EXISTENCE-UNIQUENESS OF SOLUTIONS FOR A NONLINEAR NONAUTONOMOUS SIZE-STRUCTURED POPULATION MODEL: AN UPPER-LOWER SOLUTION APPROACH

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ABSTRACT. Existence and uniqueness of weak solutions to a nonlinear nonautonomous size-structured population model are established using the upper-lower solution technique and a comparison principle.

1. Introduction. In this paper we study the following nonlinear and nonlocal first order hyperbolic initial boundary value problem that describes the dynamics of a size-structured population:

\begin{align}
\frac{du}{dt} + (g(t, x)u)_{x} &= -m(t, x, P(u(t, \cdot)))u, \quad 0 < t < T, \; a < x < b, \\
g(t, a)u(t, a) &= C(t) + \int_{a}^{b} q(t, x)u(t, x) \, dx, \quad 0 < t < T, \\
u(0, x) &= u_0(x), \quad a \leq x \leq b,
\end{align}

where \( P(u(t, \cdot)) = \int_{a}^{b} \eta(y)u(t, y) \, dy \). The function \( u(t, x) \) in problem (1.1) represents the density of individuals in the size class \([x, x + dx)\) at time \( t \). The parameters \( q(t, x) \) and \( g(t, x) \) are the time and size-dependent reproduction and growth rates, respectively. The function \( m(t, x, P) \) represents the mortality rate of an individual of size \( x \) at time \( t \) which depends on the population measure, \( P \), and \( C(t) \) represents the inflow of \( a \)-size individuals from an external source (e.g., seeds carried by wind).

Linear and nonlinear problems similar to (1.1) have been discussed extensively in the literature. Existence-uniqueness results have been

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A.S. ACKLEH AND K. DENG established using the characteristic method with fixed point argument or the semi-group of linear operators theoretic approach (e.g., [4], [5], [6]). Recently, in [2], [3], we developed a different approach to the study of existence and uniqueness of solutions to linear and nonlinear nonautonomous size-structured models similar to (1.1). This approach is based on the construction of monotone sequences of upper and lower solutions. As is well known, such a monotone approximation heavily relies on a comparison principle. In [1] we proved that a comparison principle works for the case $m_P \leq 0$. Moreover, we showed via a counter example that, without the restriction $m_P \leq 0$, the comparison principle fails. To overcome this difficulty, in [3], we introduced a new definition of coupled upper and lower solutions to extend this method to the case $m_P > 0$. However, therein we considered only the case $P(u(t, \cdot)) = \int_a^b u(t, x) \, dx$, i.e., $\eta(x) \equiv 1$. The goal of this paper then is twofold. On the one hand, we will extend the results of [3] to the case where no restriction on the sign of $m_P$ is imposed. On the other hand, we will consider a general function $\eta(x)$. This allows the theory to hold, for example, for the total population mass $P = \int_a^b yu(t, y) \, dy$. It is worth mentioning that the monotone sequences developed in [3] do not work for the present case. Hence, in this paper we introduce a new definition of upper and lower solutions. Based on this definition, we are able to establish a comparison result and thus construct monotone sequences of upper and lower solutions which will lead to the existence of solutions by passing to the limit.

We organize our paper as follows. In Section 2 we establish a comparison result and show the uniqueness of the weak solution to problem (1.1). In Section 3 we then construct two monotone sequences of upper and lower solutions and show their convergence to the weak solution of (1.1).

2. Comparison and uniqueness results. Throughout the discussion we assume that the parameters in (1.1) satisfy the following:

(A1) $g(t, x)$ is continuously differentiable with respect to $t$ and $x$. Furthermore, $g(t, x) > 0$ for $(t, x) \in [0, T] \times [a, b)$ and $g(t, b) = 0$ for $t \in [0, T]$.

(A2) $q(t, x) (\geq 0)$ is continuous with respect to $t$ and $x$ for $(t, x) \in [0, T] \times [a, b]$. 


(A3) \( m(t, x, P) \geq 0 \) is continuous with respect to \( t \) and \( x \) and continuously differentiable with respect to \( P \) for \((t, x, P) \in [0, T] \times [a, b] \times [0, \infty)\). In addition, there exists a constant \( M > 0 \) such that \( M + m_P(t, x, P) \geq 0 \).

(A4) \( C(t) \geq 0 \) is continuous for \( t \in [0, T] \).

(A5) \( \eta \in L^\infty(a, b) \) and \( \eta \geq 0 \) almost everywhere in \((a, b)\).

(A6) \( u_0 \in L^\infty(a, b) \) and \( u_0 \geq 0 \) almost everywhere in \((a, b)\).

Note that \( g(t, b) = 0 \) in (A1) is a natural assumption from the biological point of view. It implies that the growth of individuals ceases at size \( b \) (see, e.g., [4], [5], [6]). Furthermore, from a mathematical point of view, the following comparison result cannot be established without this assumption.

Let \( D_T = (0, T) \times (a, b) \), and we introduce the following definition of a pair of coupled upper and lower solutions of problem (1.1).

**Definition 2.1.** A pair of functions \( u(t, x) \) and \( v(t, x) \) are called an upper and a lower solution of (1.1) on \( D_T \), respectively, if all the following hold:

(i) \( u, v \in L^\infty(D_T) \).

(ii) \( u(0, x) \geq u_0(x) \geq v(0, x) \) almost everywhere in \((a, b)\).

(iii) For every \( t \in (0, T) \) and every nonnegative \( \xi \in C^1(D_T) \),

\[
\int_a^b u(t, x) \xi(t, x) \, dx \geq \int_a^b u(0, x) \xi(0, x) \, dx \\
+ \int_0^t \xi(s, a) \left( C(s) + \int_a^b q(s, x) u(s, x) \, dx \right) \, ds \\
+ \int_0^t \int_a^b [\xi_x(s, x) + g(s, x) \xi_x(s, x)] u(s, x) \, dx \, ds \\
- \int_0^t \int_a^b \xi(s, x) [m(s, x, P(v(s, \cdot))) \\
+ M(P(v(s, \cdot)) - MP(u(s, \cdot))] u(s, x) \, dx \, ds
\]
Theorem 2.2. Suppose that (A1)–(A6) hold. Let \( u \) and \( v \) be a nonnegative upper solution and a nonnegative lower solution of (1.1), respectively. Then \( u \geq v \) almost everywhere in \( D_T \).

Proof. Let \( w = v - u \). Then \( w \) satisfies

\[
(2.3) \quad w(0, x) = v(0, x) - u(0, x) \leq 0 \quad \text{a.e. in} \quad (a, b),
\]

and

\[
(2.4) \quad \int_a^b w(t, x)\xi(t, x) \, dx
\]

\[
\leq \int_a^b w(0, x)\xi(0, x) \, dx + \int_0^t \int_a^b q(s, x)w(s, x) \, dx \, ds
\]

A function \( u(t, x) \) is called a weak solution of (1.1) on \( D_T \) if \( u \) satisfies (2.1) with \( \geq \) replaced by \( = \) and \( P(v(s, \cdot)) \) by \( P(u(s, \cdot)) \). Such a weak solution definition can be formally derived from multiplying (1.1) by \( \xi \) and integrating the resulting equation by parts. Conversely, if a weak solution with enough regularity exists, then one can show that it also satisfies (1.1) in the classical sense.
\begin{align*}
&+ \int_0^t \int_a^b [\xi_s(s,x) + g(s,x)\xi_x(s,x)] w(s,x) \, dx \, ds \\
&- \int_0^t \int_a^b \xi(s,x) m(s,x, P(u(s, \cdot))) w(s,x) \, dx \, ds \\
&+ \int_0^t \int_a^b \xi(s,x) u(s,x)(M + A(s,x)) \int_a^b \eta(y) w(s,y) \, dy \, dx \, ds \\
&+ \int_0^t \int_a^b \xi(s,x) Mv(s,x) \int_a^b \eta(y) w(s,y) \, dy \, dx \, ds,
\end{align*}

where \( A(t,x) = m_P(t,x, \theta(t)) \) with \( \theta(t) \) between \( P(u(t, \cdot)) \) and \( P(v(t, \cdot)) \).

Following [1], we let \( \xi(t,x) = e^{\lambda t} \zeta(t,x) \) where \( \zeta \in C^1(D_T) \) and \( \lambda > 0 \) is chosen so that \( \lambda - m \geq 0 \) on \( D_T \times [P, \overline{P}] \) where \( P = \min \{ \inf_{[0,T]} P(u(t, \cdot)), \inf_{[0,T]} P(v(t, \cdot)) \} \) and \( \overline{P} = \max \{ \sup_{[0,T]} P(u(t, \cdot)), \sup_{[0,T]} P(v(t, \cdot)) \} \). Then we find

\begin{equation}
\int_a^b w(t,x) \zeta(t,x) \, dx 
\end{equation}

\begin{align*}
&\leq \int_a^b w(0,x) \zeta(0,x) \, dx + \int_0^t \int_a^b e^{\lambda s} \zeta(s,a) \int_a^b q(s,x) w(s,x) \, dx \, ds \\
&+ \int_0^t \int_a^b w(s,x) e^{\lambda s} [\xi_s(s,x) + g(s,x)\xi_x(s,x)] \, dx \, ds \\
&+ \int_0^t \int_a^b e^{\lambda s} \zeta(s,x)(\lambda - m)w(s,x) \, dx \, ds \\
&+ \int_0^t \int_a^b e^{\lambda s} \zeta(s,x) u(s,x)(M + A) \int_a^b \eta(y) w(s,y) \, dy \, dx \, ds \\
&+ \int_0^t \int_a^b e^{\lambda s} \zeta(s,x) Mv(s,x) \int_a^b \eta(y) w(s,y) \, dy \, dx \, ds.
\end{align*}

We now set up a backward problem as follows:

\begin{equation}
\zeta_s + g \zeta_x = 0, \quad 0 \leq s \leq t, \quad a < x < b,
\end{equation}

\begin{equation}
\zeta(s,b) = 0, \quad 0 < s \leq t,
\end{equation}

\begin{equation}
\zeta(t,x) = \varphi(x), \quad a \leq x \leq b.
\end{equation}

Here \( \varphi(x) \in C^\infty_0(a,b), \ 0 \leq \varphi \leq 1. \)
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The existence of $\zeta \in C^1(D_T)$ follows from the fact that by the variable change $\tau = t - s$, (2.6) can be written into

$$
\begin{align*}
\zeta_\tau - g\zeta_x &= 0, \quad 0 < \tau < t, \quad a < x < b, \\
\zeta(\tau, b) &= 0, \quad 0 < \tau < t, \\
\zeta(0, x) &= \varphi(x), \quad a \leq x \leq b,
\end{align*}
$$

and thus (2.7) can be solved by the characteristic method. Note that the initial and boundary values for $\zeta$ imply that $0 \leq \zeta \leq 1$ on $D_T$.

Substituting such a $\zeta$ in (2.5) yields

$$
\int_a^b w(t, x) \varphi(x) \, dx \leq \int_a^b w(0, x)^+ \, dx + \nu \int_0^t \int_a^b w(s, x)^+ \, dx \, ds,
$$

where $w(t, x)^+ = \max\{w(t, x), 0\}$ and $\nu = \max_{D_T^c}\{q(t, x) + (\lambda - m) + (b - a)|u(t, x)(M + A(t, x)) + Mv(t, x)|\eta(x)\}$. From the condition on initial data in (2.3), we then have

$$
\int_a^b w(t, x) \varphi(x) \, dx \leq \nu \int_0^t \int_a^b w(s, x)^+ \, dx \, ds.
$$

Since this inequality holds for every $\varphi \in C_0^\infty(a, b)$ with $0 \leq \varphi \leq 1$, we can choose a sequence $\{\varphi_n\}$ on $(a, b)$ converging to

$$
\chi = \begin{cases} 
1 & \text{if } w(t, x) > 0, \\
0 & \text{otherwise.}
\end{cases}
$$

Consequently, we find that

$$
\int_a^b w(t, x)^+ \, dx \leq \nu \int_0^t \int_a^b w(s, x)^+ \, dx \, ds,
$$

which by Gronwall’s inequality leads to

$$
\int_a^b w(t, x)^+ \, dx = 0.
$$

Thus, the proof is completed.
Remark 1. From the proof of Theorem 2.2, it easily follows that for any function \( w \in L^\infty(D_T) \) if \( w(0, x) \leq 0 \) almost everywhere in \((a, b)\) and the following inequality holds for every nonnegative \( \xi \in C^1(D_T) \),

\[
\int_a^b w(t, x)\xi(t, x) \, dx \\
\leq \int_a^b w(0, x)\xi(0, x) \, dx \\
+ \int_0^t \xi(s, a) \int_a^b q(s, x)w(s, x) \, dx \, ds \\
+ \int_0^t \int_a^b [\xi_x(s, x) + g(s, x)\xi_x(s, x)]w(s, x) \, dx \, ds \\
- \int_0^t \int_a^b \xi(s, x)F(s, x)w(s, x) \, dx \, ds
\]

with \( F \in L^\infty(D_T) \), then \( w(t, x) \leq 0 \) almost everywhere in \( D_T \). Such a result will be used in Section 3.

The following uniqueness result easily follows from Theorem 2.2.

**Corollary 2.3.** If \( u(t, x) \) is a nonnegative weak solution of problem (1.1) and if \( P(u(t, \cdot)) \in C[0, T] \), then \( u \) is unique.

*Proof.* Suppose that \( u_1(t, x) \) and \( u_2(t, x) \) are two nonnegative solutions of (1.1). Clearly, if \( P(u_1(t, \cdot)) = P(u_2(t, \cdot)) \) for \( 0 \leq t \leq T \), \( u_1(t, x) \equiv u_2(t, x) \). For that reason, we may assume that \( P(u_1(t, \cdot)) = P(u_2(t, \cdot)) \) for \( 0 \leq t \leq t_0 \) and \( P(u_1(t, \cdot)) > P(u_2(t, \cdot)) \) for \( t_0 < t \leq t_1 \), \( 0 \leq t_0 < t_1 \leq T \). Then from assumption (A3), we find that

\[
-m(t, x, P(u_1(t, \cdot))) = -[m(t, x, P(u_1(t, \cdot))) + MP(u_1(t, \cdot))] \\
+ MP(u_1(t, \cdot)) \\
\leq -m(t, x, P(u_2(t, \cdot))) - MP(u_2(t, \cdot)) \\
+ MP(u_1(t, \cdot))
\]

and

\[
-m(t, x, P(u_2(t, \cdot))) \\
\geq -m(t, x, P(u_1(t, \cdot))) - MP(u_1(t, \cdot)) + MP(u_2(t, \cdot)).
\]
Hence, \( u_1 \) and \( u_2 \) are a lower and an upper solution of (1.1) on \( D_{t_1} \), respectively. By Theorem 2.2 we have that \( u_1(t,x) \leq u_2(t,x) \) almost everywhere in \( D_{t_1} \), and hence \( P(u_1(t,\cdot)) \leq P(u_2(t,\cdot)) \) for \( 0 \leq t \leq t_1 \), which is a contradiction.

3. Monotone sequences and existence-uniqueness of solutions. We begin this section by constructing a pair of nonnegative lower and upper solutions of (1.1). Let \( \alpha^0(t,x) = 0 \). Choose a constant \( \gamma \) large enough such that
\[
\max_{D_T} \frac{q(t,x)}{\min_{[0,T]} g(t,a)} \leq \gamma/2.
\]
Fix this \( \gamma \) and choose \( \delta \) large enough such that
\[
\max \left\{ \|u_0\|_\infty, \max_{[0,T]} C(t)/\min_{[0,T]} g(t,a) \right\} \leq (\delta/2) \exp(-\gamma b).
\]
Now choose \( \sigma \) large enough such that
\[
\sigma \geq 2M\delta\|\eta\|_\infty \exp(-\gamma a)/\gamma + \left( \gamma \max_{D_T} g(t,x) + \max_{D_T} |g_x(t,x)| \right).
\]
Let \( \beta^0(t,x) = \delta \exp(\sigma t) \exp(-\gamma x) \). Then it can be easily shown that \( \alpha^0 \) and \( \beta^0 \) are a pair of coupled lower and upper solutions of (1.1) on \( [0,T_0] \times [a,b] \) with \( T_0 = \min\{T,(\ln 2/\sigma)\} \). We then define two sequences \( \{\alpha^k\}_{k=0}^\infty \) and \( \{\beta^k\}_{k=0}^\infty \) as follows.

For \( k = 1, 2, \ldots \),
\[
\begin{align*}
\alpha^k_t + q(t,x)\alpha^k_x &= -D^{k-1}(t,x)\alpha^k, & 0 < t < T_0, & a < x < b, \\
g(t,a)\alpha^k(t,a) &= \tilde{R}^{k-1}(t), & 0 < t < T_0, \\
\alpha^k(0,x) &= u_0(x), & a \leq x \leq b,
\end{align*}
\]
where
\[
D^{k-1}(t,x) = m(t,x,P(\beta^{k-1}(t,\cdot)))+MP(\beta^{k-1}(t,\cdot))-MP(\alpha^{k-1}(t,\cdot))
\]
\[
\tilde{R}^{k-1}(t) = C(t) + \int_a^b q(t,x)\alpha^{k-1}(t,x) \, dx,
\]
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and

\[ \beta^k_t + (g(t, x)\beta^k)_x = -E^{k-1}(t, x)\beta^k, \quad 0 < t < T_0, \quad a < x < b, \]

\( (3.2) \quad g(t, a)\beta^k(t, a) = \hat{B}^{k-1}(t), \quad 0 < t < T_0, \]

\[ \beta^k(0, x) = u_0(x), \quad a \leq x \leq b, \]

where

\[ E^{k-1}(t, x) = m(t, x, P(\alpha^{k-1}(t, \cdot))) + MP(\alpha^{k-1}(t, \cdot)) - MP(\beta^{k-1}(t, \cdot)) \]

\[ \hat{B}^{k-1}(t) \equiv C(t) + \int_a^b q(t, x)\beta^{k-1}(t, x) \, dx. \]

The existence of solutions to problems (3.1) and (3.2) follows from the fact that \( \hat{R}^{k-1} \) and \( \hat{B}^{k-1} \) are given functions. In particular, using the method of characteristics \([7], [8], [9], [11]\), one can find explicitly solutions to these problems as follows. Consider the equation for the characteristic curves given by

\[ \begin{cases} 
\frac{dt}{ds} t(s) = 1 \\
\frac{dx}{ds} x(s) = g(t(s), x(s)).
\end{cases} \quad (3.3) \]

The solution \( \alpha^k \) of (3.1) along a characteristic curve \((t(s), x(s))\) satisfies the following equation

\[ \frac{d}{ds} \alpha^k(s) = -(g_x(t(s), x(s)) + D^{k-1}(t(s), x(s)))\alpha^k(s). \]

Clearly, for any initial point \((t(s_0), x(s_0))\), the existence of a unique solution of (3.3) follows from the assumption (A1). Parametrizing the characteristic curves with the variable \( t \), then a characteristic curve passing through \((\hat{t}, \hat{x})\) is given by \((t, X(t; \hat{t}, \hat{x}))\) where \( X \) satisfies

\[ \frac{d}{dt} X(t; \hat{t}, \hat{x}) = g(t, X(t; \hat{t}, \hat{x})) \]

and \( X(t; \hat{t}, \hat{x}) = \hat{x} \). From (A1) it follows that the function \( X \) is strictly increasing. Hence, a unique inverse function \( \tau(x; \hat{t}, \hat{x}) \) exists. Now define \( G(x) = \tau(x; 0, a) \) where \((G(x), x)\) represents the characteristic
curve passing through \((0, a)\) and dividing the \((t, x)\)-plane into two parts. Then for any point \((t, x)\) with \(t \leq G(x)\), the solution \(\alpha^k(t, x)\) is determined through the initial condition by

\[
\alpha^k(t, x) = u_0(X(0; t, x)) \cdot \exp \left( - \int_0^t (g_x(s, X(s; t, x)) + D^{k-1}(s, X(s; t, x))) \, ds \right),
\]

and for any point \((t, x)\) with \(t > G(x)\) the solution is determined via the boundary condition by

\[
\alpha^k(t, x) = R^{k-1}(\tau(a; t, x)) \cdot \exp \left( - \int_{\tau(a; t, x)}^t (g_x(s, X(s; t, x)) + D^{k-1}(s, X(s; t, x))) \, ds \right),
\]

where \(R^{k-1}(t) = (\hat{R}^{k-1}(t)/g(t, a))\). Note that a representation similar to (3.4)–(3.5) can be obtained for the solution \(\beta^k\) by interchanging \(\alpha^k\) with \(\beta^k\) and replacing \(D^{k-1}\) with \(E^{k-1}\) and \(R^{k-1}\) with \(B^{k-1}\) where \(B^{k-1} = (\hat{B}^{k-1}/g(t, a))\).

Next we show that the sequences \(\{\alpha^k\}_{k=0}^\infty\) and \(\{\beta^k\}_{k=0}^\infty\) are monotone. To this end we first let \(w = \alpha^0 - \alpha^1\). Then \(w\) satisfies (2.9) with \(F(t, x) = D^0(t, x)\). Hence by Remark 1 \(w \leq 0\), which implies \(\alpha^0 \leq \alpha^1\). Similarly, it can be seen that \(\beta^0 \geq \beta^1\). From this, in view of (A3) it easily follows that \(\alpha^1\) and \(\beta^1\) are a lower and an upper solution, respectively, and hence \(\alpha^1 \leq \beta^1\).

Assume that for some \(k > 1\), \(\alpha^k\) and \(\beta^k\) are a lower and an upper solution of (1.1), respectively. By similar reasoning, we can show that \(\alpha^k \leq \alpha^{k+1} \leq \beta^{k+1} \leq \beta^k\) and that \(\alpha^{k+1}\) and \(\beta^{k+1}\) are also a lower and an upper solution of (1.1), respectively. Thus, by induction, we obtain two monotone sequences that satisfy

\[
\alpha^0 \leq \alpha^1 \leq \cdots \leq \alpha^k \leq \beta^k \leq \cdots \leq \beta^1 \leq \beta^0 \quad \text{a.e. in } D_{T_0}
\]

for each \(k = 0, 1, 2, \ldots\). Hence it follows from the monotonicity of the sequences \(\{\alpha^k\}_{k=0}^\infty\) and \(\{\beta^k\}_{k=0}^\infty\) that there exist functions \(\alpha\) and \(\beta\) such that \(\alpha^k \to \alpha\) and \(\beta^k \to \beta\) pointwise in \(D_{T_0}\). Clearly \(\alpha \leq \beta\) almost everywhere in \(D_{T_0}\).
Upon establishing the monotonicity of our sequences, we can prove the following convergence result.

**Theorem 3.1.** Suppose that (A1)–(A6) hold. Then the sequences \(\{\alpha^k\}_{k=0}^{\infty}\) and \(\{\beta^k\}_{k=0}^{\infty}\) converge uniformly along characteristic curves to a limit function \(u\). Moreover, the function \(u\) is the unique solution of problem (1.1) on \([0, T_0] \times [a, b]\).

**Proof.** Consider first the sequence \(\{\alpha^k\}_{k=0}^{\infty}\). From the solution representation for \(\alpha^k\) given in (3.4)–(3.5), the fact that \(\alpha^0 \leq \alpha^k \leq \beta^0\), and the monotonicity of the sequence \(\{\alpha^k\}\), we obtain by arguing as in [10, p. 189] that, along the characteristic curve passing through \((0, x_0)\), the solution
\[
\alpha^k(t, X(t; 0, x_0)) = u_0(x_0) \exp \left( - \int_0^t (g_x(s, X(s; 0, x_0)) + D^k(s, X(s; 0, x_0))) \, ds \right)
\]
converges to
\[
\alpha(t, X(t; 0, x_0)) = u_0(x_0) \exp \left( - \int_0^t (g_x(s, X(s; 0, x_0)) + D(s, X(s; 0, x_0))) \, ds \right)
\]
uniformly and monotonically for \(0 \leq t \leq T_0\), where
\[
D(t, x) = m(t, x, P(\beta(t, \cdot))) + MP(\beta(t, \cdot)) - MP(\alpha(t, \cdot)).
\]

On the other hand, since \(R^k(t)\) is monotone and uniformly bounded for \(0 \leq t \leq T_0\), along the characteristic curve passing through \((t_0, a)\), the solution
\[
\alpha^k(t, X(t; t_0, a)) = R^k(t_0) \exp \left( - \int_{t_0}^t (g_x(s, X(s; t_0, a)) + D^k(s, X(s; t_0, a))) \, ds \right)
\]
converges to
\[
\alpha(t, X(t; t_0, a)) = R(t_0) \exp \left( - \int_{t_0}^t (g_x(s, X(s; t_0, a)) + D(s, X(s; t_0, a))) \, ds \right)
\]
uniformly and monotonically for $t_0 \leq t \leq T_0$, where

$$R(t) = \frac{1}{g(t, a)} \left( C(t) + \int_a^b q(t, x) \alpha(t, x) \, dx \right).$$

Consequently, the limit $\alpha$ has the following implicit representation. For any point $(t, x)$ with $t \leq G(x)$,

$$\alpha(t, x) = u_0(X(0; t, x)) \cdot \exp \left( - \int_0^t (g_x(s, X(s; t, x)) + D(s, X(s; t, x))) \, ds \right)$$

and, for any point $(t, x)$ with $t > G(x)$,

$$\alpha(t, x) = R(\tau(a; t, x)) \cdot \exp \left( - \int_{\tau(a; t, x)}^t (g_x(s, X(s; t, x)) + D(s, X(s; t, x))) \, ds \right).$$

A similar convergence argument can be established for $\beta$, and we find that the limit $\beta$ can be represented as follows. For any point $(t, x)$ with $t \leq G(x)$,

$$\beta(t, x) = u_0(X(0; t, x)) \exp \left( - \int_0^t (g_x(s, X(s; t, x)) + E(s, X(s; t, x))) \, ds \right)$$

and for any point $(t, x)$ with $t > G(x)$,

$$\beta(t, x) = B(\tau(a; t, x)) \cdot \exp \left( - \int_{\tau(a; t, x)}^t (g_x(s, X(s; t, x)) + E(s, X(s; t, x))) \, ds \right),$$

where

$$E(t, x) = m(t, x, P(\alpha(t, \cdot))) + MP(\alpha(t, \cdot)) - MP(\beta(t, \cdot))$$

and

$$B(t) = \frac{1}{g(t, a)} \left( C(t) + \int_a^b q(t, x) \beta(t, x) \, dx \right).$$
We now show that $\alpha = \beta$. To this end, let $w = \beta - \alpha$. Since $\beta \geq \alpha$, $w(t, x) \geq 0$ and $w(0, x) = 0$. In view of (2.4), by choosing $\xi(t, x) \equiv 1$, we have that
\begin{equation}
\int_a^b w(t, x) \, dx = \int_0^t \int_a^b q(s, x)w(s, x) \, dx \, ds
- \int_0^t \int_a^b m(s, x, P(\beta(s, \cdot)))w(s, x) \, dx \, ds
+ \int_0^t \int_a^b \beta(s, x)(M + A(s, x)) \int_a^b \eta(y)w(s, y) \, dy \, dx \, ds
+ \int_0^t \int_a^b M\alpha(s, x) \int_a^b \eta(y)w(s, y) \, dy \, dx \, ds
\leq C_0 \int_0^t \int_a^b w(s, x) \, dx \, ds,
\end{equation}
where $C_0 = \max_{D_{t_0}} \left[ q(t, x) + \|\eta\|_\infty \beta(t, x)(M + A(t, x)) + M \|\eta\|_\infty \alpha(t, x) \right]$. Hence it follows from Gronwall’s inequality that $w(t, x) = 0$ almost everywhere in $D_{t_0}$, i.e., $\alpha = \beta$. Defining this common limit by $u$, we find that $u$ satisfies the following. For any point $(t, x)$ with $t \leq G(x)$,
\begin{equation}
u(t, x) = u_0(X(0; t, x)) \cdot \exp \left( - \int_0^t (g_x(s, X(s; t, x)) + m(s, X(s; t, x), P(u(s, \cdot)))) \, ds \right)
\end{equation}
and for any point $(t, x)$ with $t > G(x)$,
\begin{equation}
u(t, x) = R(\tau(a; t, x)) \cdot \exp \left( - \int_{\tau(a; t, x)}^t (g_x(s, X(s; t, x)) + m(s, X(s; t, x), P(u(s, \cdot)))) \, ds \right).
\end{equation}
Using arguments such as those in [6], one can establish that $P(u(t, \cdot))$ is continuous. Hence, by Corollary 2.3, $u(t, x)$ is the unique weak solution of problem (1.1).

Remark 2. Mathematically all results in Sections 2 and 3 hold if we assume instead of (A3) that $M - mP(t, x, P) \geq 0$ and define another
pair of coupled upper and lower solutions by replacing Definition 2.1(iii) with the following inequalities:

\begin{align}
\int_a^b u(t,x)\xi(t,x)\,dx \\
\geq \int_a^b u(0,x)\xi(0,x)\,dx + \int_0^t \xi(s,a)\left(C(s) + \int_a^b q(s,x)u(s,x)\,dx\right)\,ds \\
+ \int_0^t \int_a^b [\xi_s + g(s,x)\xi_x]u(s,x)\,dx\,ds \\
- \int_0^t \int_a^b \xi(s,x)[m(s,x,P(u(s,\cdot))) - MP(u(s,\cdot))] \\
+ MP(v(s,\cdot))]u(s,x)\,dx\,ds.
\end{align}

\begin{align}
\int_a^b v(t,x)\xi(t,x)\,dx \\
\leq \int_a^b v(0,x)\xi(0,x)\,dx + \int_0^t \xi(s,a)\left(C(s) + \int_a^b q(s,x)v(s,x)\,dx\right)\,ds \\
+ \int_0^t \int_a^b [\xi_s + g(s,x)\xi_x]v(s,x)\,dx\,ds \\
- \int_0^t \int_a^b \xi(s,x)[m(s,x,P(v(s,\cdot))) - MP(v(s,\cdot))] \\
+ MP(u(s,\cdot))]v(s,x)\,dx\,ds.
\end{align}

Biologically, however, it is more relevant to consider the case \( M + m_P(t,x,P) \geq 0 \), for this allows the mortality to increase dramatically without a bound as the population becomes unbounded, i.e., \( P \to \infty \).

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