PLANE WAVE DIFFRACTION BY A SLIT IN AN INFINITE PENETRABLE SHEET

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ABSTRACT. The acoustic diffraction of a plane wave by a slit in an infinite penetrable plane is discussed. An approximate boundary condition is used which depends upon the thickness and material constants which constitute the slit. The mathematical problem which is solved is an approximate model for a noise barrier which is not perfectly rigid and therefore transmits sound. It is found that the diffracted field is the sum of the fields produced by the two edges of the planes formed by the slit and an interaction field.

1. Introduction. In recent years noise has become a serious issue of environmental concern. Noise abatement has therefore attracted the attention of many scientists. An effective method of noise reduction is to use barriers in heavily built-up areas [1, 7]. In most of the calculations with noise barriers, the field in the shadow region of the barrier is assumed to be solely due to diffraction at the edge. This assumption supposes that the barrier is perfectly rigid and therefore does not transmit sound. However, most practical barriers are made of wood or plastic and will consequently transmit some of the noise through the barriers. Yeh [12] considered the problem of diffraction by a penetrable parabolic cylinder and obtained the solution in the complicated form of infinite series of parabolic cylinder functions. Another approximate approach for parabolic cylinder coordinates was used by Shmoys [11] to present the results in terms of Fresnel integrals. Pistol’kors [9] used the Kirchhoff-Huygens integral equation approach to solve the general problem of diffraction by a penetrable strip. Later on, Khrebet [6] extended this analysis to a dielectric half plane. The approximate boundary condition used by Pistol’kors [9] and Khrebet [6] was only good in describing a perfectly penetrable half plane, i.e., no loss within the material which comprises the half plane. For a smooth transition from a perfectly penetrable half plane to a nonpenetrable half plane, Rawlins [10] used an alternative boundary condition and calculated the diffracted field due to a line source incidence. It seems that no attempt has been made so far to discuss the acoustic diffraction.
from a slit in an infinite penetrable plane. This situation arises when there is a finite opening in an infinite barrier intercepting the line of sight from the noise source to receiver. Therefore, it is a worthwhile attempt to consider the diffraction of an acoustic wave from a slit in an infinite penetrable plane. Related studies are in [2, 3, 5].

In this paper we discuss the diffraction of a plane wave by a slit in an infinite penetrable plane. It is found that the two edges of the planes give rise to two diffracted fields, one from each edge, and an interaction field (double diffraction of the two edges). The field due to a slit in an infinite rigid barrier can be recorded as a special case by taking the values of the parameters \( \alpha \) and \( \beta \) equal to zero.

2. Formulation of the problem. Let \((x, y)\) define a system of Cartesian coordinates with origin 0. The penetrable planes are at the positions \( x < x_1, x > x_2 \) and a slit is at \( x_1 \leq x \leq x_2 \) as shown in Figure 1. The penetrable planes are assumed to be of negligible thickness and satisfying the penetrable boundary conditions on both sides of the surfaces. The system responds to an incident plane wave \( \psi_i \) given by

\[
\psi_i = \exp(-ikx \cos \theta_0 - iky \sin \theta_0),
\]

where \( \theta_0 \) is the angle measured from the \( x \)-axis. The time dependence is assumed to be of harmonic nature \( e^{-i\omega t} \), \( \omega \) is the low angular frequency, with the free space wave number of the form

\[
k = \frac{\omega}{c} = k_1 + ik_2,
\]
where \( c \) is the speed of sound. In Equation (1), \( k \) has a small positive imaginary part which has been introduced to ensure the convergence (regularity) of the Fourier transform integrals defined subsequently, Equation (11b). On suppressing the time harmonic factor, the wave equation satisfied by the total velocity potential \( \psi \) is given by

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + k^2 \psi = 0.
\]

The approximate boundary conditions for a penetrable medium of width \( 2h \) are given by [10]

\[
\pm \frac{\partial}{\partial y} \psi(x, \pm h) + i k \alpha \psi(x, \pm h) + i k \beta \psi(x, \mp h) = 0,
\]

where

\[
\alpha = \left[ \frac{T^2 e^{2ik h \sin \vartheta_0} + (e^{-2ik h \sin \vartheta_0} - R^2 e^{2ik h \sin \vartheta_0})}{(e^{-ik h \sin \vartheta_0} + Re^{ik h \sin \vartheta_0})^2 - T^2 e^{2ik h \sin \vartheta_0}} \right] \sin \vartheta_0,
\]

\[
\beta = \frac{-2T \sin \vartheta_0}{(e^{-ik h \sin \vartheta_0} + Re^{ik h \sin \vartheta_0})^2 - T^2 e^{2ik h \sin \vartheta_0}}.
\]

In Equations (3a), (3b), \( R \) and \( T \) are the reflection and transmission coefficients and are given by [10]

\[
R = \frac{(1 - N^2) \sin 2K_1 h e^{-2ik h \sin \vartheta_0}}{(1 + N^2) \sin 2K_1 h + 2iN \cos 2K_1 h},
\]

\[
T = \frac{2iN e^{-2ik h \sin \vartheta_0}}{(1 + N^2) \sin 2K_1 h + 2iN \cos 2K_1 h},
\]

\[
N = \frac{K_1 \rho}{(k \rho_1 \sin \vartheta_0)},
\]

\[
n = \frac{c}{c_1},
\]

\[
K_1 = \sqrt{n^2 - \cos^2 \vartheta_0},
\]

and \( \rho, c \) and \( \rho_1, c_1 \) are the density and sonic velocity of the media \(|y| > h\) and \(|y| < h\), respectively. For the penetrable half plane of
negligible thickness, $2kh \ll 1$ \cite{10}, Equations (3) for $x < x_1$, $y = 0$ and $x > x_2$, $y = 0$ take the form

$$
\frac{\partial}{\partial y} \psi(x, 0^+) + i k \alpha \psi(x, 0^+) + i k \beta \psi(x, 0^-) = 0,
$$

\begin{equation}
(4)
\end{equation}

$$
\frac{\partial}{\partial y} \psi(x, 0^-) - i k \alpha \psi(x, 0^-) - i k \beta \psi(x, 0^+) = 0.
$$

The boundary conditions on $x_1 \leq x \leq x_2$, $y = 0$, are given by

$$
\begin{cases}
\psi(x, 0^+) = \psi(x, 0^-), \\
\frac{\partial}{\partial y} \psi(x, 0^+) = \frac{\partial}{\partial y} \psi(x, 0^-).
\end{cases}
$$

It is appropriate to split the total field $\psi$ as

$$
\psi = \begin{cases}
\psi_i + \psi_r + \phi & y \geq 0, \\
\phi & y \leq 0,
\end{cases}
$$

where the reflected wave is

$$
\psi_r = \exp(-ikx \cos \theta_0 + iky \sin \theta_0),
$$

and $\phi$ is the diffracted field.

In addition, we insist that $\phi$ represents an outward travelling wave as $r = (x^2 + y^2)^{1/2} \to \infty$ and satisfies the edge conditions \cite{4}

$$
\psi(x, 0) = O(1),
$$

\begin{equation}
(7)
\end{equation}

$$
\frac{\partial}{\partial y} \psi(x, 0) = O(|x - x_i|^{-1/2})
$$

as $x \to x_i$, $i = 1, 2$.

The boundary value problem can now be reformulated in terms of the diffracted field $\phi$ through Equation (6) as

$$
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + k^2 \phi = 0,
$$

\begin{equation}
(8)
\end{equation}
subject to the boundary conditions

\begin{align}
\left( \frac{\partial}{\partial y} + i k \alpha \right) \phi(x, 0^+) + i k \beta \phi(x, 0^-) + 2 i k a e^{-i k x \cos \vartheta_0} &= 0, \\
\left( \frac{\partial}{\partial y} - i k \alpha \right) \phi(x, 0^-) - i k \beta \phi(x, 0^+) - 2 i k \beta e^{-i k x \cos \vartheta_0} &= 0,
\end{align}

(9) \quad x < x_1, \quad x > x_2

\phi(x, 0^+) - \phi(x, 0^-) = -2 e^{-i k x \cos \vartheta_0},

(10) \quad \frac{\partial}{\partial y} \phi(x, 0^+) = \frac{\partial}{\partial y} \phi(x, 0^-), \quad x_1 \leq x \leq x_2.

3. Solution of the problem. We define the Fourier transform pair by [8]

\begin{align}
\tilde{\phi}(\nu, y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x, y) e^{i\nu x} \, dx, \\
\phi(x, y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\phi}(\nu, y) e^{-i\nu x} \, dx,
\end{align}

(11a)

where \( \nu \) is a complex variable. In order to accommodate three-part boundary conditions on \( y = 0 \), we split \( \tilde{\phi}(\nu, y) \) as

\begin{align}
\tilde{\phi}(\nu, y) &= \tilde{\phi}_+(\nu, y) e^{i\nu x_2} + \tilde{\phi}_-(\nu, y) e^{i\nu x_1} + \tilde{\phi}_1(\nu, y),
\end{align}

(11b)

where

\begin{align}
\tilde{\phi}_+(\nu, y) &= \frac{1}{\sqrt{2\pi}} \int_{x_2}^{\infty} \phi(x, y) e^{i\nu(x - x_2)} \, dx, \\
\tilde{\phi}_1(\nu, y) &= \frac{1}{\sqrt{2\pi}} \int_{x_1}^{x_2} \phi(x, y) e^{i\nu x} \, dx, \\
\tilde{\phi}_-(\nu, y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_1} \phi(x, y) e^{i\nu(x - x_1)} \, dx.
\end{align}

In Equation (11b), \( \tilde{\phi}_+(\nu, y) \) is regular for \( \text{Im} \nu > -\text{Im} k \), \( \tilde{\phi}_-(\nu, y) \) is regular for \( \text{Im} \nu < \text{Im} k \) and \( \tilde{\phi}_1(\nu, y) \) is an integral function and is
therefore analytic in the common region \(-\text{Im} \, k < \text{Im} \, \nu < \text{Im} \, k\). For this we recall that \(k\) is complex and \(\phi\) represents an outward travelling wave of the form \(|\phi| < \exp(-k_2|x|)\) as \(|x| \to \infty\) for any fixed \(y\). Taking Fourier transform of Equation (8), we obtain

\[
\frac{d^2}{dy^2} \tilde{\phi}(\nu, y) - \gamma^2 \tilde{\phi}(\nu, y) = 0,
\]

where \(\gamma = (\nu^2 - k^2)^{1/2}\) and the \(\nu\)-plane is cut such that \(\text{Re} \, \gamma > 0\). The solution of Equation (12) satisfying the radiation condition is given by

\[
\tilde{\phi}(\nu, y) = \begin{cases} 
A_1(\nu) e^{-\gamma y} & y \geq 0, \\
A_2(\nu) e^{\gamma y} & y \leq 0.
\end{cases}
\]

Transforming boundary conditions (9) and (10), we get

\[
\tilde{\phi}_-^\prime(\nu, 0^+) + ik[\alpha \tilde{\phi}_-(\nu, 0^+) + \beta \tilde{\phi}_-(\nu, 0^-)] + \frac{2k\alpha e^{-i k \cos \theta_0 x_1}}{\sqrt{2\pi(\nu-k \cos \theta_0)}} = 0,
\]

\[
\tilde{\phi}_-^\prime(\nu, 0^-) - ik[\alpha \tilde{\phi}_-(\nu, 0^-) + \beta \tilde{\phi}_-(\nu, 0^+)] - \frac{2k\beta e^{-i k \cos \theta_0 x_1}}{\sqrt{2\pi(\nu-k \cos \theta_0)}} = 0,
\]

\[
\tilde{\phi}_+^\prime(\nu, 0^+) + ik[\alpha \tilde{\phi}_+(\nu, 0^+) + \beta \tilde{\phi}_+(\nu, 0^-)] - \frac{2k\alpha e^{-i k \cos \theta_0 x_2}}{\sqrt{2\pi(\nu-k \cos \theta_0)}} = 0,
\]

\[
\tilde{\phi}_+^\prime(\nu, 0^-) - ik[\alpha \tilde{\phi}_+(\nu, 0^-) + \beta \tilde{\phi}_+(\nu, 0^+)] + \frac{2k\beta e^{-i k \cos \theta_0 x_2}}{\sqrt{2\pi(\nu-k \cos \theta_0)}} = 0,
\]

\[
\tilde{\phi}_1(\nu, 0^+) - \tilde{\phi}_1(\nu, 0^-) = 2iG(\nu),
\]

\[
\tilde{\phi}_1^\prime(\nu, 0^+) = \tilde{\phi}_1^\prime(\nu, 0^-),
\]

where

\[
G(\nu) = \frac{1}{\sqrt{2\pi(\nu-k \cos \theta_0)}} \{e^{i(\nu-k \cos \theta_0)x_2} - e^{i(\nu-k \cos \theta_0)x_1}\},
\]

and the primes denote differentiation with respect to \(y\).
Using the boundary conditions (14)–(16) in Equation (13) and after eliminating $\bar{\phi}'_+$ and $\bar{\phi}'_-$, we get

\begin{equation}
\tag{17}
e^{i\nu x_2} \bar{\chi}_+(\nu, 0) + \frac{\bar{\phi}'_1(\nu, 0)}{[\gamma - i k(\alpha - \beta)]} + e^{i\nu x_1} \bar{\chi}_-(\nu, 0) = -iG(\nu),
\end{equation}

where

\[ \bar{\phi}_+(\nu, 0^+) - \bar{\phi}_+(\nu, 0^-) = 2\bar{\chi}_+(\nu, 0), \]
\[ \bar{\phi}_-(\nu, 0^+) - \bar{\phi}_-(\nu, 0^-) = 2\bar{\chi}_-(\nu, 0). \]

Equation (17) is the standard Wiener-Hopf functional equation. For the solution of this equation, we make the following factorizations

\begin{equation}
\tag{18}\gamma = K_+(\nu)K_-(\nu) = (\nu + \kappa)^{1/2}(\nu - k)^{1/2},
\end{equation}

and

\begin{equation}
\tag{19}\left[ 1 - \frac{i k(\alpha - \beta)}{\gamma} \right] = L_+(\nu)L_-(\nu) = L(\nu),
\end{equation}

where $L_+(\nu)$ and $K_+(\nu)$ are regular for $\text{Im}\, \nu > -\text{Im}\, k$ and $L_-(\nu)$ and $K_-(\nu)$ are regular for $\text{Im}\, \nu < \text{Im}\, k$. The factorization (19) has been discussed by Noble [8] and is given by

\begin{equation}
\tag{20}L_\pm(\nu) = 1 - \frac{i(\alpha - \beta)}{\pi}[(\nu/k)^2 - 1]^{-1/2}\cos^{-1}(\pm\nu/k).
\end{equation}

Using Equations (18) and (19) in Equation (17), we obtain

\begin{equation}
\tag{21}e^{i\nu x_2} \bar{\chi}_+(\nu, 0) + \frac{\bar{\phi}'_1(\nu, 0)}{S_+(\nu)S_-(\nu)} + e^{i\nu x_1} \bar{\chi}_-(\nu, 0) = -iG(\nu),
\end{equation}

where $S_+(\nu) = K_+(\nu)L_+(\nu)$ is regular for $\text{Im}\, \nu > -\text{Im}\, k$ and $S_-(\nu) = K_-(\nu)L_-(\nu)$ is regular for $\text{Im}\, \nu < \text{Im}\, k$.

With the help of Equations (11b), (13) and (14a)–(16c), the unknown functions $A_1(\nu)$ and $A_2(\nu)$ are given by

\begin{equation}
\tag{22a}2A_1(\nu) = e^{i\nu x_2}(\bar{\phi}_+(\nu, 0^+) - \bar{\phi}_+(\nu, 0^-)) + e^{i\nu x_1}(\bar{\phi}_-(\nu, 0^+) - \bar{\phi}_-(\nu, 0^-)) + 2iG(\nu)
\end{equation}

\[ + \frac{i k(\alpha + \beta)}{\gamma} \{e^{i\nu x_2}(\bar{\phi}_+(\nu, 0^+) + \bar{\phi}_+(\nu, 0^-))
\end{equation}

\[ + e^{i\nu x_1}(\bar{\phi}_-(\nu, 0^+) + \bar{\phi}_-(\nu, 0^-)) + 2iG(\nu), \]
\[ (22b) \]
\[-2A_2(\nu) = e^{ivx_2}(\tilde{\phi}_+(\nu, 0^+) - \tilde{\phi}_+(\nu, 0^-)) + e^{ivx_1}(\tilde{\phi}_-(\nu, 0^+) - \tilde{\phi}_-(\nu, 0^-)) + 2iG(\nu) \]
\[ - \frac{ik(\alpha + \beta)}{\gamma} \{ e^{ivx_2}(\tilde{\phi}_+(\nu, 0^+) + \tilde{\phi}_+(\nu, 0^-)) + e^{ivx_1}(\tilde{\phi}_-(\nu, 0^+) + \tilde{\phi}_-(\nu, 0^-)) + 2iG(\nu) \}. \]

We assert that \( k(\alpha + \beta)/\gamma \) is very, very small provided \( |\nu/k| \) is not too near to 1. In order to justify this argument, we note that the constants \((\alpha \pm \beta)\) which appear in Equations (22) and \( L(\nu) \), respectively, are related to the physical dimensions and material constants of the slit, from Equations (3), by the remarkably simple expressions

\[ (22c) \]
\[ \alpha + \beta = -iN \tan K_1 h \sin \theta_0, \]
\[ \alpha - \beta = -iN \cot K_1 h \sin \theta_0. \]

From Equation (22c), it is clear that \((\alpha + \beta)\) is proportional to \(kh\) and, for \(kh\) small, \((\alpha + \beta)\) is small because \(2kh \ll 1\), but \((\alpha - \beta)\) need not be small. As we continue with the calculation, we shall find that some terms are proportional to \((\alpha + \beta)\) and can be dropped, while others containing \((\alpha - \beta)\) need not be small and are retained. We could just set \((\alpha + \beta)\) to zero at this point, but, by carrying it through the calculation, the different roles of the barrier thickness and absorption become clearer. Moreover, though we are assuming that \((\alpha - \beta)\) is not small, it can be set to zero to recover the case of a rigid barrier. Thus, using this approximation, Equations (22a), (22b) yield

\[ (23) \]
\[ 2A_1(\nu) = -2A_2(\nu) = e^{ivx_2}(\tilde{\phi}_+(\nu, 0^+) - \tilde{\phi}_+(\nu, 0^-)) + e^{ivx_1}(\tilde{\phi}_-(\nu, 0^+) - \tilde{\phi}_-(\nu, 0^-)) + 2iG(\nu). \]

Now, multiplying Equation (21) by \( S_+(\nu)e^{-ivx_2} \) and using the general decomposition theorem \([8, \text{p. } 13]\), we obtain

\[ (24) \]
\[ S_+(\nu)\bar{X}_+(\nu, 0) + \frac{ie^{-ik\cos \theta_0 x_2}}{\sqrt{2\pi(\nu - k \cos \theta_0)}}(S_+(\nu) - S_+(k \cos \theta_0)) + U_+(\nu) + V_+(\nu) \]
\[ = \frac{-ie^{-ik\cos \theta_0 x_2}S_+(k \cos \theta_0)}{\sqrt{2\pi(\nu - k \cos \theta)}} \]
\[ - \frac{e^{-ivx_2}\tilde{\phi}_1(\nu, 0)}{S_-(\nu)} - U_-(\nu) - V_-(\nu), \]
where

$$S_+ (\nu) \tilde{X}_- (\nu, 0) e^{-i \nu (x_2 - x_1)} = U (\nu) = U_+ (\nu) + U_- (\nu),$$

$$\frac{-ie^{-i \nu (x_2 - x_1)} - ik \cos \vartheta_0 x_1 S_+ (\nu)}{\sqrt{2 \pi (\nu - k \cos \vartheta_0)}} = V (\nu) = V_+ (\nu) + V_- (\nu),$$

and

$$U_+ (\nu) = \frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{U (\xi)}{(\xi - \nu)} d\xi, \quad \text{Im} \, \nu > 0,$$

$$U_- (\nu) = -\frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{U (\xi)}{(\xi - \nu)} d\xi, \quad \text{Im} \, \nu < 0,$$

$$V_+ (\nu) = \frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{V (\xi)}{(\xi - \nu)} d\xi, \quad \text{Im} \, \nu > 0,$$

$$V_- (\nu) = -\frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{V (\xi)}{(\xi - \nu)} d\xi, \quad \text{Im} \, \nu < 0.$$  

Next, multiplying Equation (21) by $S_- (\nu) e^{-i \nu x_1}$, we get

$$S_- (\nu) \tilde{X}_- (\nu, 0) - \frac{ie^{-ik \cos \vartheta_0 x_1} S_- (\nu)}{\sqrt{2 \pi (\nu - k \cos \vartheta_0)}} + R_- (\nu) - Q_- (\nu)$$

$$= -\frac{e^{-i \nu x_1} \tilde{\vartheta}_1 (\nu, 0)}{S_+ (\nu)} - R_+ (\nu) + Q_+ (\nu),$$

where

$$S_- (\nu) \tilde{X}_+ (\nu, 0) e^{i \nu (x_2 - x_1)} = R (\nu) = R_+ (\nu) + R_- (\nu),$$

$$\frac{-ie^{i \nu (x_2 - x_1)} - ik \cos \vartheta_0 x_2 S_- (\nu)}{\sqrt{2 \pi (\nu - k \cos \vartheta_0)}} = Q (\nu) = Q_+ (\nu) + Q_- (\nu),$$

and

$$R_+ (\nu) = \frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{R (\xi)}{(\xi - \nu)} d\xi, \quad \text{Im} \, \nu > 0,$$

$$R_- (\nu) = -\frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{R (\xi)}{(\xi - \nu)} d\xi, \quad \text{Im} \, \nu < 0,$$

$$Q_+ (\nu) = \frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{Q (\xi)}{(\xi - \nu)} d\xi, \quad \text{Im} \, \nu > 0,$$

$$Q_- (\nu) = -\frac{1}{2 \pi i} \int_{-\infty}^{\infty} \frac{Q (\xi)}{(\xi - \nu)} d\xi, \quad \text{Im} \, \nu < 0.$$
Let \( f_1(\nu) \) define a function equal to both sides of Equation (24) since the lefthand side of Equation (24) is regular for \( \text{Im}\nu > -\text{Im}\, k \) and the righthand side is regular for \( \text{Im}\nu < \text{Im}(k \cos \vartheta_0) \), respectively. Therefore, by analytic continuation, the definition of \( f_1(\nu) \) can be extended throughout the complex \( \nu \) plane. The form of \( f_1(\nu) \) is ascertained by examining the asymptotic behavior of the terms in Equation (24) as \( \nu \to \infty \). From Equation (20) we note that \( |L_{\pm}(\nu)| \sim O(1) \) as \( |\nu| \to \infty \) and, with the help of the edge conditions, we find that \( \tilde{x}_+(\nu, 0) \) and \( \tilde{x}_-(\nu, 0) \) must be at least of \( O(|\nu|^{-1/2}) \) as \( |\nu| \to \infty \). Using the extended form of Liouville's theorem, it can be seen from Equation (24) that \( f_1(\nu) \sim O(|\nu|^{-1/2}) \) and, therefore, the polynomial representing \( f_1(\nu) \) can only be a constant equal to zero. Hence, from Equation (24), we obtain

\[
S_+(\nu)\tilde{x}_+^*(\nu, 0) + \frac{1}{2\pi i} \int_{-\infty + ic}^{\infty + ic} \frac{K_+(\xi)\tilde{x}_+^*(\xi, 0)e^{-i\xi(x_2-x_1)}L(\xi)}{\left(\xi - \nu\right)L_-^*(\xi)} d\xi - \frac{ie^{-ik\cos \vartheta_0\varphi_2}S_+(k \cos \vartheta_0)}{\sqrt{2\pi}(\nu - k \cos \vartheta_0)} = 0,
\]

where

\[
\tilde{x}_+(\nu, 0) + \frac{ie^{-ik\cos \vartheta_0\varphi_2}}{\sqrt{2\pi}(\nu - k \cos \vartheta_0)} = \tilde{x}_+^*(\nu, 0),
\]

\[
\tilde{x}_-(\nu, 0) - \frac{ie^{-ik\cos \vartheta_0\varphi_1}}{\sqrt{2\pi}(\nu - k \cos \vartheta_0)} = \tilde{x}_-^*(\nu, 0).
\]

Similarly, from Equation (25), we have

\[
S_-(\nu)\tilde{x}_-^*(\nu, 0)
\]

\[
- \frac{1}{2\pi i} \int_{-\infty + id}^{\infty + id} \frac{K_-(\xi)\tilde{x}_-^*(\xi, 0)e^{i\xi(x_2-x_1)}L(\xi)}{\left(\xi - \nu\right)L_+^*(\xi)} d\xi = 0.
\]

The unknown functions \( \tilde{x}_+(\nu, 0) \) and \( \tilde{x}_-(\nu, 0) \) appearing in Equations (26) and (27) have been determined by using the procedure discussed by Noble [8] and are given by

\[
2\tilde{x}_+(\nu, 0) = \tilde{\phi}_+(\nu, 0^+) - \tilde{\phi}_+(\nu, 0^-)
\]

\[
= -\frac{2i}{\sqrt{2\pi S_+(\nu)}}(G_1(\nu) + C_1(k)T(\nu)),
\]

where \( G_1(\nu) \) and \( C_1(k)T(\nu) \) are given by...
\[ 2\tilde{\chi}_-(\nu,0) = \tilde{\phi}_-(\nu,0^+) - \tilde{\phi}_-(\nu,0^-) \]
\[ = -\frac{2i}{\sqrt{2\pi}S_-(\nu)} \left\{ (G_2(-\nu) + C_2(k)T(-\nu)) \right. \]
\[ - \frac{ik(\alpha - \beta)e^{-ik\cos\vartheta_0x_1}}{(\nu - k\cos\vartheta_0)S_+(k\cos\vartheta_0)} \right\}, \]

where

\[ C_1(k) = \frac{1}{D} \left( G_1(k)T(k) + G_2(k)S_+(k)L_+(k) \right. \]
\[ \left. + \frac{ik(\alpha - \beta)e^{-ik\cos\vartheta_0x_1}}{(k + k\cos\vartheta_0)S_+(k\cos\vartheta_0)} \right), \]
\[ D = S_+(k)\frac{L_+^2(k)}{k^2} - T^2(k), \]
\[ C_2(k) = \frac{1}{D} \left( G_2(k)T(k) + G_1(k)S_+(k)L_+(k) \right. \]
\[ \left. + \frac{ik(\alpha - \beta)e^{-ik\cos\vartheta_0x_1}}{(k + k\cos\vartheta_0)S_+(k\cos\vartheta_0)} \right), \]

(30a)
\[ G_1(\nu) = P_1(\nu)e^{-ik\cos\vartheta_0x_1} - \left(R_1(\nu)/L_+(k)\right)e^{-ik\cos\vartheta_0x_2}, \]

(30b)
\[ G_2(\nu) = P_2(\nu)e^{-ik\cos\vartheta_0x_1} - \left(R_2(\nu)/L_+(k)\right)e^{-ik\cos\vartheta_0x_2}, \]

\[ P_1(\nu) = \frac{1}{(\nu - k\cos\vartheta_0)} \left( S_+(\nu) - S_+(k\cos\vartheta_0) \right), \]
\[ P_2(\nu) = \frac{1}{(\nu + k\cos\vartheta_0)} \left( S_+(\nu) - \frac{K_-(k\cos\vartheta_0)}{L_+(k\cos\vartheta_0)} \right), \]
\[ R_{1,2}(\nu) = \frac{E_0(W_0[-i(k\pm k\cos\vartheta_0)(x_2-x_1)] - W_0[-i(k+\nu)(x_2-x_1)])}{2\pi i(\nu \mp k\cos\vartheta_0)}, \]
\[ T(\nu) = \frac{1}{2\pi i}E_0W_0[-i(k + \nu)(x_2 - x_1)], \]
\[ E_0 = 2e^{i\pi/2} \frac{e^{ik(x_2 - x_1)}}{(x_2 - x_1)^{1/2}}, \]
\[ W_0(m) = \Gamma(3/2)e^{m/2}(m)^{-1/4}W_{3/4,1/4}(m), \]
$W_{i,j}$ is a Whittaker function and $m = -i(k + \nu)(x_2 - x_1)$. Substitution of Equations (28) and (29) in Equation (23) yields

$$
A_{1,2}(\nu) = -\frac{\text{isgn}(y)}{\sqrt{2\pi}} \left\{ \frac{e^{i\nu x_2}}{S_+(\nu)} (G_1(\nu) + C_1(k)T(\nu)) \right. \\
+ \frac{e^{i\nu x_1}}{S_-(\nu)} \left( G_2(-\nu) + C_2(k)T(-\nu) \\
- \frac{ik(\alpha - \beta)e^{-ik\cos \theta_0 x_1}}{(\nu - k \cos \theta_0)S_+(k \cos \theta_0)} \right) \right\} + \text{isgn}(y)G(\nu).
$$

(31)

Now, substituting the value of $A_{1,2}(\nu)$ in Equation (13) and using Equations (30a) and (30b), the field $\phi(x, y)$ can be written as

$$
\phi = \phi^{\text{sep}}(x, y) + \phi^{\text{int}}(x, y),
$$

where

$$
\phi^{\text{sep}}(x, y) = \frac{\text{isgn}(y)}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{K_+(k \cos \theta_0)L_+(k \cos \theta_0)e^{i(\nu - k \cos \theta_0)x_2}}{K_+(\nu)L_+(\nu)(\nu - k \cos \theta_0)} - \frac{K_-(k \cos \theta_0)e^{i(\nu - k \cos \theta_0)x_1}}{K_-(\nu)L_-(\nu)(\nu - k \cos \theta_0)L_+(k \cos \theta_0)} \\
+ \frac{ik(\alpha - \beta)e^{i(\nu - k \cos \theta_0)x_1}}{(\nu - k \cos \theta_0)S_+(k \cos \theta_0)S_-(\nu)} \right\} e^{-i\nu x - \gamma y} d\nu,
$$

(32)

$$
\phi^{\text{int}}(x, y) = \frac{\text{isgn}(y)}{2\pi} \int_{-\infty}^{\infty} \left\{ \left( \frac{R_1(\nu)}{L_+(k)} e^{-ik \cos \theta_0 x_1} \right) e^{i\nu x_2} \\
- C_1(k)T(\nu) \frac{e^{i\nu x_2}}{K_+(\nu)L_+(\nu)} \\
+ \left( \frac{R_2(-\nu)}{L_+(k)} e^{-ik \cos \theta_0 x_2} \right) e^{i\nu x_1} \left( C_2(k)T(-\nu) \frac{e^{i\nu x_1}}{K_-(\nu)L_-(\nu)} \right) \right\} e^{-i\nu x - \gamma y} d\nu.
$$

(33)

Here $\phi^{\text{sep}}(x, y)$ represents the field diffracted by the edges at $x = x_2$ and $x = x_1$, and $\phi^{\text{int}}(x, y)$ gives the interaction of one edge upon the other. The integrals appearing in Equations (32) and (33) can
be evaluated asymptotically by using the steepest descent method [4]. For that, we put \( x = r \cos \vartheta, \ y = r \sin \vartheta \) and deform the contour by the transformation \( \nu = -k \cos (\vartheta + iq), \ 0 < \vartheta < \pi, \ -\infty < q < \infty. \) Hence, for large \( kr, \)

\[
\phi_{\text{sep}}(x, y) = \frac{i \sin \vartheta \text{sgn}(y)}{\sqrt{2\pi kr}} \mathcal{F}_1(-k \cos \vartheta) e^{i(kr^{-\pi/4})}, \\
\phi_{\text{int}}(x, y) = \frac{ik \sin \vartheta \text{sgn}(y)}{\sqrt{2\pi kr}} \mathcal{F}_2(-k \cos \vartheta) e^{i(kr^{-\pi/4})},
\]

where

\[
\mathcal{F}_1(-k \cos \vartheta) = \begin{cases} 
K_+(k \cos \vartheta_0)L_+(k \cos \vartheta_0)e^{-ik(\cos \vartheta + \cos \vartheta_0)x_2} \\
\frac{K_+(-k \cos \vartheta)L_+(-k \cos \vartheta)(\cos \vartheta + \cos \vartheta_0)}{S_-(-k \cos \vartheta)L_+(k \cos \vartheta_0)(\cos \vartheta + \cos \vartheta_0)} \\
- \frac{S_+(k \cos \vartheta_0)(\cos \vartheta + \cos \vartheta_0)}{(\cos \vartheta + \cos \vartheta_0)S_+(k \cos \vartheta_0)S_-(-k \cos \vartheta)}
\end{cases}
\]

\[
\mathcal{F}_2(-k \cos \vartheta) = -\left( \frac{R_1(-k \cos \vartheta)}{L_+(k)} e^{-ik \cos \vartheta_0 x_1} - C_1(k)T(-k \cos \vartheta) \right) e^{-ik \cos \vartheta x_2} \\
\times \frac{K_+(-k \cos \vartheta)L_+(-k \cos \vartheta)}{K_-(-k \cos \vartheta)L_-(k \cos \vartheta)} \\
- \left( \frac{R_2(k \cos \vartheta)}{L_+(k)} e^{-ik \cos \vartheta_0 x_2} - C_2(k)T(k \cos \vartheta) \right) e^{-ik \cos \vartheta x_1} \\
\times \frac{K_+(-k \cos \vartheta)L_+(-k \cos \vartheta)}{K_-(-k \cos \vartheta)L_-(k \cos \vartheta)}.
\]

4. Concluding remarks. We have solved a new canonical diffraction problem of a plane wave by a slit in an infinite penetrable sheet. The problem solved in this paper takes into account the material properties and thickness of the planes. It is worth looking from Equations (34) and (35) that the field corresponds to a rigid barrier if the material comprising the planes forming the slit becomes very dense, i.e., \( \text{Im} (n) > 0, |n| \rightarrow \infty, K_1h \rightarrow \infty. \) Further, the results of diffraction by
a slit in an infinite absorbing sheet can be obtained as a special case of this problem by taking $\beta = 0$ and $\alpha = \rho_0 c / z_\alpha$, $\rho_0$ is the density of the undisturbed stream, $c$ is the speed of sound and $z_\alpha$ is the acoustic impedance of the surface.

It is also of interest to note how the parameters $(\alpha \pm \beta)$ enter the calculation. The parameter $(\alpha + \beta)$ represents essentially the thickness of the barrier and appears in the calculation separated from the other terms, while $(\alpha - \beta)$ represents the absorption of the barrier and is intimately included in the calculation through its role in the terms $L_{\pm}$ and $L$. It is also worth noting that the approximate boundary condition which is used to solve the problem is insensitive to the variation of the angle of incidence, and therefore presumably the type of source, when the planes forming the slit are dense, $|K_1 h| \to \infty$, or transmissive, $|K_1 h| \to 0$. This is because the factors $\alpha$ and $\beta$ become independent of $\vartheta_0$.

The present work also has applications in electromagnetism when considering diffraction by a slit in an infinite dielectric sheet. For this, we take $n = [(\varepsilon_1 \sigma_1)/(\varepsilon \sigma)]^{1/2}$, $N = [(K_1 \sigma)/(k \varepsilon \sin \vartheta_0)]$, for $\psi = H_z$ magnetic vector parallel to the z-axis, $N = [(K_1 \sigma)/(k \sigma_1 \sin \vartheta_0)]$, for $\psi = E_z$ electric vector parallel to the z-axis, where $\sigma$, $\varepsilon$ and $\sigma_1$, $\varepsilon_1$ are the permeability and permittivity of the media $|y| > h$ and $|y| < h$, respectively.

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