ABSTRACT. This paper analyzes a model of the blood production system proposed by Rey and Mackey, that is, a maturity structured population model with time delay. Existence, uniqueness, regularity, invariance and asymptotic behavior of solutions are studied.

1. Introduction. The objective of this paper is to analyze a model of the blood production system proposed by Rey and Mackey in [8]. This model describes the production of proliferative stem and precursor cells in the bone marrow. The model distinguishes normal and abnormal blood production systems in the following way: normal systems stabilize independently of initial values, whereas abnormal systems exhibit sensitive dependence on initial values. The abnormal systems correspond to aplastic anemias, which are believed to result from injury or destruction of common pluripotential stem cells affecting all subsequent cell populations.

The model in [8] is a maturity structured population model with time delay. The maturity structure variable is used to distinguish primitive and mature cell types. The equations are

\[ u_t(x,t) + (xu(x,t))_x = \mu u(\alpha x, t-\tau)(1-u(\alpha x, t-\tau)), \quad t > 0 \]

\[ u(x,t) = \varphi(x,t), \quad -\tau \leq t \leq 0. \]

Here \( u(x,t) \) is the population density of cells with respect to maturity \( x \) at time \( t \) and \( \mu, \alpha, \tau \) are parameters satisfying \( \mu > 0, \ 0 < \alpha < 1, \ \tau > 0 \). The maturity variable \( x \) has values in \([0,1]\) and may relate to...
the intra-cellular hemoglobin content of individual cells. The maturity transport term \((g(x)u(x,t))_x\) assumes that all cells have maturation rate \(g(x) = x\). The time delay \(\tau\) and maturity displacement \(\alpha\) arise when a maturity-age structured model in [8] is simplified by assuming that all cells divide at exactly the same age. In this model the cell division process is not modeled directly, but there is a production of new cells of all maturity values as governed by a logistic nonlinear dependence on the population density.

The model (E) has been analyzed in the case \(\tau = 0\) and \(\alpha = 1\) by Lasota [4], Brunowsky [1], Brunowsky and Komornik [2], and Rudnicki [9]. In these studies it is shown that if \(0 < \mu < 1\), then all solutions converge to 0. If \(\mu > 1\), then the behavior of solutions is dependent upon the value of \(\varphi(0)\). If \(\varphi(0) > 0\), then the solution converges to \(1 - 1/\mu\). If \(\varphi(0) = 0\) then the solution is unstable in the space of all such initial functions. In [8] numerical studies indicate similar behavior for the case \(\tau > 0\) and \(0 < \alpha < 1\). In this paper we will prove that such behavior does occur in this case. In the interpretation of this model the case that \(\varphi(0, t) > 0\) corresponds to a sufficient supply of the primitive cell type and to a normal blood production system. The case \(\varphi(0, t) = 0\) corresponds to an insufficient supply of the most primitive cell types and to an aplastic anemia.

The organization of this paper is as follows. In Section 2 we develop the semigroup generated by the linear operator \((fw)(x) = -\delta w(x) - \tau x w'(x)\), in Section 3 we treat the existence and regularity of solutions to (E), in Section 4 we treat the uniqueness of solutions, in Section 5 we treat the invariance of the solutions, in Section 6 we treat the asymptotic behavior of solutions and in Section 7 we treat the existence of equilibrium solutions and the sensitive dependence of solutions upon initial values. In Sections 8 and 9 we study the case \(\tau = 0\).

2. The semigroup generated by the operator \(fw(x) = -\delta w(x) - \tau x w'(x)\). We look first at the semigroup generated by the linear part of equation (E). Let \(r\) and \(\delta\) be positive constants. Consider the operator \(f\),

\[
f : D_f \subset C([0, 1]) \longrightarrow C([0, 1])
\]
defined as follows

\[ D_f = \{ w \in C([0,1]), xw(x) \in C^1([0,1]) \} \]

\[ (fw)(x) = -r(xw(x))' - (\delta - r)w(x). \]

In our case \( \delta = r = 1 \). Note that

**Lemma 2.1.** Let \( w \in C([0,1]) \) be differentiable on \([0,1]\) and such that \( \lim_{x \to 0^+} xw'(x) = b, -\infty \leq b \leq +\infty \). Then \( b = 0 \).

**Proof.** From Lagrange's theorem, for \( x \in ]0,1] \),

\[ w(x) = w(0) + w'(\xi_x)x, \quad \text{where } 0 < \xi_x < x; \]

and so

\[ 0 = \lim_{x \to 0} |w(x) - w(0)| = \lim_{x \to 0} |\xi_x w'(\xi_x)| \left| \frac{x}{\xi_x} \right| \geq \lim_{x \to 0} |\xi_x w'(\xi_x)| = |b|; \]

hence \( b = 0 \). \( \square \)

It follows that an equivalent definition of \( f \) is the following.

\[ D_f = \{ w \in C([0,1]), w \text{ is differentiable on } ]0,1], \]
\[ w' \in C([0,1]) \text{ and } \lim_{x \to 0} xw'(x) = 0 \} \]

\[ fw(x) = -\delta w(x) - rxw'(x), \quad 0 < x \leq 1 \]
\[ fw(0) = -\delta w(0). \]

Note that \( C^1([0,1]) \subset D_f \subset C^1([0,1]) \), and the inclusions are strict. An example of function which does not belong to \( C^1([0,1]) \) but belongs to \( D_f \) is \( x^\alpha, 0 < \alpha < 1 \). A function which belongs to \( C([0,1]) \cap C^1([0,1]) \) but does not belong to \( D_f \) is \( w(x) = x \sin(1/x), 0 < x \leq 1, w(0) = 0 \).

We prove that \( f \) is the generator of a \( C_0 \) contraction semigroup and write explicitly the semigroup generated by \( f \).
Consider the operator $A$,

$$D_A = D_f = \{ w \in C([0,1]), (xw(x))' \in C([0,1]) \}$$

$$(Aw)(x) = -rxw'(x), \quad 0 < x \leq 1$$

$$(Aw)(0) = 0.$$

**Proposition 2.2.** $A$ is the generator of a contraction semigroup.

**Proof.** $D_f$ contains $C^1([0,1])$, hence $\overline{D_f} = C([0,1])$.

Denote by $| \cdot |_\infty$ the norm in $C([0,1])$, $|u|_\infty = \sup_{x \in [0,1]} |u(x)|$. Let $\lambda > 0$ and $v \in C([0,1])$. We must prove that the equation

$$ (I - \lambda A)w = v $$

has a unique solution $w \in D_A$ and that $|w|_\infty \leq |v|_\infty$.

So consider the equation

$$ w(x) + \lambda rxw'(x) = v(x), \quad 0 < x \leq 1, $$

that is, the equation

$$ w'(x) + \frac{1}{\lambda r x} w(x) = \frac{1}{\lambda r} \frac{1}{x} v(x), \quad 0 < x \leq 1, $$

whose solutions are the functions

$$ w(x) = e^{-\int_1^x (1/(\lambda r))(1/t) \, dt} \left\{ w(1) + \int_1^x \frac{1}{\lambda r \, t} v(t) e^{\int_1^t (1/(\lambda r))(1/s) \, ds} \, dt \right\} $$

$$ = x^{-1/(\lambda r)} \left\{ k + \int_0^x \frac{1}{\lambda r} v(t)t^{(1/(\lambda r))-1} \, dt \right\} $$

for each $k \in \mathbb{R}$. As

$$ \lim_{x \to 0^+} x^{-1/(\lambda r)} \int_0^x \frac{1}{\lambda r} v(t)t^{(1/(\lambda r))-1} \, dt = v(0), $$

if we want $w$ continuous at 0 we must choose $k = 0$. And the function

$$ w(x) = \begin{cases} x^{-1/(\lambda r)} \int_0^x 1/(\lambda r) v(t)t^{(1/(\lambda r))-1} \, dt & 0 < x \leq 1 \\ v(0) & x = 0, \end{cases} $$
is the unique solution of (2.1). Also, if $0 < x \leq 1$,

$$|w(x)| \leq \frac{1}{\lambda^r} |v|\infty x^{-1/(\lambda^r)} \int_0^x t^{(1/(\lambda^r)-1)} dt = |v|\infty$$

and so $|w|\infty \leq |v|\infty$. \qed

Consider now the linear operators $S(t)$, $t \geq 0$, $S(t) : C([0,1]) \rightarrow C([0,1])$,

$$(S(t)\varphi)(x) = \varphi(e^{-rt} x).$$

It is easy to verify that $\{S(t)\}_{t \geq 0}$ is a $C_0$ contraction semigroup. In fact, $|S(t)\varphi(x)| \leq |\varphi|\infty$ and so $|S(t)\varphi|\infty \leq |\varphi|\infty$, and $||S(t)|| \leq 1$. Also

i) $S(0)\varphi(x) = \varphi(x)$ so $S(0) = I$.

ii) $S(t)S(s)\varphi(x) = S(t)\varphi(e^{-rs} x) = \varphi(e^{-rt}e^{-rs} x) = \varphi(e^{-(r(t+s))} x)$, so $S(t+s) = S(t)S(s)$.

iii) $\varphi(e^{-rt} x)$ is uniformly continuous in $[0,t_0+1] \times [0,1]$ and so

$|\varphi(e^{-rt} x) - \varphi(e^{-rt_0} x)| \leq \varepsilon$ for $|t-t_0| < \delta_\varepsilon$, for any $x \in [0,1]$, and

$|S(t)\varphi - S(t_0)\varphi|\infty \leq \varepsilon$ for $|t-t_0| < \delta_\varepsilon$; so $S(t)\varphi$ is continuous in $[0,\infty)$ for all $\varphi \in C([0,1])$.

**Proposition 2.3.** The infinitesimal generator of the semigroup $\{S(t)\}_{t \geq 0}$ is the operator $A$.

**Proof.** Let $B$ be the infinitesimal generator of $S(t)_{t \geq 0}$. We prove that $D_B \subset D_B$ and $B = A|_{D_B}$. This implies $A = B$. Let $w \in D_B$, that is, suppose that $\lim_{t \to 0^+} (S(t)w - w)/t$ exists in $C([0,1])$. Let $x \neq 0$. Then

$$\lim_{t \to 0^+} \frac{S(t)w(x) - w(x)}{t} = \lim_{t \to 0^+} \frac{w(e^{-rt} x) - w(x)}{t} = \lim_{h \to 0^-} \frac{w(x+h) - w(x)}{h} \lim_{t \to 0^+} \frac{e^{-rt} - 1}{t}$$

$$= -rx \lim_{h \to 0^-} \frac{w(x+h) - w(x)}{h} = (Bw)(x).$$

Hence, if $w \in D_B$ and $x \in ]0,1]$, $w$ is differentiable on the left in $x$ and

$$(Bw)(x) = -rx \frac{d^- w(x)}{dx}. $$
But, as \( Bw \in C([0,1]) \), \( d^-w(x)/dx \) is continuous in \([0,1]\) and so \( d^-w(x)/dx = dw(x)/dx \); also \( S(t)w(0) - w(0) = 0 \); and we have

\[
Bw(x) = \begin{cases} 
-rwx'(x) & 0 < x \leq 1 \\
0 & x = 0, 
\end{cases}
\]

and so \( w \in D_A \) and \( Bw = Aw \).

**Proposition 2.4.** \( f \) is the infinitesimal generator of the semigroup \( \{T(t)\}_{t \geq 0} \),

\[
T(t)\varphi(x) = e^{-\delta t}\varphi(e^{-\tau t}x), \quad 0 \leq x \leq 1,
\]

and so \( \|T(t)\| \leq e^{-\delta t} \).

**Proof.** \( A - \delta I \) is the generator of the semigroup \( \{e^{-\delta t}S(t)\}_{t \geq 0} \), that is, \( f \) is the generator of the semigroup \( \{T(t)\}_{t \geq 0} \); and \( \|T(t)\| = e^{-\delta t}\|S(t)\| \leq e^{-\delta t} \).

Note that if \( w \in D_f \cap C_0([0,1]) \), where \( C_0([0,1]) = \{\varphi \in C([0,1]), \varphi(0) = 0\} \), then \( fw \in C_0([0,1]) \). It follows easily that the restriction of \( f \) to \( C_0([0,1]) \) is the infinitesimal generator of the semigroup \( \{T_0(t)\}_{t \geq 0} \), where \( T_0(t) \) is the restriction of \( T(t) \) to \( C_0([0,1]) \).

3. **Existence and regularity of solutions when \( \tau > 0 \).** The abstract form of the equation (E)

\[
u_t(x,t) + (\nu u(x,t))_x = \mu u(ax,t - \tau)(1 - u(ax,t - \tau)), \quad \text{for} \quad 0 \leq x \leq 1, \quad t > 0,
\]

\( u(x,t) = \varphi(x,t), \quad 0 \leq x \leq 1, \quad -\tau \leq t \leq 0 \)

is the functional differential equation in the Banach space \( C([0,1]) \)

\[
du(t)/dt = f(u(t)) + g(u_t), \quad t > 0
\]

\( u(t) = \varphi(t), \quad -\tau \leq t \leq 0 \)

where \( f \) is the operator studied in Section 1,

\[
D_f = \{w \in C([0,1]), xw(x) \in C^1([0,1]) \}
\]

\( (fw)(x) = -(xw(x))' \),
$g$ is the operator

$$g : C([-\tau, 0], C([0, 1])) \rightarrow C([0, 1]),$$

$$g(u)(x) = \mu a(\alpha x, -\tau)(1 - u(\alpha x, -\tau)),$$

$u_t \in C([-\tau, 0], C([0, 1]))$ is defined pointwise by $u_t(\theta) = u(t + \theta)$ and $\varphi(t)(x) = \varphi(x, t)$.

The equation (FDE) where $f$ is the generator of a linear semigroup, $T(t)$, $t \geq 0$, and $g$ is nonlinear, together with its mild or integrated form

$$u(t) = T(t)\varphi(0) + \int_0^t T(t - s)g(u_s)\, ds, \quad t > 0$$

$$u(t) = \varphi(t), \quad -\tau \leq t \leq 0$$

has been studied by many authors using the semigroup theory. We only recall that the case that $g$ is Lipschitz continuous was first considered by Webb in [10] and that in recent papers Dyson and Villella-Bressan consider the case in which $f$ is nonlinear and $g$ locally Lipschitz continuous [3]. However, here we can write explicitly $T(t)\varphi$ and hence the integration version of (E),

$$u(x, t) = e^{-t} \varphi(e^{-t} x, 0)$$

$$+ \int_0^t e^{-(t-s)} \mu u(e^{-(t-s)} \alpha x, s - \tau)(1 - u(e^{-(t-s)} \alpha x, s - \tau))\, ds,$$

$$0 \leq x \leq 1, \quad t \geq 0$$

$$u(x, t) = \varphi(x, t), \quad 0 \leq x \leq 1, \quad -\tau \leq t \leq 0,$$

and derive directly existence and asymptotic results.

We prove the following existence and regularity results.

**Theorem 3.1.** i) **Suppose that the initial data $\varphi(x, t)$ is continuous in $[0, 1] \times [-\tau, 0]$. Then there exists a unique mild solution of (E). That is, there exists a unique continuous function $u(x, t)$, $0 \leq x \leq 1, t \geq -\tau$, which satisfies (E)**.

ii) **Suppose also that $\partial \varphi(x, t)/\partial x$ exists for $0 < x \leq 1$ and $-\tau \leq t \leq 0$ and that for each $0 < \beta < 1$ and $-\tau \leq t \leq 0$ there exists $m(t, \beta) \in \mathbb{R}$ such that**

$$|\partial \varphi(x, t)/\partial x| \leq m(t, \beta) \quad \text{for } \beta \leq x \leq 0$$
where \( m(t, \beta) \in L^1(-\tau, 0) \). Then \( \partial u(x, t)/\partial t \) and \( \partial u(x, t)/\partial x \) exist for \( t > 0 \) and \( 0 < x \leq 1 \) and \( u(x, t) \) is a solution of (E) for \( 0 < x \leq 1 \) and \( t \geq -\tau \).

iii) If, moreover, for each \( -\tau \leq t \leq 0 \) there exists \( m(t) \in \mathbb{R} \) such that

\[
\left| x \frac{\partial \varphi(x, t)}{\partial x} \right| \leq m(t) \quad \text{for } 0 < x \leq 1,
\]

where \( m(t) \in L^1(-\tau, 0) \), and

\[
\lim_{x \to 0} x \frac{\partial \varphi(x, t)}{\partial x} = 0 \quad \text{for } -\tau \leq t \leq 0,
\]

then

\[
\lim_{x \to 0} x \frac{\partial u(x, t)}{\partial x} = 0 \quad \text{for } t > 0,
\]

\((\partial/\partial x)(xu(x, t))\) exists also for \( x = 0 \) and \( u(x, t) \) is a solution of (E) for \( 0 \leq x \leq 1 \) and \( t \geq -\tau \).

iv) If also \( \varphi(t) \in D_f \) for \( -\tau \leq t \leq 0 \), then \( u(t, .) \in D_f \) for all \( t > 0 \); and \( \partial u(x, t)/\partial t \) is continuous for \( t > 0, 0 \leq x \leq 1 \).

v) If also \( \varphi'(0) = f(\varphi(0)) + g(\varphi) \), then \((\partial u/\partial t)(x, 0)\) exists and \((\partial u/\partial t)(x, t)\) is continuous at \( t = 0 \).

Proof. The proof is very simple and uses elementary techniques. This is due to the fact that the delay in \( t \) is strict, that is, \( g(\varphi) = g(\varphi|_{[-\tau, \eta]} \) with \( \gamma < 0 \), and in fact we have \( g(\varphi) = \mu \varphi(\alpha x, -\tau)(1 - \varphi(\alpha x, -\tau)) \);

and so we can use the method of steps.

i) From (E)' for \( 0 \leq t \leq \tau, 0 \leq x \leq 1 \), we have

\[
u(x, t) = e^{-t} \varphi(e^{-t} x, 0) + \int_0^t e^{-(t-s)} \mu \varphi(e^{-(t-s)} \alpha x, s - \tau)(1 - \varphi(e^{-(t-s)} \alpha x, s - \tau)) \, ds
\]

and the continuity of \( \varphi(x, t) \) in the compact set \([0, 1] \times [-\tau, 0]\) implies the continuity of \( u(x, t) \) in the compact set \([0, 1] \times [0, \tau]\). Also \( u(x, 0) = \)


the method of steps gives existence and uniqueness of a continuous solution of (E)' also for $t \geq \tau$, $0 \leq x \leq 1$.

ii) Formally, for $0 \leq t \leq \tau$, $0 < x < 1$,

$$
\frac{\partial u(x, t)}{\partial t} = -e^{-t}\varphi(e^{-t}x, 0) - e^{-t}e^{-t}x \frac{\partial \varphi (e^{-t}x, 0)}{\partial x} \\
- e^{-t} \int_0^t e^{s}\mu \varphi(e^{-(t-s)}\alpha x, s - \tau)(1 - \varphi(e^{-(t-s)}\alpha x, s - \tau)) \, ds \\
+ e^{-t}e^{t}\mu \varphi(\alpha x, t - \tau)(1 - \varphi(\alpha x, t - \tau)) \\
- e^{-t} \int_0^t e^{s}\mu e^{-(t-s)}\alpha x \frac{\partial \varphi}{\partial x} \\
(e^{-(t-s)}\alpha x, s - \tau)(1 - 2\varphi(e^{-(t-s)}\alpha x, s - \tau)) \, ds
$$

and

$$
\frac{\partial u(x, t)}{\partial x} = e^{-t}e^{-t}x \frac{\partial \varphi (e^{-t}x, 0)}{\partial x} \\
+ \int_0^t e^{-(t-s)}\mu e^{-(t-s)}\alpha x \frac{\partial \varphi}{\partial x} \\
(e^{-(t-s)}\alpha x, s - \tau)(1 - 2\varphi(e^{-(t-s)}\alpha x, s - \tau)) \, ds.
$$

(3.4)

Let $\beta > 0$. Then for $\beta \leq x \leq 1$ and $0 \leq t \leq \tau$,

$$
\left| e^{-(t-s)}\mu e^{-(t-s)}\alpha x \frac{\partial \varphi (e^{-(t-s)}\alpha x, s - \tau)(1 - 2\varphi(e^{-(t-s)}\alpha x, s - \tau))}{\partial x} \right| \\
\leq \text{const} \, m(s - \tau, e^{-\tau}\alpha \beta) = m_1(t, \beta) \in L^1(0, \tau),
$$

and so both $\partial u(x, t)/\partial t$ and $\partial u(x, t)/\partial x$ exist for $0 < x < 1$ and $0 \leq t \leq \tau$ and satisfy

$$
(3.5) \quad \frac{\partial u(x, t)}{\partial t} = -u(x, t) - x \frac{\partial u(x, t)}{\partial x} + \mu u(\alpha x, t - \tau)(1 - u(\alpha x, t - \tau))
$$

for $0 < x \leq 1$ and $0 \leq t \leq \tau$. 
Also, from (3.4), for $\beta \leq x \leq 1$,
\[
\left| \frac{\partial u(x, t)}{\partial x} \right| \leq \left| \frac{\partial \varphi}{\partial x}(e^{-t}x, 0) \right| \\
+ \int_0^t \text{const} \left| \frac{\partial \varphi}{\partial x}(e^{-(t-s)}ax, s - \tau) \right| \, ds \\
\leq m(0, e^{-\tau}\beta) + \int_{-\tau}^0 \text{const} \, m(s, e^{-\tau}a\beta) \, ds \\
\leq m(0, e^{-\tau}\beta) + \text{const} \in L^1(0, \tau),
\]
and the method of steps gives the result.

iii) From (3.4), for $0 \leq t \leq \tau$,
\[
x \frac{\partial u(x, t)}{\partial x} = e^{-t}e^{-t}x \frac{\partial \varphi(e^{-t}x, 0)}{\partial x} \\
+ \int_0^t e^{-(t-s)}\mu e^{-(t-s)x} \frac{\partial \varphi(e^{-(t-s)}ax, s - \tau)}{\partial x} \\
(1 - 2\varphi(e^{-(t-s)}ax, s - \tau)) \, ds
\]
so
\[
\left| x \frac{\partial u(x, t)}{\partial x} \right| \leq \left| e^{-t}x \frac{\partial \varphi(e^{-t}x, 0)}{\partial x} \right| \\
+ \int_0^t \text{const} \left| \alpha e^{-(t-s)x} \frac{\partial \varphi(e^{-(t-s)}ax, s - \tau)}{\partial x} \right| \, ds
\]
and by the Lebesgue dominated convergence theorem,
\[
\lim_{x \to 0} x \frac{\partial u(x, t)}{\partial t} = 0;
\]
also
\[
\left| x \frac{\partial u(x, t)}{\partial x} \right| \leq m(0) + \int_{-\tau}^0 \text{const} \, m(s) \, ds \leq \text{const},
\]
and $u(x, t)$ satisfies analogous conditions to (3.1) and (3.2) for $0 \leq t \leq \tau$; and again the method of steps proves that $u(x, t)$ satisfies (3.3).
If $x > 0$, 
\[
\frac{\partial}{\partial x} (x u(x, t)) = u(x, t) + x \frac{\partial}{\partial x} u(x, t)
\]
and from (3.3), 
\[
\lim_{x \to 0} \frac{\partial}{\partial x} (x u(x, t)) = u(0, t).
\]
So $(\partial/\partial x)(x u(x, t))$ exists also for $x = 0$ and is equal to $u(0, t)$. But from (E)' with $x = 0$ we have 
\[
(3.6) \quad \frac{\partial u}{\partial t}(0, t) = -u(0, t) + \mu u(0, t - \tau)(1 - u(0, t - \tau)),
\]
so (E) is satisfied also for $x = 0$.

iv) From i), for each $t$, $u(x, t)$ is continuous on $[0, 1]$, from (3.4) 
$(\partial u/\partial x)(x, t)$ is continuous on $]0, 1]$ if $(\partial \varphi/\partial x)(x, t)$ is continuous on $]0, 1]$ and from (3.3) \(\lim_{x \to 0} x (\partial u/\partial x)(x, t) = 0\), hence $u(\cdot, t) \in D_f$ also for $t > 0$; also from (3.4) $(\partial u/\partial t)(x, t)$ is continuous on $]0, 1]$ and from (3.5) and (3.6), \(\lim_{x \to 0} (\partial u/\partial t)(x, t) = (\partial u/\partial t)(0, t)\) and $(\partial u/\partial t)(x, t)$ is continuous on $[0, 1]$.

v) From (3.4) for $0 < x \leq 1$
\[
\lim_{t \to 0} \frac{\partial u}{\partial t}(x, t) = -\varphi(x, 0) - x \frac{\partial \varphi}{\partial x}(x, 0)
\]
\[
- \mu \varphi(\alpha x, -\tau)(1 - \varphi(\alpha x, -\tau))
\]
\[
= -f(\varphi(0))(x) - g(\varphi)(x),
\]
as required. \(\Box\)

4. **Uniqueness of solutions when $\tau > 0$.** As we have seen in 
Section 3, given $\varphi \in C([0, 1] \times [-\tau, 0])$ there exists a unique solution of 
(E)' with initial data $\varphi$. And, in fact, if $\varphi_1$ and $\varphi_2 \in C([0, 1] \times [-\tau, 0])$ and 
$$
\varphi_1(x, t) = \varphi_2(x, t) \quad \text{for} \quad x \in [0, b] \text{ and } t \in [-\tau, 0]$
$$
where $0 < b \leq 1$, then 
$$
u_1(x, t) = u_2(x, t) \quad \text{for} \quad x \in [0, b] \text{ and } t \geq 0,$$
where $u_1$ and $u_2$ are the solutions with initial data $\varphi_1$ and $\varphi_2$, respectively.
We can prove more.

**Theorem 4.1.** Let \( \varphi_1, \varphi_2 \in C([0,1] \times [-\tau,0]) \) and suppose that there is a \( b, \) \( 0 < b < 1 \) such that

\[
\varphi_1(x,t) = \varphi_2(x,t) \quad \text{for} \ x \in [0,b] \ \text{and} \ t \in [-\tau,0].
\]

Suppose that \( u_1 \) and \( u_2 \) are solutions of \((E)\)' with initial data \( \varphi_1 \) and \( \varphi_2 \) respectively. Then there is a \( \bar{t} \) such that

\[
u_1(x,t) = u_2(x,t) \quad \text{for} \ x \in [0,1] \ \text{and} \ t \geq \bar{t}.
\]

We can choose \( \bar{t} = \log(1/b) + (\log b / \log \alpha)\tau.\)

**Proof.** Set \( t_0 = 0, t_{n+1} = t_n + \log \alpha^{-1} + \tau = \log \alpha^{-(n+1)} + (n + 1)\tau, \) and make the induction hypothesis for \( n \) such that \( b\alpha^{-n} \leq 1, \)

\[
u_1(x,t) = u_2(x,t) \quad \text{for} \ x \in [0,b\alpha^{-n}] \ \text{and} \ t \geq t_n - \tau.
\]

This is true for \( n = 0. \) Suppose true for \( n. \) If \( i \geq t_n, \)

\[
u_i(x,t) = e^{-(t-t_n)}u_i(e^{-(t-t_n)}x,t_n)
\]

\[
+ \int_{t_n}^{t} e^{-(t-s)}\mu u_i(e^{-(t-s)}\alpha x,s-\tau)(1-u_i(e^{-(t-s)}\alpha x,s-\tau)) \, ds,
\]

\[i = 1, 2.
\]

Let \( x \in [0,b\alpha^{-(n+1)}]. \) Then \( e^{-(t-s)}\alpha x \in [0,b\alpha^{-n}] \) and so

\[
u_1(e^{-(t-s)}\alpha x,s-\tau) = u_2(e^{-(t-s)}\alpha x,s-\tau) \quad \text{for} \ s \geq t_n.
\]

Also if \( t \geq t_{n+1} - \tau, \) then

\[
e^{-(t-t_n)} \leq e^{-(t_{n+1} - \tau - t_n)} = e^{-\log \alpha^{-1}} = \alpha,
\]

so \( e^{-(t-t_n)}x \in [0,b\alpha^{-n}] \) and so \( u_1(e^{-(t-t_n)}x,t_n) = u_1(e^{-(t-t_n)}x,t_n). \)

It follows that \( u_1(x,t) = u_2(x,t) \) also for \( x \in [0,b\alpha^{-(n+1)}] \) and \( t \geq t_{n+1} - \tau. \)
Let $N$ be such that $b\alpha^{-N} < 1 \leq b\alpha^{-(N+1)}$ so that $\log(1/b) > \log \alpha^{-N}$. Take $x \in [0, 1]$ and $t \geq \log(1/b) + N\tau$. Then $t \geq t_N$ and
\[
e^{-(t-t_N)x} \leq e^{\log b - N\tau + \log \alpha^{-N} + N\tau} = b\alpha^{-N},
\]
and so $u_1(e^{-(t-t_N)x}, t_N) = u_1(e^{-(t-t_N)x}) = u_1(e^{-(t-s)x}, s-N\tau) = u_2(e^{-(t-s)x}, s-N\tau)$ for $s \geq t_N$. So, as above, $u_1(x, t) = u_2(x, s)$ for $x \in [0, 1]$ and $t \geq \log(1/b) + N\tau$. And as $\log(1/b) > N\log(1/\alpha)$ we can choose $\bar{t} = \log(1/b) + (\log b/\log \alpha)\tau$.
\[\square\]

This theorem has useful applications when proving asymptotic results. Generally asymptotic results are proved for $x \in [0, b]$ for some $0 < b < 1$ and then extended to $x \in [0, 1]$ using the above result.

We shall use the following corollary.

**Corollary 4.2.** Let $0 < b \leq 1$ and $\bar{t} \geq 0$. Let $u_1(x, t)$ and $u_2(x, t)$ be solutions of equation (E)' such that
\[u_1(x, t) = u_2(x, t) \text{ for } x \in [0, b] \text{ and } \bar{t} - \tau \leq t \leq \bar{t}.
\]

Then
\[u_1(x, t) = u_2(x, t) \text{ for } x \in [0, 1] \text{ and } t \geq \bar{t} + \log \frac{1}{b} + \frac{\log b}{\log \alpha}\tau.
\]

**5. Invariance when $\tau > 0$**. Suppose that $V \subset C([-\tau, 0]; C([0, 1]))$. We say that $V$ is **flow-invariant** or simply **invariant** if given $\varphi \in V$ then $u_t \in V$ for all $t \geq 0$, where $u(x, t)$ is the solution of (E)' with initial data $\varphi$.

**Theorem 5.1.** Let $0 < b \leq 1$ and $\bar{t} = \log(1/b) + (\log b/\log \alpha)\tau$.

i) If $0 < R \leq 1$ and $0 < \mu \leq 4R$ or $0 < \mu \leq 1$, then the set
\[U_{R,b} = \{\varphi \in C([-\tau, 0]; C([0, 1])), 0 \leq \varphi(x, t) \leq R \text{ for } x \in [0, b]\}
\]
is invariant.
In particular, if $0 < \mu \leq 4$ and $0 \leq \varphi(x,t) \leq 1$ for $x \in [0,b]$, then $0 \leq u(x,t) \leq 1$ for $x \in [0,b]$ and $t \geq 0$.

Also, if $\varphi \in U_{R,b}$ then $u_t \in U_{R,1}$ for $t \geq \bar{t} + \tau$.

ii) If $0 < \mu < 1$ and $0 < \rho \leq 1/\mu - 1$, then the set

$$V_{\rho,b} = \{ \varphi \in C([-\tau,0];C[0,1]), |\varphi(x,t)| \leq \rho \text{ for } x \in [0,b] \}$$

is invariant.

Also, if $\varphi \in V_{\rho,b}$ then $u_t \in V_{\rho,1}$ for $t \geq \bar{t} + \tau$.

iii) If $1 < \mu \leq 2$ and $\beta$ and $\gamma$ are such that $0 \leq \beta \leq 1 - 1/\mu$ and $1 - 1/\mu \leq \gamma \leq 1/2$, then the set

$$W_{\beta,\gamma,b} = \{ \varphi \in C([-\tau,0];C[0,1]), \beta \leq \varphi(x,t) \leq \gamma \text{ for } x \in [0,b] \}$$

is invariant. So, in particular, the sets

$$W_{0,1-1/\mu,b} = \{ \varphi \in C([\tau,0];C[0,1]), 0 \leq \varphi(x,t) \leq 1 - 1/\mu \text{ for } x \in [0,b] \}$$

and

$$W_{1-1/\mu,1/2,b} = \{ \varphi \in C([\tau,0];C[0,1]), 1 - 1/\mu \leq \varphi(x,t) \leq 1/2 \text{ for } x \in [0,b] \}$$

are invariant.

Also, if $\varphi \in W_{\beta,\gamma,b}$, then $u_t \in W_{\beta,\gamma,1}$ for $t \geq \bar{t} + \tau$.

Proof. i) Let $0 < \mu \leq 4R$. For all $\xi \in \mathbb{R}$, $\mu \xi(1 - \xi) \leq \mu(1/4) \leq 4R(1/4) = R$ and if $0 \leq \xi \leq 1$, then $\xi(1 - \xi) \geq 0$. From (E)' we have for $t \in [0,\tau]$, $x \in [0,b]$,

$$u(x,t) = e^{-t} \varphi(e^{-t}x,0)$$

$$+ \int_0^t e^{-(t-s)} \mu \varphi(e^{-(t-s)}x, s-\tau)(1-\varphi(e^{-(t-s)}x, s-\tau)) ds$$

and so if $0 \leq \varphi(x,t) \leq R \leq 1$,

$$0 \leq u(x,t) \leq e^{-t}R + \int_0^t e^{-(t-s)}R ds = R$$
and, using induction, we prove that \(0 \leq u(x, t) \leq R\) for \(x \in [0, b]\) and \(t \in [0, n\tau]\) for \(n \in \mathbb{N}\). Hence for all \(t \geq 0\), \(u_t \in U_{R,b}\).

Let now \(0 < \mu \leq 1\). If \(0 < \xi \leq R \leq 1\), then \(0 \leq \mu\xi(1 - \xi) \leq \xi \leq R\) and as above we prove that if \(\varphi \in U_{R,b}\) then \(u_t \in U_{R,b}\) for all \(t \geq 0\).

Set
\[
\tilde{\varphi}(x, t) = \begin{cases} 
\varphi(x, t) & x \in [0, b], \ t \in [-\tau, 0] \\
\varphi(b, t) & x \in [b, 1], \ t \in [-\tau, 0],
\end{cases}
\]
so that \(\tilde{\varphi} \in U_{R,1}\) and so \(0 \leq \tilde{u}(x, t) \leq R\) for \(x \in [0, 1]\) and \(t \geq 0\).

From Theorem 4.1, \(u(x, t) = \tilde{u}(x, t)\) for \(x \in [0, 1]\) and \(t \geq \log(1/b) + (\log b/\log \alpha)\tau\). And so \(u_t = \tilde{u}_t\) for \(t \geq \tilde{t} + \tau\).

ii) For \(x \in [0, b]\) and \(t \in [0, \tau]\) we have, for
\[
||\varphi||_b = \sup_{(x,t) \in [0,b] \times [-\tau,0]} |\varphi(x, t)|,
\]

\[
|u(x, t)| \leq e^{-t}||\varphi||_b + \int_0^t e^{-(t-s)}\mu||\varphi||_b(1 + ||\varphi||_b) \, ds
\]

\[
\leq e^{-t}||\varphi||_b + \int_0^t e^{-(t-s)}||\varphi||_b \, ds = ||\varphi||_b,
\]
as \(\mu(1 + \rho) \leq 1\). And by induction we can prove that \(|u(x, t)| \leq ||\varphi||_b\) for \(x \in [0, b]\) and \(t \geq 0\).

As above, we also have \(u_t \in V_{\rho,1}\) for \(t \geq \tilde{t} + \tau\).

iii) The proof depends essentially on the fact that for \(\xi \in [\beta, \gamma]\), \(\xi(1 - \xi)\) is increasing. Take \(t \in [0, \tau]\), \(x \in [0, b]\) and use (5.1) to get
\[
\beta e^{-t} + (1 - e^t)\mu\beta(1 - \beta) \leq u(x, t) \leq \gamma e^{-t} + (1 - e^{-t})\mu\gamma(1 - \gamma).
\]

But under the given conditions on \(\beta\) and \(\gamma\), \(\beta \leq \mu\beta(1 - \beta)\) and \(\gamma \geq \mu\gamma(1 - \gamma)\) so that
\[
\beta \leq u(x, t) \leq \gamma \quad \text{for } x \in [0, b] \text{ and } t \in [0, \tau].
\]

Continue by induction to prove the result for \(t \in [0, n\tau]\) and \(n \in \mathbb{N}\).

As above, we also have \(u_t \in W_{\beta,\gamma,1}\) for \(t \geq \tilde{t} + \tau\). \(\square\)

**Corollary 5.2.** Let \(0 < b < 1\) and \(\tilde{t} = \log(1/b) + (\log b/\log \alpha)\tau\).
i) Let $0 < \mu \leq 1$ and $0 < R \leq 1$. If
\[ 0 \leq u(x, t) \leq R \quad \text{for} \ x \in [0,b] \ \text{and} \ \tilde{t} - \tau \leq t \leq \tilde{t}, \]
then
\[ 0 \leq u(x, t) \leq R \quad \text{for} \ x \in [0,1] \ \text{and} \ t \geq \tilde{t} + \tilde{\tau}, \]
that is, if $u_t, \in U_{R,b}$ then $u_t \in U_{R,1}$ for $t \geq \tilde{t} + \tilde{\tau} + \tau$.

ii) Let $0 < \mu < 1$ and $\rho < 1/\mu - 1$. If
\[ |u(x, t)| \leq \rho \quad \text{for} \ x \in [0,b] \ \text{and} \ \tilde{t} - \tau \leq t \leq \tilde{t}, \]
then
\[ |u(x, t)| \leq \rho \quad \text{for} \ x \in [0,1] \ \text{and} \ t \geq \tilde{t} + \tilde{\tau}, \]
that is, if $u_t, \in V_{R,b}$ then $u_t \in V_{R,1}$ for $t \geq \tilde{t} + \tilde{\tau} + \tau$.

iii) Let $1 < \mu \leq 2$ and $\beta$ and $\gamma$ such that $0 \leq \beta \leq 1 - 1/\mu$ and $1 - 1/\mu \leq \gamma \leq 1/2$. If
\[ \beta \leq u(x, t) \leq \gamma \quad \text{for} \ x \in [0,b] \ \text{and} \ \tilde{t} - \tau \leq t \leq \tilde{t}, \]
then
\[ \beta \leq u(x, t) \leq \gamma \quad \text{for} \ x \in [0,1] \ \text{and} \ t \geq \tilde{t} + \tilde{\tau}, \]
that is, if $u_t, \in W_{\beta,\gamma,b}$ then $u_t \in W_{\beta,\gamma,1}$ for $t \geq \tilde{t} + \tilde{\tau} + \tau$.

Note that the space $X_0 = \{ \varphi \in C([0,1] \times [-\tau,0]), \varphi(0, t) = 0 \}$ is invariant for all $\mu > 0$. This is not true for the cones $X^+ = \{ \varphi \in C([0,1] \times [-\tau,0]), \varphi \geq 0 \}$ and $X^+_0 = X_0 \cup X^+$; the sets $U_{1,1}$ and $W_{\beta,\gamma,1}$ are subsets of $X^+$ which are invariant respectively for $0 < \mu \leq 4$ and $1 < \mu \leq 2$.

6. Asymptotic behavior when $\tau > 0$. We shall assume throughout that the initial data $\varphi$ belong to $C([0,1] \times [-\tau,0])$ and denote by $u(x, t)$ the solution of (E)' with initial data $\varphi$.

We first consider the case $0 < \mu < 1$.

**Theorem 6.1.** Let $0 < \mu < 1$. Suppose that there exists $0 < b \leq 1$ such that either
\[(6.1) \quad 0 \leq \varphi(x, t) \leq 1 \quad \text{for} \ x \in [0,b] \ \text{and} \ t \in [-\tau,0] \]
(6.2) \[ |\varphi(x, t)| \leq 1/\mu - 1 \quad \text{for } x \in [0, b] \text{ and } t \in [-\tau, 0]. \]

Then \( u(x, t) \) tends to zero exponentially as \( t \) tends to infinity, uniformly for \( x \in [0, 1] \).

**Proof.** Suppose that (6.1) holds. From i) of Theorem 5.1,

\[ 0 \leq u(x, t) \leq 1 \quad \text{for } x \in [0, b] \text{ and } t \geq -\tau, \]

and so \( 0 \leq u(e^{(t-s)\alpha x}, t-\tau)(1 - u(e^{(t-s)\alpha x}, t-\tau)) \leq 1 \) for \( x \in [0, b] \) and \( t \geq 0 \). And so, for \( t \geq 0 \),

\[ u(x, t) \leq e^{-t} + \int_0^t e^{-(t-s)}\mu \, ds = e^{-t}(1 - \mu) + \mu. \]

Let \( \mu < \lambda < 1 \) and \( T > \tau \) such that \( e^{-(T-\tau)}(1 - \mu) = \lambda - \mu \), that is, \( T = \log[(1 - \mu)/(\lambda - \mu)] + \tau \). So

\[ u(x, t) \leq \lambda \quad \text{for } x \in [0, b] \text{ and } t \geq T - \tau. \]

We prove by induction that

\[ u(x, t) \leq \lambda^n \quad \text{for } x \in [0, b] \text{ and } t \geq nT - \tau. \]

It is true for \( n = 1 \). Suppose it is true for \( n \). Let \( t \geq (n + 1)T - \tau \). Denote by \( |\cdot|_b \) the sup norm on \( C([0, b]) \). Then

\[ u(x, t) \leq e^{-(t-nT)}|u(nT)|_b + \int_{nT}^t e^{-(s-nT)}\mu|u(s - \tau)|_b \, ds \]

\[ \leq e^{-(t-nT)}\lambda^n + \int_{nT}^t e^{-(s-nT)}\mu\lambda^n \, ds \]

\[ = \lambda^n(e^{-(t-nT)}(1 - \mu) + \mu) \]

\[ \leq \lambda^n(e^{-(T-\tau)}(1 - \mu) + \mu) = \lambda^{n+1}. \]

Then from i) of Corollary 5.2

\[ u(x, t) \leq \lambda^n \quad \text{for } x \in [0, 1] \text{ and } t \geq nT + \log \frac{1}{b} + \frac{\log b}{\log \alpha} \tau. \]
Let \( t \geq \log(1/b) + (\log b / \log \alpha)\tau \) and let \( n \) be such that

\[
nT \leq t - \log \frac{1}{b} - \frac{\log b}{\log \alpha} \tau < (n + 1)T.\]

Then \( u(x, t) \leq \lambda^n \leq (1/\lambda)^{t/T - (\log(1/b) + (\log b / \log \alpha)\tau)(1/T)} \) and so

\[
0 \leq u(x, t) \leq Me^{\beta t} \quad \text{for } t \geq \log \frac{1}{b} + \frac{\log b}{\log \alpha} \tau,
\]

where

\[
(6.3) \quad M = \lambda^{-1-(\log(1/b) + (\log b / \log \alpha)\tau)(1/T)} \quad \text{and } \quad \beta = \frac{\log \lambda}{T},
\]

and \( \beta < 0 \) as \( \lambda < 1 \).

Suppose now that (6.2) holds, that is, suppose \( ||\varphi||_b < 1/\mu - 1 \) so that \( \nu = \mu(1 + ||\varphi||_b) < 1 \). Choose \( \lambda \) and \( T \) as follows: \( \nu < \lambda < 1 \) and \( T = \log((1 - \nu)/(\lambda - \nu)) + \tau \). It is easy to adapt the above proof to show that

\[
|u(x, t)| \leq ||\varphi||_b \lambda^n \quad \text{for } x \in [0, 1] \text{ and } t \geq nT + \log \frac{1}{b} + \frac{\log b}{\log \alpha} \tau,
\]

and hence that

\[
|u(x, t)| \leq ||\varphi||_b Me^{\beta t} \quad \text{for } t \geq \log \frac{1}{b} + \frac{\log b}{\log \alpha} \tau,
\]

where \( M \) and \( \beta \) are as in (6.3). \( \Box \)

The following corollary can be proved using a simple compactness argument.

**Corollary 6.2.** Let \( 0 < \mu < 1 \). If \( 1 - 1/\mu < \varphi(0, t) < \max\{1, 1/\mu - 1\} \) for \( t \in [-\tau, 0] \), in particular if \( \varphi \in X_0 \), then \( u(x, t) \) tends to zero exponentially as \( t \) tends to infinity.

If \( \mu = 1 \) and \( \varphi \in X_0^+ \), then i) of Theorem 5.1 shows that \( u(x, t) \) tends to zero as \( t \) tends to infinity.
We now consider the case $\mu > 1$.

**Proposition 6.3.** Let $1 < \mu \leq 2$ and suppose that there exists $b$, $0 < b \leq 1$ such that $\varphi(x, t) = \psi(t)$ for $x \in [0, b]$ and $t \in [-\tau, 0]$ where $\psi$ is independent of $x$ and $0 < \psi(t) < 1/2$. Then $u(x, t)$ tends to $1 - 1/\mu$ exponentially as $t$ tends to infinity.

**Proof.** There exist $\beta$ and $\gamma$ such that $0 < \beta < 1 - 1/\mu$, $1 - 1/\mu < \gamma < 1/2$ and $\beta \leq \psi(t) \leq \gamma$ for $t \in [-\tau, 0]$. Thus, by iii) of Theorem 5.1, $\beta \leq u(x, t) \leq \gamma$ for $x \in [0, b]$. We look for solutions $u(t)$ of (E) which are independent of $x$. So we want to solve

$$u'(t) + u(t) = \mu u(t - \tau)(1 - u(t - \tau))$$

$$u(t) = \psi(t), \quad t \in [-\tau, 0].$$

This clearly has a unique solution which satisfies

$$u(t) = e^{-t}\psi(0) + \int_0^t e^{-(t-s)}\mu u(s - \tau)(1 - u(s - \tau))\, ds.$$  \hspace{1cm} (6.4)

If $\psi(t) = 1 - 1/\mu$, $1 - 1/\mu$ is the solution of (6.4), so setting $v(t) = u(t) - (1 - 1/\mu)$, we have

$$v(t) = e^{-t}\left(\psi(0) - \left(1 - \frac{1}{\mu}\right)\right) + \int_0^t e^{-(t-s)}v(s - \tau)(1 - \mu u(s - \tau))\, ds.$$ \hspace{1cm} (6.5)

But $1 - \mu\gamma \leq 1 - \mu u(s - \tau) \leq 1 - \mu\beta$, $1 - \mu\gamma > 1 - \mu/2 \geq 0$ and $1 - \mu\beta < 1$. Set $1 - \mu \beta = \nu$, and let $\lambda$ and $T$ be such that $\nu < \lambda < 1$ and $T = \log[(1 - \nu)/(\lambda - \nu)]$. So from (6.5) for $t \geq T$,

$$|v(t)| \leq e^{-t}\|v\|_\infty + \int_0^t e^{-(t-s)}\|v\|_\infty\nu\, ds = \|v\|_\infty(v + e^{-t}(1 - \nu)) \leq \lambda\|v\|_\infty,$$

where $\|v\|_\infty = \sup_{t \geq -\tau}|v(t)|$. It now follows by induction, as in the proof of Theorem 6.1, that

$$|v(t)| \leq \lambda^n\|v\|_\infty \quad \text{for} \quad t \geq nT + (n + 1)\tau,$$
and so \( u(t) \) tends to \( 1 - 1/\mu \) exponentially as \( t \) tends to \( \infty \).

Now if \( u(x,t) \) is the solution of \( (E)' \) with initial data \( \varphi \) where \( \varphi(x,t) = \psi(t) \) for \( x \in [0,b] \) and \( t \in [-\tau,0] \), then \( u(x,t) = u(t) \) for \( x \in [0,1] \) and \( t \geq \log(1/b) + (\log b/\log \alpha)\tau \). And so \( u(x,t) \) also tends to \( 1 - 1/\mu \) exponentially as \( t \) tends to \( \infty \), uniformly for \( x \in [0,1] \). \( \square \)

**Theorem 6.4.** Let \( 1 < \mu \leq 2 \), and suppose that \( \varphi(0,t) = \psi(t) \) where \( 0 < \psi(t) < 1/2 \). Then \( u(x,t) \) tends to \( 1 - 1/\mu \) exponentially as \( t \to \infty \).

**Proof.** Let \( \varepsilon > 0 \), \( \varepsilon < \min_{t \in [-\tau,0]} \{ \psi(t), 1/2 - \psi(t) \} \). Then there is a \( b \), \( 0 < b \leq 1 \), such that
\[
0 < \psi(t) - \varepsilon < \varphi(x,t) < \psi(t) + \varepsilon < 1/2 \quad \text{for } x \in [0,b].
\]
Let \( u(x,t), v(x,t) \) and \( w(x,t) \) be the solutions of \( (E)' \) with initial data \( \varphi(x,t), \psi(t) - \varepsilon \) and \( \psi(t) + \varepsilon \) respectively. Then by iii) of Theorem 5.1,
\[
0 < u(x,t), v(x,t), w(x,t) < 1/2 \quad \text{for } x \in [0,b] \text{ and } t \geq 0.
\]
Also \( \xi(1-\xi) \) is increasing for \( \xi \in [0,1/2] \), so we can use \( (E)' \) to prove by induction that for \( t \in [0,n\tau] \), \( n > 0 \),
\[
v(x,t) \leq u(x,t) \leq u(x,t) \quad \text{for } x \in [0,b].
\]
However, \( v(x,t) \) and \( w(x,t) \) tend to \( 1 - 1/\mu \) as \( t \) tends to \( \infty \), so \( u(x,t) \) tends to \( 1 - 1/\mu \) as \( t \) tends to \( \infty \) for \( x \in [0,b] \). Thus, by iii) of Corollary 5.2, \( u(x,t) \) tends to \( 1 - 1/\mu \) as \( t \) tends to \( \infty \) for \( x \in [0,1] \). \( \square \)

**7. Equilibrium solutions for the case 1 < \mu \leq 4.** The existence of equilibrium solutions requires solutions of the equation
\[
(x\varphi(x))' - \mu \varphi(\alpha x) = -\mu \varphi(\alpha x)^2, \quad 0 \leq x \leq 1.
\]
Obviously, \( \varphi = 0 \) and \( \varphi = 1 - 1/\mu \) are solutions of (7.1). We prove that when \( 1 < \mu \leq 4 \) there exists a one-parameter family of equilibrium solutions of \( (E)' \) in \( X_0^+ \).

**Theorem 7.1.** Let \( 1 < \mu \leq 4 \), \( a_1 > 0 \), and let \( r \) be the unique positive solution of \( r + 1 = \mu a_1^r \). There exists a solution \( \varphi(x) \) to (7.1) such that \( \varphi(0) = 0 \), \( 0 < \varphi(x) \leq 1 \) for \( 0 < x \leq 1 \), and \( \lim_{x \to 0+} (\varphi(x)/x^r) = a_1 \).
**Proof.** Let \( \varphi(x) = \sum_{n=1}^{\infty} a_n x^{rn} \), where \( a_2, a_3, \ldots \) will be determined below. Substitute \( \varphi \) into (7.1) to obtain

\[
\sum_{n=1}^{\infty} a_n (nr + 1 - \mu \alpha \tau^n) x^{rn} = -\mu \{ a_1^2 \alpha^{2r} x^{2r} + 2a_1a_2 \alpha^{3r} x^{3r} + (2a_1a_3 + a_2^2) \alpha^{4r} x^{4r} + (2a_1a_4 + 2a_2a_3) \alpha^{5r} x^{5r} + \cdots \}.
\]

Define \( q(n) = \mu/(nr + 1 - \mu \alpha \tau^n) \), \( n = 2, 3, \ldots \). Notice that \( q(n) > 0 \) and \( q(2) > q(3) > \cdots \), since \( nr + 1 > r + 1 = \mu \alpha \tau > \mu \alpha \tau^n \). Define

\[
a_2 = -a_1^2 \alpha^{2r} q(2), \quad a_3 = -2a_1a_2 \alpha^{3r} q(3), \ldots,
\]

\[
a_{2n} = -\left( a_n^2 + \sum_{i+j=2n, i \neq j} a_ia_j \right) \alpha^{2nr} q(2n),
\]

\[
a_{2n+1} = -\left( \sum_{i+j=2n+1} a_ia_j \right) \alpha^{(2n+1)r} q(2n+1), \ldots.
\]

Let \( K = \sup_{n \geq 2} (n - 1)q(n) \) (note that \( \lim_{n \to \infty} (n - 1)q(n) = \mu/r > 1 \)). Observe that \( |a_2| \leq a_1^2 q(2) \leq a_1^2 K, |a_3| \leq 2a_1 |a_2| q(3) \leq a_1^3 K^2, \ldots, |a_n| \leq a_1^n K^{n-1} \). Choose \( x_1 \) such that \( 0 < x_1^r < 1/(2a_1 K) < 1/(a_1(1+K)) \). For \( 0 \leq x \leq x_1 \)

\[
\sum_{n=1}^{\infty} |a_n| x^{rn} \leq \sum_{n=1}^{\infty} a_1^nk^{n-1} x_1^{rn} = \frac{a_1 x_1^r}{1 - a_1 K x_1^r}.
\]

Thus, \( \sum_{n=1}^{\infty} a_n x^{rn} \) converges uniformly for \( 0 \leq x \leq x_1 \). Define \( \varphi(x) = \sum_{n=1}^{\infty} a_n x^{rn} \), \( 0 \leq x \leq x_1 \). The series \( \sum_{n=1}^{\infty} n a_n x^{rn} \) converges uniformly on \([0, x_1]\) since, for \( 0 \leq x \leq x_1 \),

\[
\sum_{n=1}^{\infty} n|a_n| x^{rn} \leq \sum_{n=1}^{\infty} na_1^n K^{n-1} x_1^{rn} = \frac{a_1 x_1^r}{(1 - a_1 K x_1^r)^2}.
\]

Thus, \( (x \varphi(x))' = \sum_{n=1}^{\infty} a_n x^{rn+1} \) \( = \sum_{n=1}^{\infty} (rn + 1) a_n x^{rn} \) and \( \varphi(x) \) satisfies (7.1) on \([0, x_1]\). Integrate (7.1) to obtain

\[
(7.2) \quad \varphi(x) = \frac{\mu}{\alpha x} \int_0^{ax} \varphi(y)(1 - \varphi(y)) \, dy, \quad 0 < x \leq x_1.
\]
Formula (7.2) allows \( \varphi \) to be extended to \([x_1, x_1/\alpha], [x_1/\alpha, x_1/\alpha^2], \ldots \), and thus we obtain a solution of (7.1) on \([0, 1]\). Since \( x_1^* < 1/(2a_1K) < 1/(a_1(1 + K)) \), for \( 0 < x \leq x_1 \)

\[
0 < a_1 x^r \left( 1 - \frac{a_1 K x^r}{1 - a_1 K x^r} \right)
= a_1 x^r \left( 1 - \sum_{n=2}^\infty (a_1 K x^r)^{n-1} \right) \leq \varphi(x)
\leq a_1 x^r \left( 1 + \sum_{n=2}^\infty (a_1 K x^r)^{n-1} \right)
= a_1 x^r \left( 1 + \frac{a_1 K x^r}{1 - a_1 K x^r} \right) \leq 1.
\]

Since \( \mu \leq 4 \), and \( z(1 - z) \leq 1/4 \) for \( 0 \leq z \leq 1 \), we then obtain

\[
0 < (x \varphi(x))' = \mu \varphi(\alpha x)(1 - \varphi(\alpha x)) \leq 1, \quad 0 < x \leq x_1/\alpha,
\]

which implies \( 0 < x \varphi(x) \leq x \), for \( 0 < x \leq x_1/\alpha \) and \( 0 < \varphi(x) \leq 1 \), for \( 0 < x \leq x_1/\alpha \). Repeat this argument to obtain \( 0 < \varphi(x) \leq 1 \), for \( 0 < x \leq x_1/\alpha^2, \ldots \), and thus \( 0 < \varphi(x) \leq 1 \), for \( 0 < x \leq 1 \). \( \Box \)

**Theorem 7.2.** Let \( 1 < \mu \leq 4 \). The solution \( u(x, t) \) of \((E) \) is unstable in \( X_0 \) for every initial data \( \varphi \in X_0 \).

**Proof.** We denote here by \( \| \cdot \| \) the sup norm on \( C([0, 1] \times [-\tau, 0]) \). Let \( \varphi \in X_0 \) and assume that \( u(x, t) \) is stable in \( X_0 \). Let \( \tilde{\varphi} \in X_0 \) such that \( \hat{u}(x, t) = \tilde{\varphi}(x) \) is an equilibrium solution, and let \( \varepsilon = \| \tilde{\varphi} \| \). There exists \( \delta > 0 \) such that if \( \psi \in X_0 \) and \( \| \varphi - \psi \| < \delta \), then \( \| u(x, t) - v(x, t) \| < \varepsilon/2 \) for \( t \geq 0 \), where \( v \) is the solution with initial data \( \psi \). There exists \( b > 0 \) and \( \varphi_1, \varphi_2 \in X_0 \) such that \( \| \varphi - \varphi_1 \| < \delta \), \( \| \varphi - \varphi_2 \| < \delta \), and \( \varphi_1(x) = 0 \) for \( 0 \leq x \leq b \), \( \varphi_2(x) = \tilde{\varphi}(x) \) for \( 0 \leq x \leq b \). By Theorem 4.1, for \( t > - \log b + (\log b/\log \alpha) \tau \)

\[
\varepsilon = \| 0 - \tilde{\varphi} \| = \| u_1(x, t) - u_2(x, t) \|
\leq \| u_1(x, t) - u(x, t) \| + \| u_2(x, t) - u(x, t) \| < \varepsilon,
\]

which yields a contradiction. \( \Box \)
Remark. The equilibrium solutions \( \varphi(x) = \sum_{n=1}^{\infty} a_n x^r^n \) in Theorem 7.1 have coefficients \( a_2 = -a_1^2 \alpha^2 r q(2) \), \( a_3 = -2a_1 a_2 a_3 r q(3) \), \( a_4 = -(2a_2 a_3 + a_4^2) \alpha^4 r q(4) \), ... for \( \alpha < 1 \). If \( \alpha = 1 \), then \( r = \mu - 1 \), and \( a_2 = -\mu a_1^2 / r \), \( a_3 = \mu^2 a_1^3 / r^2 \), \( a_4 = -\mu^3 a_1^4 / r^3 \), ... Thus, for \( 0 \leq x < (\mu - 1)/(a_1 \mu) \),

\[
\sum_{n=1}^{\infty} a_n x^r^n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\mu^{n-1} a_1^n x^r^n}{r^{n-1}} = a_1 x^r \frac{r}{r + \mu a_1 x^r} = \frac{(\mu - 1) a_1 x^{\mu-1}}{(\mu - 1) + \mu a_1 x^{\mu-1}}.
\]

This last expression provides a one-parameter family of equilibrium solutions when \( \mu > 1 \) and \( \alpha = 1 \). These equilibria when \( \alpha = 1 \) are thus compatible with the equilibria when \( \alpha < 1 \) of Theorem 7.1 as \( \alpha \to 1^- \).

8. Existence and uniqueness when \( \tau = 0 \). We now study the case \( \tau = 0 \) and discuss and compare behavior of solutions with the case \( \tau > 0 \).

Consider the equation

\[
(E_0)
\]
\[
\begin{align*}
  u_t(x,t) + (ux(x,t))_x &= \mu u(\alpha x, t)(1 - u(\alpha x, t)), & 0 \leq x \leq 1, & t > 0 \\
  u(x,0) &= \varphi(x), & 0 \leq x \leq 1,
\end{align*}
\]

and the integrated form

\[
\begin{align*}
  u(x,t) &= e^{-t} \varphi(e^{-t} x, 0) \\
  &+ \int_0^t e^{-(t-s)} \mu u(e^{-(t-s)} \alpha x, s)(1 - u(e^{-(t-s)} \alpha x, s)) ds, & 0 \leq x \leq 1, & t \geq 0 \\
  u(x,0) &= \varphi(x), & 0 \leq x \leq 1.
\end{align*}
\]

We shall consider existence and uniqueness of continuous solutions \( u(x,t) \) of \( (E_0)' \) for initial data \( \varphi \in C([0,1]) \). As for the case \( \tau > 0 \) we have uniqueness of solutions.

**Proposition 8.1.** If there exists a continuous solution of \( (E_0)' \) for given \( \varphi \in C([0,1]) \), then it is unique.
Proof. Suppose that $u(x, t)$ and $v(x, t)$ are two solutions of $(E_0)'$ with initial data $\varphi$. Fix $T > 0$. Then there exists $K$ such that $|u(x, t)|, |v(x, t)| \leq K$ for $x \in [0, 1]$ and $t \in [0, T]$. Then from $(E_0)'$,

$$|u(x, t) - v(x, t)| \leq \int_0^t e^{-(t-s)} \mu |u(\alpha x e^{-(t-s)}, s) - v(\alpha x e^{-(t-s)}, s)|(1 + 2K) \, ds,$$

so taking the supremum over $x \in [0, 1]$

$$|u(t) - v(t)|_{\infty} \leq \int_0^t e^{-(t-s)} \mu |u(s) - v(s)|_{\infty}(1 + 2K) \, ds.$$

The result follows from Gronwall’s inequality. \qed

In fact, if the initial data are equal on $[0, b]$ where $0 < b \leq 1$, then the solutions are equal for $t \geq \log(1/b)$.

**Theorem 8.2.** Let $\varphi_1, \varphi_2 \in C([0, 1])$ and suppose that there is a $b$, $0 < b < 1$, such that

$$\varphi_1(x) = \varphi_2(x) \quad \text{for } x \in [0, b].$$

Then

$$u_1(x, t) = u_2(x, t) \quad \text{for } x \in [0, 1] \text{ and } t \geq \log(1/b),$$

where $u_1$ and $u_2$ are the solutions of $(E_0)'$ with initial data $\varphi_1$ and $\varphi_2$, respectively.

The proof is an obvious adaptation of the analogous result for $\tau > 0$.

The following local existence result is proved using standard iteration techniques.

**Theorem 8.3.** Given $\varphi \in C[0, 1]$ there exists a $\bar{t} > 0$ such that $(E_0)'$ has a continuous solution for $0 \leq t \leq \bar{t}$.

Unlike the case $\tau > 0$, in general there are not global (in time) solutions when $\tau = 0$, as we shall see when studying the case of initial
data constant. However, the delay in the maturity variable $x$ gives a global solution for $x \in [0,1]$ if there is a global solution for $x \in [0,b]$, where $0 < b < 1$.

**Proposition 8.4.** Given $\varphi \in C([0,1])$, suppose that there is a $b$ such that $(E_0)'$ has a solution for $x \in [0,b]$ and $t \in [0,T]$. Then $(E_0)'$ has a solution for $x \in [0,1]$ and $t \in [0,T]$.

**Proof.** Take $x \in [0, b \alpha^{-1}]$. Then $\alpha x e^{-(t-s)} \in [0, b]$ so $u(x,t)$ is defined for $x \in [0, b \alpha^{-1}]$. Continue by induction. □

By restricting $\mu$ and $\varphi$ we have global existence.

**Theorem 8.5.** If $0 < \mu \leq 4$ and there is a $b$, $0 < b \leq 1$ such that $0 \leq \varphi(x) \leq 1$ for $x \in [0,b]$, then there exists a solution of $(E_0)'$ for $x \in [0,1]$ and $t \geq 0$.

**Proof.** Let $x \in [0,b]$. Define the sequence $u_n(x,t)$ by

\[
\begin{aligned}
u_0(x,t) &= e^{-t}\varphi(xe^{-t}), \\
u_n(x,t) &= e^{-t}\varphi(xe^{-t}) \\
&\quad + \int_0^t e^{-(t-s)} \mu u_{n-1}(\alpha xe^{-(t-s)}, s)(1-u_{n-1}(\alpha xe^{-(t-s)}, s)) ds.
\end{aligned}
\]

Then as $\xi(1-\xi) \leq 1/4$ for all $\xi \in \mathbb{R}$,

\[u_n(x,t) \leq e^{-t} + \int_0^t e^{-(t-s)} ds = 1.\]

Also, by induction, $u_n(x,t) \geq 0$. But the operator $G$,

\[G : C[0,b] \to C[0,b], \quad (Gu)(x) = u(\alpha x)(1 - u(\alpha x)),\]

is locally Lipschitz continuous:

\[|Gu - Gv|_b \leq |u - v|_b(1 + |u|_b + |v|_b),\]
and so it is Lipschitz continuous on the ball of radius $1$. Hence $u_n(x, t)$ converges to $u(x, t)$ for $(x, t) \in [0, b] \times [0, \infty)$ uniformly on compact subsets.

The result now follows from Proposition 8.4. □

**Corollary 8.6.** If $0 < \mu \leq 4$ and $0 < \varphi(0) < 1$ or $\varphi \in X_0^+$, then there exists a solution of $(E_0)'$ for $x \in [0, 1]$ and $t \geq 0$.

**Theorem 8.7.** If $0 < \mu < 1$ and there is a $b, 0 < b \leq 1$ such that $|\varphi(x)| \leq 1/\mu - 1$ for $x \in [0, b]$, then there exists a solution of $(E_0)'$ for $x \in [0, 1]$ and $t \geq 0$.

**Proof.** We take the iterative scheme (8.1) and prove by induction that $|u_n(t)|_b \leq |\varphi|_b$ for all $t \geq 0$. This is clearly true if $n = 0$. Assume that it is true for $n$, then

$$|u_n(x, t)| \leq e^{-t}|\varphi|_b + \int_0^t e^{-(t-s)}\mu|\varphi|_b(1 + |\varphi|_b)\, ds \leq |\varphi|_b$$

as $\mu(1 + |\varphi|_b) \leq 1$. The proof now follows as in Theorem 8.5, as $G$ is Lipschitz continuous on the ball of radius $|\varphi|_b$. □

**Corollary 8.8.** If $0 < \mu < 1$ and $|\varphi(0)| < 1/\mu - 1$, and in particular if $\varphi \in X_0$, then there exists a solution of $(E_0)'$ for $x \in [0, 1]$ and $t \geq 0$.

We now consider the case of initial data constant, $\varphi = c$. In this case we can write the solution $u(x, t)$ explicitly. If $c = 0$, then $u(x, t) = 0$. If $c \neq 0$, then the solution of $(E_0)$ is

$$u(x, t) = \frac{e^{(\mu-1)t}(1 - 1/\mu)}{(1 - 1/\mu - c)/c + e^{(\mu-1)t}} \quad \text{if } \mu \neq 1$$

and

$$u(x, t) = \frac{c}{t + c} \quad \text{if } \mu = 1.$$  

If $0 < \mu < 1$ and $c < 1 - 1/\mu$ or if $\mu \geq 1$ and $c < 0$, then $t_c$ given by

$$t_c = \frac{1}{\mu - 1} \log \frac{-1 + 1/\mu + c}{c} \quad \text{if } \mu \neq 1$$
and
\[ t_c = -\frac{1}{c} \quad \text{if } \mu = 1, \]
is positive and \( \lim_{t \to t_c} u(x, t) = \infty \), and so the solution is local. It is
global for different values of \( \mu \) and \( c \). We can conclude with an existence
result which follows easily from Theorem 8.2 and Proposition 8.4.

**Proposition 8.9.** Let \( \varphi \in C([0, 1]) \) and suppose that \( \varphi(x) = c \) for
\( x \in [0, b] \), where \( 0 < b \leq 1 \). If
\[ 0 < \mu < 1 \quad \text{and} \quad c < 1 - 1/\mu \quad \text{or} \quad \mu \geq 1 \quad \text{and} \quad c < 0, \]
then the solution with initial data \( \varphi \) exists on \([0, t_c)\). If
\[ 0 < \mu < 1 \quad \text{and} \quad c \geq 1 - 1/\mu \quad \text{or} \quad \mu \geq 1 \quad \text{and} \quad c > 0, \]
then the solution \( u(x, t) \) is defined for \( t \geq 0 \) and
\[ u(x, t) = \frac{e^{(\mu-1)t}(1 - 1/\mu)}{(1 - 1/\mu - c)/c + e^{(\mu-1)t}} \quad \text{for } x \in [0, 1] \]
and
\[ t \geq \log \frac{1}{b} \quad \text{if } \mu \neq 1 \]
and
\[ u(x, t) = \frac{c}{t + c} \quad \text{for } x \in [0, 1] \quad \text{and} \quad t \geq \log \frac{1}{b} \quad \text{if } \mu = 1. \]

And so, if \( 0 < \mu \leq 1 \) and \( c > 1 - 1/\mu \), then \( \lim_{t \to \infty} u(x, t) = 0 \); if \( \mu > 1 \)
and \( c > 0 \), then \( \lim_{t \to \infty} u(x, t) = 1 - 1/\mu \).

9. Invariance and asymptotic behavior when \( \tau = 0 \). We have
analogous invariance results as for the case \( \tau > 0 \). But while results
for \( \tau > 0 \) are proved using step by step methods, those for \( \tau = 0 \) are
proved using the iterative scheme (8.1).

**Theorem 9.1.** Let \( 0 < b \leq 1 \).
i) If $0 < R \leq 1$ and $0 < \mu \leq 4R$ or $0 < \mu \leq 1$, then if the initial function $\varphi$ is an element of the set

$$U_{R,b} = \{\varphi \in C([0,1]), 0 \leq \varphi(x) \leq R \text{ for } x \in [0,b]\},$$

$u(x,t) \in U_{R,b}$ for all $t \geq 0$.

Also, if $\varphi \in U_{R,b}$, then $u(x,t) \in U_{R,1}$ for $t \geq \log(1/b)$.

ii) If $0 < \mu < 1$ and $0 < \rho \leq 1/\mu - 1$, then if the initial function $\varphi$ is an element of the set

$$V_{\rho,b} = \{\varphi \in C([0,1]), |\varphi(x)| \leq \rho \text{ for } x \in [0,b]\},$$

$u(x,t) \in V_{\rho,b}$ for all $t \geq 0$. Also, if $\varphi \in V_{\rho,b}$ then $u(x,t) \in V_{\rho,1}$ for $t \geq \log(1/b)$.

iii) If $1 < \mu \leq 2$ and $\beta$ and $\gamma$ are such that $0 \leq \beta \leq 1 - 1/\mu$ and $1 - 1/\mu \leq \gamma \leq 1/2$, then if the initial function $\varphi$ is an element of the set

$$W_{\beta,\gamma,b} = \{\varphi \in C([0,1]), \beta \leq \varphi(x) \leq \gamma \text{ for } x \in [0,b]\}$$

$u(x,t) \in W_{\beta,\gamma,b}$ for all $t \geq 0$. So, in particular, the sets

$$W_{0,1-1/\mu,b} = \{\varphi \in C([0,1]), 0 \leq \varphi(x) \leq 1 - 1/\mu \text{ for } x \in [0,b]\}$$

and

$$W_{1-1/\mu,1/2,b} = \{\varphi \in C([0,1]), 1 - 1/\mu \leq \varphi(x) \leq 1/2 \text{ for } x \in [0,b]\}$$

are invariant.

Also, if $\varphi \in W_{\beta,\gamma,b}$ then $u(x,t) \in W_{\beta,\gamma,1}$ for $t \geq \log(1/b)$.

Proof. i) The case $R = 1$ was proved in Theorem 8.5 and is similar for $0 < R < 1$. If $0 < \mu \leq 4R$, then if $0 \leq u_{n-1}(x,t) \leq R \leq 1$

$$0 \leq \mu u_{n-1}(axe^{-\mu(t-s)},s)(1 - u_{n-1}(axe^{-\mu(t-s)},s)) \leq \mu(1/4) \leq R$$

and so $0 \leq u_n(x,t) \leq R$ and the result follows.

The proof of ii) is given in the proof of Theorem 8.7.

iii) As in iii) of Theorem 5.1, the proof depends essentially on the fact that for $\xi \in [0,1/2]$ and so for $\xi \in [\beta,\gamma]$, $\xi(1 - \xi)$ is increasing.
We first prove by induction that \( 0 \leq u_n(x, t) \leq \gamma \). It is clearly true for \( n = 0 \). Assume that it is true \( n \). If \( 1 < \mu \leq 2 \) and \( 1 - 1/\mu \leq \gamma \leq 1/2 \), then for \( 0 \leq \xi \leq \gamma \), \( 0 \leq \mu \xi (1 - \xi) \leq \mu \gamma (1 - \gamma) \leq \gamma \). So

\[
\begin{align*}
  u_{n+1}(x, t) & = e^{-t} \varphi(xe^{-t}) \\
  & \quad + \int_0^t e^{-(t-s)} \mu u_n(\alpha xe^{-(t-s)}, s)(1 - u_n(\alpha xe^{-(t-s)}, s)) \, ds \\
  & \leq e^{-t} \gamma + \int_0^t e^{-(t-s)} \gamma \, ds = \gamma,
\end{align*}
\]

and clearly \( u_{n+1} \geq 0 \).

To show that \( u(x, t) \geq \beta \) we prove by induction that \( u_n(x, t) \geq \beta e^{-t} (1 + \cdots + t^n/n!) \), \( n \geq 0 \). It is clearly true for \( n = 0 \) and assume it is true for \( n \). Note that if \( 0 \leq \xi \leq 1 - 1/\mu \leq 1/2 \), then \( \mu \xi (1 - \xi) \geq \xi \). So, for \( s \geq 0 \),

\[
\begin{align*}
  \mu u_n(\alpha xe^{-(t-s)}, s)(1 - u_n(\alpha xe^{-(t-s)}, s)) & \geq \beta e^{-s} (1 + \cdots + s^n/n!),
\end{align*}
\]

and so

\[
\begin{align*}
  u_{n+1}(x, t) & \geq \beta e^{-t} + \int_0^t e^{-(t-s)} \beta e^{-s} (1 + \cdots + s^n/n!) \, ds \\
  & = \beta e^{-t} \left( 1 + \cdots + \frac{t^{n+1}}{(n+1)!} \right)
\end{align*}
\]

as required. Thus, in the limit, \( \beta \leq u(x, t) \leq \gamma \), for \( x \in [0, b] \) and \( t \geq 0 \).

\[ \square \]

The asymptotic behavior is very much the same as in the case \( \tau > 0 \), and for \( 0 < \mu < 1 \) the proofs are similar.

**Theorem 9.2.** Let \( 0 < \mu < 1 \). Suppose that there exists \( 0 < b \leq 1 \) such that either

\[
0 \leq \varphi(x) \leq 1 \quad \text{for } x \in [0, b]
\]

or

\[
|\varphi(x)| < 1/\mu - 1 \quad \text{for } x \in [0, b].
\]
Then $u(x,t)$ tends to zero exponentially as $t$ tends to infinity, uniformly for $x \in [0,1]$.

**Corollary 9.3.** Let $0 < \mu < 1$. If $1-1/\mu < \varphi(0) < \max\{1, 1/\mu - 1\}$, in particular if $\varphi \in X_0$, then $u(x,t)$ tends to zero exponentially as $t$ tends to infinity.

If $\mu = 1$ and $\varphi \in X_0^+$ then, as in the case $\tau > 0$, $u(x,t)$ tends to zero as $t$ tends to infinity.

We now consider the case $\mu > 1$ and prove the analogue of Theorem 6.4 for the case $\tau = 0$.

**Theorem 9.4.** Let $1 < \mu \leq 2$. If $\varphi(0) = c$ where $0 < c < 1/2$, then $u(x,t)$ tends to $1 - 1/\mu$ as $t$ tends to infinity.

**Proof.** Given $\varepsilon > 0$ such that $\varepsilon < \min\{c, 1/2 - c\}$, there exists $b$, $0 < b \leq 1$, such that

$$0 < c - \varepsilon < \varphi(x) < c + \varepsilon < 1/2 \quad \text{for } 0 \leq x \leq b.$$ 

Let $u,v,w$ be the solutions of $(E_0)'$ for initial data $\varphi, c - \varepsilon, c + \varepsilon$, respectively, and let $u_n,v_n,w_n$ be the respective iterates defined as in (8.1). From iii) of Theorem 9.1,

$$0 \leq v_n(x,t), u_n(x,t), w_n(x,t) \leq 1/2 \quad \text{for } x \in [0,b], \quad t \geq 0$$

and as $\xi(1 - \xi)$ is increasing on $[0,1/2]$, it follows, by induction, that

$$v_n(x,t) \leq u_n(x,t) \leq w_n(x,t) \quad \text{for } x \in [0,b], \quad t \geq 0$$

and hence

$$v(x,t) \leq u(x,t) \leq w(x,t) \quad \text{for } x \in [0,b], \quad t \geq 0.$$ 

But from Proposition 8.9 $v(x,t)$ and $w(x,t)$ tend to $1 - 1/\mu$ as $t \to \infty$. The result now follows from iii) of Theorem 9.1. □
REFERENCES


11. ———, Periodic and chaotic behaviour in structured models of cell population dynamics, to appear.

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