a(t) \) and \( f(y) \), one of Theorem 3 or Theorem 4 will be vacuous. In fact, if \( f_0 \leq f_\infty \), then Theorem 3 should be used and if \( f_\infty \leq f_0 \), then Theorem 4 should be used.

Remark. In the case when \( f \) is either superlinear (i.e., \( f_0 = 0 \) and \( f_\infty = \infty \)) or sublinear (i.e., \( f_0 = \infty \) and \( f_\infty = 0 \)), Theorems 3 and 4 say that there exists a positive solution for any positive \( \lambda \).

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Department of Mathematics, 336-K Sid Richardson Bldg., Baylor University, Waco, TX 76798 USA
E-mail address: John_M_Davis@baylor.edu

Department of Mathematics, Auburn University, Auburn, AL 36849 USA
E-mail address: hendej2@mail.auburn.edu

Department of Mathematics, Auburn University, Auburn, AL 36849 USA and Department of Applied Mathematics, Andhra University, Visakhapatnam 530003 INDIA
E-mail address: rajendra92@hotmail.com

Department of Mathematics, LaGrange College, LaGrange, GA 30240 USA
E-mail address: wyin@lgc.edu