GLOBAL STABILITY OF A COMPETITION MODEL WITH PERIODIC COEFFICIENTS AND TIME DELAYS

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ABSTRACT. A competition model with periodic coefficients and time delays is studied. Sufficient conditions are determined that guarantee the existence of a positive periodic solution which is globally asymptotically stable. Some earlier results are extended and improved.

1. Introduction. Let us consider the Lotka-Volterra population model

\[ \begin{align*}
x'(t) &= x(t)[a_1(t) - b_1(t)x(t) + c_1(t)x(t - \tau_{11}(t)) - e_1(t)y(t - \tau_{12}(t))] \\
y'(t) &= y(t)[a_2(t) - b_2(t)x(t - \tau_{21}(t)) + c_2(t)y(t - \tau_{22}(t)) - e_2(t)y(t)]
\end{align*} \tag{1.1} \]

where \( a_i(t), b_i(t), c_i(t), e_i(t) \) and \( \tau_{ij}(t), i, j = 1, 2 \), are continuous, \( \omega \)-periodic functions with \( b_i(t) > 0, e_i(t) > 0 \) and \( \tau_{ij}(t) \geq 0 \) for all \( t \geq 0 \); \( \tau_{ij}(t) \) are continuously differentiable with \( \tau_{ij}'(t) < 1 \) for all \( t \geq 0 \). System (1.1) models the competition between two species in an \( \omega \)-periodic environment. We are interested in the existence and global stability of a positive periodic solution (its precise definition will be given later) of (1.1), which is directly motivated by the work of Freedman and Wu [5]. In [5], they proposed a single-species model with periodic time delay and established sufficient conditions that ensure that there exists a positive periodic solution which is globally asymptotically stable. The results were extended to many-species case by Tang and Kuang [8]. However, the criteria of Freedman and Wu [5] involve the location of positive periodic solutions. This makes the criteria difficult to use since the periodic solution cannot be located in general. The main objective of this paper is to improve their results in this respect. We determine sufficient conditions on the parameters...
of the model that ensure the existence and global stability of positive periodic solution in (1.1). Our criteria are in explicit forms of the parameters and thus are verifiable.

2. Main results. We first set forth some notions and notations that will be used throughout this paper. Let \( R^2_+ = \{(x, y) : x \geq 0, y \geq 0\} \) and \( ||(x, y)|| = \max\{x, y\} \). We write \((x, y) > 0\) if \(x > 0\) and \(y > 0\). Define

\[
\tau = \max\{\tau_{ij}(t) : t \in R; i, j = 1, 2\}.
\]

Then let \( C^+ \) denote the space of initial functions

\[
C^+ = \{(\varphi_1, \varphi_2) : \varphi_i(\theta) \geq 0, \theta \in [-\tau, 0], i = 1, 2; (\varphi_1(0), \varphi_2(0)) > 0; \\
\varphi_i(\theta) \text{ is continuous, } i = 1, 2\}.
\]

Given \( \sigma \in R \) and \( \varphi = (\varphi_1, \varphi_2) \in C^+ \), it is easy to see that (1.1) has a unique solution \( (x(\sigma, \varphi)(t), y(\sigma, \varphi)(t)) \) through \((\sigma, \varphi)\). Moreover, \( x(\sigma, \varphi)(t) > 0 \) and \( y(\sigma, \varphi)(t) > 0 \) for all \( t \in [\sigma, \alpha) \), where \([\sigma, \alpha)\) is the maximal existence interval of the solution. Such solutions of (1.1) are called positive solutions.

Set

\[
p_{ij} = \min \left\{ \frac{1}{1 - \tau_{ij}'(t)} : t \in R \right\}
\]

\[
p_{ij}^* = \max \left\{ \frac{1}{1 - \tau_{ij}'(t)} : t \in R \right\}
\]

for \( i, j = 1, 2 \). Due to \( \tau_{ij}'(t) < 1, t \in R \) and periodicity of these functions, it is obvious that \( p_{ij} \) and \( p_{ij}^* \) are positive constants. Now we define

\[
\sigma_{ij}(t) = t - \tau_{ij}(t), \quad t \in R
\]

for \( i, j = 1, 2 \). It follows from \( \tau_{ij}'(t) < 1 \) for all \( t \in R \) that \( \sigma_{ij}(t) \) has its inverse function \( \mu_{ij} \).

Set

\[
a_* = \min\{b_1(t) - p_{11}^* | c_1(\mu_{11}(t))| : t \in R\}
\]

\[
e^* = \max\{p_{21}^* b_2(t) : t \in R\}
\]

\[
b^* = \max\{p_{12}^* e_1(t) : t \in R\}
\]

\[
f_* = \min\{e_2(t) - p_{22}^* | c_2(\mu_{22}(t))| : t \in R\}.
\]
Given a continuous $\omega$-periodic function $g(t)$, we define

$$\langle g \rangle = \int_0^\omega g(t) \, dt / \omega.$$ 

The following theorem sets forth the principal result of this paper.

**Theorem 2.1.** Suppose that system (1.1) satisfies the following assumptions:

(A1) $b_1(t) > \lvert c_1(t) \rvert$ and $e_2(t) > \lvert c_2(t) \rvert$ for all $t \in R$

(A2) $\langle a_i \rangle > 0$ for $i = 1, 2$

(A3) $a_* > 0$ and $f_* > 0$

(A4) $a_* \langle a_2 \rangle - e^* \langle a_1 \rangle > 0$

(A5) $f_* \langle a_1 \rangle - b^* \langle a_2 \rangle > 0$

Then system (1.1) has a positive periodic solution which is globally asymptotically stable.

Before the proof, we give interpretation on the assumptions of Theorem 2.1. (A1) and (A3) require that the undelayed intraspecific competitions dominate the delayed intraspecific competitions. (A2) represents positive intrinsic growth rates in the mean of the populations. By (A2), (A4) and (A5), we have $a_* f_* > e^* b^*$. Thus (A4) and (A5) may be interpreted by saying the intraspecific competition dominates the interspecific competition.

To prove our main result, we need the following lemmas.

**Lemma 2.2.** Positive solutions of system (1.1) are uniformly bounded and uniformly ultimately bounded if (A1) holds.

**Proof.** By (A1) there exist $\gamma > 1$ and $H > 1$ such that

\begin{equation}
\begin{aligned}
a_1(t) - (b_1(t) - \gamma \lvert c_1(t) \rvert) H &< -1 \\
 a_2(t) - (e_2(t) - \gamma \lvert c_2(t) \rvert) H &< -1
\end{aligned}
\end{equation}

for all $t \in R$. Define $V(x, y) = \max\{x, y\}$. Calculating the upper right derivative of $V$ along the positive solutions of (1.1), we have

\begin{equation}
V'(x(t), y(t)) \leq -1
\end{equation}
if $||x(t), y(t)|| \geq H, V(x(t+\theta), y(t+\theta)) < \gamma V(x(t), y(t))$ for $\theta \in [-\tau, 0]$. It follows from the theorem of Lyapunov-Razumikhin type [6] that the positive solutions of (1.1) are uniformly ultimately bounded.

Fix $M > H$. Let $(x(t), y(t))$ denote the solution of system (1.1) through $(\sigma, \varphi)$, where $\varphi = (\varphi_1, \varphi_2) \in C^+$ and $0 \leq \varphi_i(\theta) \leq M$ on $[-\tau, 0]$ for $i = 1, 2$. We claim that $||x(t), y(t)|| \leq M$ for all $t \geq \sigma$. Otherwise, there exists a $\bar{t} \geq \sigma$ such that

\begin{align}
(2.3) & \quad ||x(t), y(t)|| \leq M, \quad \sigma - \tau \leq t < \bar{t}, \\
(2.4) & \quad ||(x(\bar{t}), y(\bar{t}))|| = M, \\
(2.5) & \quad V'(x(\bar{t}), y(\bar{t})) \geq 0.
\end{align}

Using (2.3) and (2.4), we have from (1.1),

$$V'(x(\bar{t}), y(\bar{t})) \leq M \max\{a_1(\bar{t}) - (b_1(\bar{t}) - |c_1(\bar{t})|)M, a_2(\bar{t}) - (e_2(\bar{t}) - |c_2(t)|)M\}.$$ 

It follows from (2.1) that $V'(x(\bar{t}), y(\bar{t})) < 0$, which contradicts (2.5) and, therefore, the uniform boundedness of the positive solutions of (1.1) follows. This proves the lemma. \qed

Given $d > \delta > 0$, we define

$$C^+[\delta, d] = \{\varphi = (\varphi_1, \varphi_2) \in C^+ : \delta \leq \varphi_i(\theta) \leq d, \theta \in [-\tau, 0], i = 1, 2\}.$$

**Lemma 2.3.** Let the assumptions (A1)-(A5) hold for (1.1). Then, for given $0 < \delta < d$, there exist constants $0 < \eta < D$ such that for all $\sigma \in \mathbb{R}$, $\varphi \in C^+[\delta, d]$, we have

$$\eta \leq x(\sigma, \varphi)(t) \leq D; \quad \eta \leq y(\sigma, \varphi)(t) \leq D$$

for all $t \geq \sigma$.

**Proof.** For given $0 < \delta < d$, Lemma 2.2 implies that there exists a constant $D > 0$ such that for all $\sigma \in \mathbb{R}$ and $\varphi \in C^+[\delta, d]$, we have

\begin{align}
(2.6) & \quad x(\sigma, \varphi)(t) \leq D, \quad y(\sigma, \varphi)(t) \leq D
\end{align}
for all \( t \geq \sigma \). We show below that there exists a constant \( \eta > 0 \), such that for all \( \sigma \in R \) and \( \varphi \in C^+[\delta, d] \), we have

\[
(2.7) \quad \eta \leq x(\sigma, \varphi)(t), \quad \eta \leq y(\sigma, \varphi)(t)
\]

for all \( t \geq \sigma \). To this end, we first construct a Lyapunov functional as follows:

\[
(2.8) \quad V_1(t) = [x(t)]^{-\alpha_1} [y(t)]^{-\alpha_2} \exp \left\{ -\alpha_1 p^{*}_{11} \int_{t-\tau_{11}(t)}^{t} |c_1(\mu_{11}(s))| x(s) \, ds 
+ \alpha_1 p_{12} \int_{t-\tau_{12}(t)}^{t} e_1(\mu_{12}(s)) y(s) \, ds 
- \alpha_2 p_{21}^{*} \int_{t-\tau_{21}(t)}^{t} b_2(\mu_{21}(s)) x(s) \, ds 
- \alpha_2 p_{22}^{*} \int_{t-\tau_{22}(t)}^{t} |c_2(\mu_{22}(s))| y(s) \, ds \right\}
\]

where \( \alpha_1 \) and \( \alpha_2 \) are positive constants to be chosen.

Calculating the derivative of \( V_1 \) along the positive solution of (1.1) and simplifying, we have

\[
(2.9) \quad V'_1 \geq V_1 \{ \alpha_2 a_2(t) - \alpha_1 a_1(t) - A(t)y(t)
+ [\alpha_1 b_1(t) - \alpha_1 p^{*}_{11}|c_1(\mu_{11}(t))|] 
- \alpha_2 p_{21}^{*} b_2(\mu_{21}(t)) x(t) \}
\]

where

\[
A(t) = \alpha_2 e_2(t) - \alpha_1 p_{12} e_1(\mu_{12}(t)) + \alpha_2 p_{22}^{*} |c_2(\mu_{22}(t))|.
\]

Setting \( \alpha_1 = e^* \) and \( \alpha_2 = a_* \), we obtain from (2.9),

\[
(2.10) \quad V'_1 \geq V_1 (a_* a_2(t) - e^* a_1(t) - A(t)y(t)).
\]

In the following, we always assume that \( \alpha_1 = e^* \) and \( \alpha_2 = a_* \).
Now, we define another Lyapunov functional $V_2$ by

$$V_2(t) = (x(t))^{\beta_1}(y(t))^{-\beta_2} \exp \left\{ -\beta_1 p_1^* \int_{t-\tau_{11}(t)}^t c_1(\mu_{11}(s))x(s) \, ds - \beta_1 p_{12}^* \int_{t-\tau_{12}(t)}^t e_1(\mu_{12}(s))y(s) \, ds + \beta_2 p_{21} \int_{t-\tau_{21}(t)}^t b_2(\mu_{21}(s))x(s) \, ds - \beta_2 p_{22}^* \int_{t-\tau_{22}(t)}^t c_2(\mu_{22}(s))y(s) \, ds \right\}$$

where $\beta_1$ and $\beta_2$ are positive constants to be chosen.

Calculating the derivative of $V_2$ along the positive solutions of (1.1), we have

$$V'_2 = V_2 \{ \beta_1 a_1(t) - \beta_2 a_2(t) - B(t)x(t) + [\beta_2 e_2(t) - \beta_2 p_{22}^* c_2(\mu_{22}(t))] - \beta_1 p_{12}^* e_1(\mu_{12}(t)) y(t) \}$$

(2.12)

where

$$B(t) = \beta_1 b_1(t) + \beta_1 p_{11}^* c_1(\mu_{11}(t)) - \beta_2 p_{21} b_2(\mu_{21}(t)).$$

Setting $\beta_1 = f_*$ and $\beta_2 = b^*$, we have from (2.12),

$$V'_2 \geq V_2 \{ f_* a_1(t) - b^* e_2(t) - B(t)x(t) \}.$$  

(2.13)

We always assume that $\beta_1 = f_*$ and $\beta_2 = b^*$ in the following.

Since $A(t)$ and $B(t)$ are continuous and bounded, there exist constants $A^* > 0$ and $B^* > 0$ such that $|A(t)| \leq A^*$ and $|B(t)| \leq B^*$ for all $t \in R$.

For convenience in the proof, we set

$$R_1 = a_*(a_2) - e^*(a_1), \quad R_2 = f_*(a_1) - b^*(a_2).$$

Then since $R_1 > 0$ and $R_2 > 0$ there exist constants $h_1 > 0$ and $h_2 > 0$ such that

$$\frac{R_1}{2} - A^* h_1 > \frac{R_1}{4} \quad \text{and} \quad \frac{R_2}{2} - B^* h_2 > \frac{R_2}{4}.$$  

(2.14)
Furthermore, since \(a_1(t)\) and \(a_2(t)\) are periodic, it follows from \(R_1 > 0\) and \(R_2 > 0\) that there exists a \(T^* > 0\) such that, for all \(t_0 \in R\), \(t \geq t_0 + T\), we have

\[
\int_{t_0}^{t} \frac{a_2(s) - e^*a_1(s)}{t - t_0} \, ds > \frac{R_1}{2}
\]

\[
\int_{0}^{t} \frac{f_2 a_1(s) - b^*a_2(s)}{t - t_0} \, ds > \frac{R_2}{2}
\]

It is easy to see from (2.8) and (2.11) that there exist positive constants \(0 < s_1 < 1\), \(0 < s_2 < 1\), \(1 < S_1\) and \(1 < S_2\) such that

\[
s_1 x(t)^{-\alpha_1} y(t)^{\alpha_2} \leq V_1(t) \leq S_1 x(t)^{-\alpha_1} y(t)^{\alpha_2}
\]

\[
s_2 x(t)^{\beta_1} y(t)^{-\beta_2} \leq V_2(t) \leq S_2 x(t)^{\beta_1} y(t)^{-\beta_2}
\]

if \(0 < x(t + \theta) \leq D\) and \(0 < y(t + \theta) \leq D\) for \(\theta \in [-\tau, 0]\).

Select \(\eta_1 > 0\) and \(\eta_2 > 0\) such that

\[
\eta_1 < \min\{h_2, \delta\}, \quad \eta_2 < \min\{h_1, \delta\}.
\]

We define

\[
Q_1 = \sup_{t \in R} \{a_2|a_2(t)| + e^*|a_1(t)| + |A(t)|M_2\}
\]

\[
Q_2 = \sup_{t \in R} \{f_2|a_1(t)| + b^*|a_2(t)| + |B(t)|M_1\}.
\]

Then we choose \(H_1 > 0\) and \(H_2 > 0\) such that

\[
H_1 S_1 < s_1 D^{-\alpha_1} \eta_2^{\alpha_2} \exp(-Q_1 T^*)
\]

\[
H_2 S_2 < s_2 D^{-\beta_2} \eta_1^{\beta_1} \exp(-Q_2 T^*).
\]

We claim that if \((x(t), y(t))\) is a solution of (1.1) through \((\sigma, \varphi)\), where \(\sigma \in R\), \(\varphi = (\varphi_1, \varphi_2) \in C^+[\delta, d]\), then we have

\[
(x(t))^{-\alpha_1} (y(t))^{\alpha_2} > H_1
\]

\[
(x(t))^{\beta_1} (y(t))^{-\beta_2} > H_2
\]
for all $t \geq \sigma$. Assume, for the sake of contradiction, that (2.22) is not valid. Then, since

$$S_1(x(\sigma))^{-\alpha_1}(y(\sigma))^{\alpha_1} \geq V_1(\sigma) \geq s_1(x(\sigma))^{-\alpha_1}(y(\sigma))^{\alpha_2} \geq s_1D^{-\alpha_1}\eta_2^{\alpha_2}$$

from (2.20) we have

$$(x(\sigma))^{-\alpha_1}(y(\sigma))^{\alpha_2} > H_1.$$ 

It follows that there exists a $t_1$, $t_1 > \sigma$, such that

$$x(t_1)^{-\alpha_1}y(t_1)^{\alpha_2} = H_1,$$

$$(x(t))^{-\alpha_1}(y(t))^{\alpha_2} > H_1$$

for all $\sigma \leq t < t_1$.

If $y(t_1) \geq \eta_2$, then from (2.20) we have

$$(x(t_1))^{-\alpha_1}(y(t_1))^{\alpha_2} \geq D^{-\alpha_1}\eta_2^{\alpha_2} > H_1.$$ 

We are led to a contradiction and hence $y(t_1) < \eta_2$.

Define

$$t^* = \sup\{t : y(t) = \eta_2; \sigma < t < t_1\}.$$ 

Notice that the existence of $t^*$ is assured by $y(\sigma) \geq \delta > \eta_2$ and $y(t_1) < \eta_2$. Furthermore, it is clear that $y(t^*) = \eta_2$ and $y(t) < \eta_2$ for $t^* < t < t_1$.

By (2.10) we have (2.24)

$$V_1(t) \geq V_1(t^*)\exp\left\{(t - t^*)\left(\int_{t^*}^{t}\frac{(a_2(a_2(s) - e^*a_1(s)))ds}{t - t^*} - A^*\eta_2\right)\right\}.$$ 

If $t_1 \geq t^* + T^*$, it follows from (2.24) that

$$V_1(t_1) \geq s_1D^{-\alpha_1}\eta_2^{\alpha_2}\exp\left\{T^*\left(\frac{R_1}{2} - A^*\eta_2\right)\right\}$$

$$> s_1D^{-\alpha_1}\eta_2^{\alpha_2} > H_1S_1.$$ 

But from (2.16) we see that

$$V_1(t_1) \leq S_1(x(t_1))^{-\alpha_1}(y(t_1))^{\alpha_2} = S_1H_1.$$
Thus we are led to a contradiction and hence $t_1 \leq t^* + T^*$.

By (2.10) and (2.18) we see that

$$V_1(t) \geq V_1(t^*) \exp(-Q_1(t - t^*))$$

for $t^* \leq t \leq t_1$, which yields

$$V_1(t_1) > V_1(t^*) \exp(-Q_1 T^*)$$

$$\geq s_1 D^{-\alpha_1} \eta_2^{\alpha_2} \exp(-Q_1 T^*)$$

$$> H_1 S_1 \geq V_1(t_1).$$

Thus we are led to a contradiction. As a consequence, we can conclude that (2.22) is valid. By a similar method, we can also show that (2.23) is valid.

It is easy to see from (2.22) and (2.23) that

$$\begin{align*}
-\alpha_1 \ln x(t) + \alpha_2 \ln y(t) &> \ln H_1 \\
\beta_1 \ln x(t) - \beta_2 \ln y(t) &> \ln H_2.
\end{align*}$$

(2.25)

Recall that $\alpha_1 = e^*$, $\alpha_2 = a^*$, $\beta_1 = f^*$ and $\beta_2 = b^*$. From (A2)–(A5) we see that $\beta_1 \alpha_2 > \alpha_1 \beta_2$. As a consequence, from (2.25) we have

$$x(t) > (H_1^{\beta_2} H_2^{\alpha_2})^{1/(\alpha_2 \beta_1 - \alpha_1 \beta_2)} = \varepsilon_1$$

$$y(t) > (H_1^{\beta_1} H_2^{\alpha_1})^{1/(\alpha_2 \beta_1 - \alpha_1 \beta_2)} = \varepsilon_2$$

for all $t \geq \sigma$. Set $\eta = \min(\varepsilon_1, \varepsilon_2)$. Since $\eta > 0$ is independent of the choice of the solution $(x(t), y(t))$, we actually verify (2.7). This completes the proof of Lemma 2.3. \qed

**Lemma 2.4.** Let the assumptions (A1)–(A5) hold for system (1.1). Then there exist constants $0 < \varepsilon < M$ such that, for given $0 < \delta < d$, there exists a constant $T = T(\delta, d) > 0$ such that

$$\varepsilon \leq x(\sigma, \varphi)(t) \leq M; \quad \varepsilon \leq y(\sigma, \varphi)(t) \leq M$$

for $t \geq \sigma + T$, $\sigma \in R$, and $\sigma \in C^+[\delta, d]$.

**Proof.** By Lemma 2.2, there exists an $M > 0$ such that for given $0 < \delta < d$, there exists a $T_1 = T_1(d) > 0$ such that

$$x(\sigma, \varphi)(t) \leq M, \quad y(\sigma, \varphi)(t) \leq M$$

(2.26)
for all \( t \geq \sigma + T_1 \), \( \sigma \in R \) and \( \varphi \in C^+[\delta, d] \). Furthermore, by Lemma 2.3, there exists an \( \eta > 0 \) such that

\[
(2.27) \quad x(\sigma, \varphi)(t) \geq \eta, \quad y(\sigma, \varphi)(t) \geq \eta
\]

for all \( t \geq \sigma \), \( \sigma \in R \) and \( \varphi \in C^+[\delta, d] \).

Fix \( 0 < \delta < d \) and \( \eta < \min\{h_1, h_2\} \) in the following. Then select a \( T_2 \geq T^* \) such that

\[
(2.28) \quad s_1 M^{-\alpha_1} \eta^{\alpha_2} \exp \frac{T_2 R_1}{4} > S_1 \eta^{-\alpha_1} h_1^{\alpha_2}
\]

\[
(2.29) \quad s_2 \eta^\beta_1 M^{-\beta_2} \exp \frac{T_2 R_1}{4} > S_2 h_2^{\beta_1} \eta^{-\beta_2}
\]

Let \((x(t), y(t))\) denote the solution of system (1.1) through \((\sigma, \varphi)\), where \( \sigma \in R \) and \( \varphi \in C^+[\delta, d] \); we claim that there is a \( t_2, \sigma + T_1 \leq t_2 \leq \sigma + T_1 + T_2 \) such that \( y(t_2) > h_1 \). Assume, for the sake of contradiction, that \( y(t) \leq h_1 \) for all \( \sigma + T_1 \leq t \leq \sigma + T_1 + T_2 \). It follows from (2.10) that

\[
(2.30) \quad V_1(t) \geq V_1(\sigma + T_1)
\]

\[
\cdot \exp \left\{ (t - \sigma - T_1) \left[ \int_{\sigma + T_1}^{t} \frac{(a_2(s) - e_s a_1(s))}{t - \sigma - T_1} ds - A^* h_1 \right] \right\}
\]

for all \( \sigma + T_1 \leq t \leq \sigma + T_1 + T_2 \). Since \( T_2 \geq T^* \), from (2.30), (2.13), (2.14), (2.26) and (2.28), we have

\[
(2.31) \quad V_1(\sigma + T_1 + T_2) \geq s_1 M^{-\alpha_1} \eta^{\alpha_2} \exp \frac{T_2 R_1}{4} > S_1 \eta^{-\alpha_1} h_1^{\alpha_2}
\]

But, from (2.27), we have

\[
V_1(t) \leq S_1 (x(t))^{-\alpha_1} (y(t))^{\alpha_2} \leq S_1 \eta^{-\alpha_1} h_1^{\alpha_2}
\]

for all \( \sigma + T_1 \leq t \leq \sigma + T_1 + T_2 \), which yields

\[
V_1(\sigma + T_1 + T_2) < S_1 \eta^{-\alpha_1} h_1^{\alpha_2}
\]

which contradicts (2.31) and, therefore, there exists a \( t_2, \sigma + T_1 \leq t_2 \leq \sigma_1 + T_1 + T_2 \), such that \( y(t_2) > h_1 \). Choose \( G_1 > 0 \) such that

\[
G_1 S_1 < s_1 M^{-\alpha_1} h_1^{\alpha_2} e^{-Q_1 T^*}
\]
As shown in the proof of Lemma 2.3, the solution \((x(t), y(t))\) satisfies

\[(2.32) \quad (x(t))^{-\alpha_1}(y(t))^{\alpha_2} > G_1\]

for all \(t \geq t_2\). In a similar way one can show that there exists a \(t_3\), \(\sigma + T_1 \leq t_3 \leq \sigma + T_1 + T_2\), such that \(x(t_3) > h_2\). Then choose \(G_2 > 0\) such that

\[G_2 s_2 < s_2 h_2^{\beta_1} M^{-\beta_2} e^{-Q_2 T^*}.\]

As in the proof of Lemma 2.3, we can show that

\[(2.33) \quad (x(t))^{\beta_1}(y(t))^{-\beta_2} > G_2\]

for all \(t \geq t_3\). It follows from (2.32) and (2.33) that

\[(2.34) \quad x(t) > (G_1^{\beta_2} G_2^{\alpha_2})^{1/(\alpha_2 \beta_1 - \alpha_1 \beta_2)} = \epsilon_1^*\]

\[y(t) > (G_1^{\beta_1} G_2^{\alpha_1})^{1/(\alpha_2 \beta_1 - \alpha_1 \beta_2)} = \epsilon_2^*\]

for all \(t \geq \sigma + T_1 + T_2\).

Set \(T = T_1 + T_2\) and \(\epsilon = \min\{\epsilon_1^*, \epsilon_2^*\}\). It follows from (2.34) that \(x(t) \geq \epsilon\) and \(y(t) \geq \epsilon\) for all \(t \geq \sigma + T\). From the proof we see that \(\epsilon\) and \(T\) are independent of the choice of \(\delta, d\) and the solution. Thus, we actually verify Lemma 2.4, and the proof is completed. 

At this time we able to give the proof of Theorem 2.1.

Proof of Theorem 2.1. By the change of variables \(X = \ln(x), Y = \ln(y)\), we transform (1.1) into a system with respect to \(X\) and \(Y\). Lemma 2.3 and Lemma 2.4 imply that this system is uniformly bounded and uniformly ultimately bounded. It follows from Theorem 4.2 of [6, page 91] or Theorem 2.3 in paper [7] or papers [2] and [4] that the system has an \(\omega\)-periodic solution, and therefore, there exists a positive periodic solution \((x_0(t), y_0(t))\) in (1.1). We show below that it is globally asymptotically stable. To this end, let

\[u(t) = \ln \frac{x(t)}{x_0(t)}, \quad v(t) = \ln \frac{y(t)}{y_0(t)}.\]
Then we have

\[
\begin{align*}
    u'(t) &= -b_1(t)x_0(t)(e^{u(t)} - 1) \\
    &\quad + c_1(t)x_0(t - \tau_{11}(t))(e^{u(t - \tau_{11}(t))} - 1) \\
    &\quad - e_1(t)y_0(t - \tau_{12}(t))(e^{v(t - \tau_{12})} - 1) \\
    v'(t) &= -e_2(t)y_0(t)(e^{v(t)} - 1) \\
    &\quad - b_2(t)x_0(t - \tau_{21}(t))(e^{v(t - \tau_{21}(t))} - 1) \\
    &\quad + c_2(t)y_0(t - \tau_{22}(t))(e^{v(t - \tau_{22}(t))} - 1).
\end{align*}
\]

(2.35)

Since \(x_0(t)\) and \(y_0(t)\) are bounded above and below by positive constants, it suffices to show that the trivial solution of (2.35) is globally asymptotically stable.

From

\[ a_* f_* - b^* e^* > 0, \]

we see that there is \( \alpha > 0 \) and \( \beta > 0 \) such that

\[
(2.36) \quad e^* \beta - a_* \alpha = -\varepsilon, \quad b^* \alpha - f_* \beta = -\varepsilon
\]

where \( \varepsilon > 0 \) is fixed. In the following, we always assume that \( \alpha \) and \( \beta \) satisfy (2.36).

Define a Lyapunov functional \( V \) by

\[
V = \alpha |u(t)| + \beta |v(t)| \\
+ \alpha \rho_{11}^* \int_{t - \tau_{11}(t)}^{t} x_0(s)|c_1(\mu_{11}(s))(e^{u(t)} - 1)| \, ds \\
+ \alpha \rho_{12}^* \int_{t - \tau_{12}(t)}^{t} y_0(s)e_1(\mu_{12}(s))(e^{v(s)} - 1) \, ds \\
+ \beta \rho_{21}^* \int_{t - \tau_{21}(t)}^{t} x_0(s)b_2(\mu_{21}(s))(e^{u(s)} - 1) \, ds \\
+ \beta \rho_{22}^* \int_{t - \tau_{22}(t)}^{t} y_0(s)c_2(\mu_{22}(s))(e^{v(s)} - 1) \, ds.
\]
Calculating the upper right derivative of $V$ along the solutions of (2.35) we have

\[
D^+V|_{(2.35)} \leq -[\alpha b_1(t) - \alpha p_{11}^* \mu_2(\mu_1(t))] \\
\quad - \beta p_{21}^* b_2(\mu_21(t)) x_0(t) e^{u(t)} - 1 \\
\quad - [\beta e_2(t) - \beta p_{22}^* c_2(\mu_2(t))] \\
\quad - \alpha p_{12}^* e_1(\mu_1(t)) y_0(t) e^{v(t)} - 1 \\
\leq -[\alpha a_\star - \beta e_\star x_0(t)] e^{u(t)} - 1 \\
\quad - [\beta f_\star - \alpha b_\star y_0(t)] e^{v(t)} - 1 \\
\leq -\varepsilon \gamma (|e^{u(t)} - 1| + |e^{v(t)} - 1|)
\]

where $\gamma$ is a positive constant such that $\gamma < \min_{t \in R} \{x_0(t), y_0(t)\}$. It follows from Theorem 2.1 of [6, page 105] and paper [3] that the trivial solution of (2.34) is globally asymptotically stable. This proves the theorem. □

To illustrate our results, we set $\tau_{ij}(t) = 0$ and $c_i(t) = 0$ for $t \in R$, $i, j = 1, 2$. Then (1.1) reduces to

\begin{align*}
(2.37) \quad x'(t) &= x(t) [a_1(t) - b_1(t) x(t) - e_1(t) y(t)] \\
y'(t) &= y(t) [a_2(t) - b_2(t) x(t) - e_2(t) y(t)].
\end{align*}

Set

\[
\begin{align*}
a_i^m &= \min_t a_i(t); & a_i^M &= \max_t a_i(t) \\
b_i^m &= \min_t b_i(t); & b_i^M &= \max_t b_i(t) \\
e_i^m &= \min_t e_i(t); & e_i^M &= \max_t e_i(t).
\end{align*}
\]

By applying Theorem 2.1 to (2.37), we obtain the following corollary.

**Corollary 2.5.** In (2.36), if

\[
\begin{align*}
\langle a_1 \rangle &> 0; & \langle a_2 \rangle &> 0 \\
\langle b_1 \rangle \langle a_1 \rangle - \langle b_2 \rangle \langle a_2 \rangle &> 0; & \langle b_1 \rangle \langle a_2 \rangle - \langle b_2 \rangle \langle a_1 \rangle &> 0
\end{align*}
\]

then (2.36) has a unique positive periodic solution which is globally asymptotically stable.
Remark. Paper [1] has shown that if
\[
\begin{align*}
    a_1^n &> 0; & a_2^n &> 0 \\
    e_2^m a_1^m - e_1^m a_2^m &> 0; & a_2^m b_1^m - b_2^m a_1^m &> 0
\end{align*}
\]
then (2.37) has a unique positive periodic solution which is globally asymptotically stable. Clearly the conclusion of our corollary is more general.

REFERENCES


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