THREE-DIMENSIONAL INVERSION SCHEME FOR MISE-À-LA-MASSE PROSPECTING DATA USING SIMULATED ANNEALING

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1. Introduction. Electromagnetic methods are used extensively in the field of mineral exploration. These techniques involve generating a constant, sinusoidal or transient electromagnetic field and measuring the response caused by a buried conductive anomaly. The induced fields are then used to infer details of the conductivity structure of the subsurface. Electromagnetic techniques which utilize a constant current source (DC) are referred to as resistivity methods.

A common resistivity technique for surveying vein-type ore bodies is the mise-à-la-masse prospecting method which was pioneered by Schlumberger in the 1920’s [19]. In this method, a constant current source is applied directly to the ore body, either through a drill hole which intersects the body or through an exposed outcropping. The current source is grounded at a large distance from the body to effectively isolate the source. Measurements of potential or potential gradient (electric field) are then made at numerous points on the surface to generate a 2-D response profile. Interpretation of the resultant surface fields represents a difficult inverse problem. The difficulty arises from the fact that the system is highly nonlinear, nonunique, and possesses a high degree of dimensionality. Discussion of the inverse problems associated with electromagnetic prospecting in geophysics can be found in [17].

Solution of the forward problem (i.e., knowing the conductive properties of the medium and solving for the surface fields) is achievable numerically. Suitable numerical solution techniques include finite differences [7], finite elements [16] or integral equation methods [10, 24]. While these researchers consider full three dimensional models of the conductivity substructure, none attempt to solve the inverse problem; that is, recovering the conductivity structure through boundary measurements. Not only does the work presented here utilize a full 3D
model but successful inversion using the surface fields is achieved.

A class of reconstruction methods phrase the solution to an inverse problem in terms of a multidimensional cost function [8, 18]. The cost function in this case is based on fitting the measured surface field data in a least squares sense. Using the chosen numerical forward solver one could find an approximate solution to the inverse problem using a guess and improve method. The global minimum of the cost function now represents the best solution to the inverse problem. Finding the global minimum is a task ideally suited to the optimization technique, simulated annealing [1, 5, 6, 14]. Inversions in geophysics have been attempted before using the simulated annealing technique. For example, in [20], simulated annealing was used in the inversion of nonlinear seismic soundings for a 1D earth model.

This paper will formulate the mathematical model for the case of mise-à-la-masse prospecting. A numerical method using a finite difference approximation is proposed. A least squares fit to the surface field is suggested as a suitable method to approximate the inverse problem including geophysically motivated assumptions which regularize the problem. The value of the least squares fit is assigned to a cost function where the minimum value of the cost function is the best fit of the surface fields and hence the best approximation to the inverse problem. The search for this best-fit is attempted with the use of simulated annealing. A proof of principle example using a synthetically generated data set is furnished to validate the technique. Further work includes an example which incorporates the dimensions from a real-life vein-type ore body. Extensions to the model which investigate the effects of random data noise and conductive overburden are also considered.

2. Formulation. The mise-à-la-masse prospecting method can be modelled by the electrostatic problem of an electrode buried in an inhomogeneous infinite half-space. The conservation of electric charge dictates that the current density \( \mathbf{J} \) obeys

\[
\nabla \cdot \mathbf{J} = 0,
\]

when no charge flux is contained. In a linear isotropic medium the relation between current density and electric field \( \mathbf{E} \) is given by

\[
\mathbf{J} = \sigma \mathbf{E},
\]
where $\sigma$ is the conductance of the inhomogeneous medium. Let

$$
(2.3) \quad \sigma = \sigma(x, y, z),
$$

be represented in rectangular coordinates where $z$ increases vertically downward (Figure 1a). The electric field can be written in terms of a scalar potential $\phi$ as

$$
(2.4) \quad \mathbf{E} = \nabla \phi.
$$

In terms of the scalar potential (2.1) yields

$$
(2.5) \quad \nabla \cdot (\sigma \nabla \phi) = 0.
$$

For a homogeneous conductivity field, $\sigma$ would be constant and (2.5) reduces to Laplace’s equation.

The boundary conditions consist of the potential field asymptotically approaching zero at infinity. That is,

$$
(2.6a) \quad \lim_{(x, y) \to \pm \infty} \phi(x, y, z) = 0,
$$

and

$$
(2.6b) \quad \lim_{z \to +\infty} \phi(x, y, z) = 0.
$$

The boundary condition at the surface arises from the continuity of $\mathbf{J}$ at the interface. At any interface the normal component of the current density must be continuous. In terms of the electric field, this yields

$$
(2.7) \quad \mathbf{n} \cdot (\sigma_{\text{air}} \mathbf{E}_{\text{air}} - \sigma_{\text{ground}} \mathbf{E}_{\text{ground}}) = 0.
$$

Since the air layer is nonconductive (i.e., $\sigma_{\text{air}} = 0$), a no-flux condition is established on the surface given by

$$
(2.8) \quad \frac{\partial \phi}{\partial z}_{|z=0} = 0.
$$

The buried electrode is assumed to be maintained at a fixed potential. The corresponding boundary condition is

$$
(2.9) \quad \phi(0, 0, L) = \phi_0,
$$
FIGURE 1a) Coordinate system used for mise-à-la-masse model. The z-direction points vertically downward into the earth. b) The idealized ore body with length scales shown. The source electrode lies buried within the ore body at the position $(0,0,L)$. A finite region $R$ is defined on the surface where the surface field measurements are compiled.

where $\phi_0$ is some constant.

In the mise-à-la-masse prospecting method, one attempts to reconstruct a buried conductive anomaly using either the potential or potential gradient (electric field) measured on the surface. The solution to this inverse problem is highly nonlinear and most certainly nonunique. As is typical in the solution of inverse problems, certain assumptions must be made to regularize the problem [13, 17, 8]. That is, physical
insight is applied to limit the set of solutions. For example, negative values of conductivity are not considered nor are values of conductivity which exceed that of known rock structures. In the proposed model, geophysical insight is used to construct the two following regularizing assumptions as to the details of the ore body.

Firstly, the ore body is assumed to be characterized by a high value of conductivity. Outside of the body, the conductivity is some lesser value. Both the ore body and the surrounding medium are assumed to be conductively homogeneous; the conductivity is constant and known. For vein type bodies a high conductivity contrast between ore bearing and non-ore bearing rock is common. A typical ratio of conductivities is on the order of $10^4 : 1$ [16]. Hence, this conductivity ratio can be supplied as a parameter rather than found as a result of the calculation. This limits some of the degrees of freedom of the inversion; the algorithms is responsible for finding the extent of the ore body, not the conductivity of it. This assumption is similar to that found in [13] when incomplete boundary data is known.

Secondly, the shape of the ore body is assumed to be a rectangular prism with the buried source lying somewhere within this prism. This box-type structure is found commonly in modelling [7] as well as in practice [23]. The choice of a box structure provides an important uniqueness result. Theoretical results from [4] show that the solution for the potential problem for convex polyhedrons of constant conductivity is unique. Hence, two different high conductance boxes cannot produce the same external potential. It is interesting to note that a similar situation in 2-D does not hold. That is, two distinct polygons can be constructed that generate the same external potential [21].

The box of high conductance can be represented uniquely by eight parameters; three values $(x_0, y_0, z_0)$ denote the position of the box center, three values $(L_1, L_2, L_3)$ give the length of each box dimension and two parameters represent the dip and strike angles $(\gamma, \theta)$ (see Figure 2). The dip angle is the angle between the box's $L_1$ axis and the $xy$-plane while the strike angle represents the rotation about the $z$-axis. As is the practice in prospecting, these are the fundamental angles which describe an ore vein. Note that the range of the angles is only between $-45^\circ$ and $45^\circ$. This limited rotation is allowed by the fact that a box has constant conductivity. For example, no difference would be found between a unit cube and a unit cube rotated $90^\circ$ about any axis.
perpendicular to a face. There is a possibility for a third angle which would describe the rotation about the $x$-axis but this angle is generally ignored.

By using the forward solver, potential values $\phi = \phi(x, y, z)$ can be found for any box configuration. The electric field at the surface can then be calculated from

$$E^c = \nabla \phi = \left( \frac{\partial \phi}{\partial x}(x, y, 0), \frac{\partial \phi}{\partial y}(x, y, 0), 0 \right),$$

where the superscript $c$ denotes calculated field. Note that $E^c_x = 0$ from the no-flux boundary condition in (2.7). Now one wishes to somehow assign a value of merit to any combination of the eight box parameters. This cost value is based on how well the observed electric fields on the surface are approximated by the calculated electric fields. The minimum cost corresponds to the best approximation to the inverse problem.

A suitable cost function $C$ is given by the least squares difference between the observed electric field and the calculated electric field.
FIGURE 3. Diagram of the grid used for the finite-difference discretization. Within the region $R$, uniform grid spacing is used while outside this region a nonuniformly spaced grid is used.

summed at points on the surface. That is,

$$C = \left| \mathbf{E}^m - \mathbf{E}^c \right|^2 \quad \text{on } R,$$

which can be written

$$C = \sum_{(x,y) \in R} \left[ (E_{x}^m - E_{x}^c)^2 + (E_{y}^m - E_{y}^c)^2 \right],$$

where

$$\mathbf{E}^m = (E_x^m, E_y^m, 0),$$

is the observed electric field and $R$ represents a finite two-dimensional region on the earth's surface where field measurements have been compiled (Figure 3). The cost is a function of the eight box parameters,

$$C = C(x_0, y_0, z_0, L_1, L_2, L_3, \gamma, \theta).$$
The norm type definition of $C$ admits the following two properties: $C \geq 0$ and $C = 0$ if the reconstructed box is exactly the generating box. Now the solution to the inverse problem becomes a search for the global minimum of the cost function.

An alternative formulation for the cost function of (2.11) using the surface potential $\phi$ rather than the potential gradient was also investigated. The new cost function would appear like

\[(2.15) \quad C = |\phi^m(x, y, 0) - \phi^c(x, y, 0)|^2 \quad \text{on } R.\]

The inversion results were similar using the cost calculation of (2.15) to that of (2.11) and hence are absent from discussion.


3.1. Solution of forward problem. The forward problem denotes the solution of (2.5) given the conductivity field of the inhomogeneous halfspace. The solution is attained numerically using a finite difference approximation for the derivatives on a three-dimensional rectangular grid. A fixed potential was applied at the electrode.

Special consideration must be given to the remaining boundary conditions. The difficulty arises from the fact that the boundary conditions are prescribed at infinity which is impossible to facilitate given a finite grid. Fortunately, the asymptotic behavior of the potential is

\[(3.1.1) \quad \phi \sim 1/r,\]

where $r$ is the distance from the source. Thus, cut-off lengths can be introduced where the potential, for practical purposes, will lose details of the near source conductivity structure and approach its asymptotic behavior. The cut-off lengths were given in each dimension by $L_x, L_y, L_z$ thus defining a large finite box which encloses the problem (Figure 1b). Assuming that

\[(3.1.2) \quad \phi = A/r,\]

for $A$ some constant, then one derivative yields [7],

\[(3.1.3) \quad \frac{\partial \phi}{\partial \hat{n}} = -\frac{A}{r^2} \hat{e}_r \cdot \hat{n} \]
where \( \mathbf{n} \) is the outward normal of any face of the cut-off box and \( \mathbf{e}_r \) is the unit radial vector. Using (3.1.2) in (3.1.3) gives

\[
(3.1.4) \quad \frac{\partial \phi}{\partial \mathbf{n}} = -\frac{\cos(\beta)\phi}{r},
\]

where \( \beta \) is the angle between the unit radial vector and the normal vector to a face. The Dirichlet boundary condition at infinity in (2.6) is now approximated by the Robin boundary condition of (3.1.4) at a finite distance from the electrode using the asymptotic behavior of the potential. The only inherent length scale to the problem as opposed to the artificially imposed cut-off lengths, is the depth of the electrode below the surface given by \( L \). This value is used to scale all the length dimensions.

A nonuniform grid was implemented to assist with the boundary conditions at infinity. The aim is to choose a large domain size on whose fringe the boundary conditions can be imposed while still maintaining reasonable resolution in the center. This task is accomplished by using a uniformly spaced rectangular grid nested within a nonuniformly spaced grid (see Figure 3). The grid spacing in the nonuniform region was chosen to be exponentially increasing. The spacing is given by

\[
(3.1.5) \quad \Delta x_i = \Delta x_0 \exp \left( \frac{(i - Ni_u)}{\alpha_x} \right),
\]

where \( \Delta x_0 \) is the uniform grid spacing \( \alpha_x \) a scaling constant and \( Ni_u \) is the number of uniformly spaced points in the \( x \)-direction. A similar expression exists for the \( y \) and \( z \) directions. The rate of increase of the nonuniform spacing is not without restrictions. This can be seen by examining the central finite difference formula for the second derivative of a function, \( \phi \). The formula with uniform grid spacing is given by

\[
(3.1.6) \quad \phi''(x_i) = \frac{\phi(x_{i-1}) - 2\phi(x_i) + \phi(x_{i+1})}{\Delta x^2} + O(\Delta x^2),
\]

where \( \Delta x \) is the distance between the uniform grid points. The truncation error is second order. The expression for nonuniformly
spaced grid points, however, is found to be
\[
\phi''_i(x_i) = \frac{\Delta x_i \phi(x_{i-1}) - 2 \Delta \tilde{x} \phi(x_i) + \Delta x_{i+1} \phi(x_{i+1})}{\Delta x_i \Delta x_{i+1} \Delta \tilde{x}} \\
+ \frac{\phi''''(x_i)}{3} (\Delta x_i - \Delta x_{i+1}) \\
+ \frac{\phi^{(iv)}(x_i)}{12} (\Delta x_i^2 - \Delta x_i \Delta x_{i+1} + \Delta x_{i+1}^2) + \cdots ,
\]
where \(\Delta x_i\) is defined in (3.1.5) and \(\Delta \tilde{x} = (\Delta x_{i+1} + \Delta x_i)/2\). The grid spacing is no longer a constant but rather a function of \(i\), the number of grid points from the source. Accordingly, the truncation error changes from second order in the uniform case to something more complicated in the nonuniform case as demonstrated by the second term in (3.1.7). The error is a product of the third derivative of the potential with the rate at which the grid spacing increases. Far from the electrode one expects (3.1.2) to hold, implying that \(\phi'' \sim 1/r^4\). The difference in grid spacing yields
\[
\Delta x_i - \Delta x_{i+1} = \Delta x_0 \exp \left( \frac{i - N \nu}{\alpha_x} \right) \left( 1 - \exp \left( \frac{1}{\alpha_x} \right) \right),
\]
which is a growing exponential. The task is to choose a reasonably large \(\alpha_x\) such that the truncation error in (3.1.7) is kept sufficiently small. One can easily verify that as \(\alpha_x \to \infty\), the grid spacing \(\Delta x_i \to \Delta x_0\) and hence the truncation error returns to second order.

The finite-difference discretization is performed on the nonhomogeneous Laplace's equation of (2.5). Expanding (2.5) for Cartesian coordinates yields
\[
\frac{\partial}{\partial x} \left( \sigma \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \sigma \frac{\partial \phi}{\partial y} \right) + \frac{\partial}{\partial z} \left( \sigma \frac{\partial \phi}{\partial z} \right) = 0.
\]
The first term in (3.1.9) can be discretized for node \((i,j,k)\) using a central difference formula, such as
\[
\frac{\partial}{\partial x} \left( \sigma \frac{\partial \phi}{\partial x} \right)_{i,j,k} = \frac{1}{2 \Delta \tilde{x}} \left( \frac{\Delta x_i}{\Delta x_{i+1}} \right) \left( \frac{\partial \phi}{\partial x} \right)_{i+1/2,j,k} \\
+ \frac{\Delta \tilde{x} \Delta \tilde{x}}{\Delta x_i \Delta x_{i+1}} \left( \frac{\partial \phi}{\partial x} \right)_{i,j,k} \\
- \frac{1}{2 \Delta \tilde{x}} \left( \frac{\Delta x_{i+1}}{\Delta x_i} \right) \left( \frac{\partial \phi}{\partial x} \right)_{i-1/2,j,k},
\]
where $\Delta \tilde{x}$ is the average grid spacing as above and $\Delta \tilde{x} = \Delta x_{i+1} - \Delta x_i$ is the difference between grid spacing. The value of the conductivity $\sigma$ at the midpoint is chosen to be the average value. That is,

\begin{equation}
\sigma_{i+1/2,j,k} = \frac{\sigma_{i+1,j,k} + \sigma_{i,j,k}}{2}.
\end{equation}

The derivatives at the midpoints in (3.1.10) can be expanded in terms of the central difference formulat to give

\begin{equation}
\left( \sigma \frac{\partial \phi}{\partial x} \right)_{i+1/2,j,k} = \sigma_{i+1/2,j,k} \left( \frac{\phi_{i+1} - \phi_i}{\Delta x_{i+1}} \right)
\end{equation}

\begin{equation}
\left( \sigma \frac{\partial \phi}{\partial x} \right)_{i-1/2,j,k} = \sigma_{i-1/2,j,k} \left( \frac{\phi_i - \phi_{i-1}}{\Delta x_i} \right)
\end{equation}

\begin{equation}
\left( \sigma \frac{\partial \phi}{\partial x} \right)_{i,j,k} = \sigma_{i,j,k} \left\{ \frac{(\Delta x_i)^2 \phi_{i+1,j,k} + 2 \Delta \tilde{x} \Delta \tilde{x} \phi_{i,j,k} - (\Delta x_{i+1})^2 \phi_{i-1,j,k}}{2 \Delta \tilde{x} \Delta x_{i+1} \Delta \tilde{x}} \right\}.
\end{equation}

The terms in (3.1.12) can be substituted in (3.1.10) to give the complete expression for the $x$-derivative. This process can be repeated for the $y$ and $z$ derivatives to give the complete discretized equation for (3.1.9). For a uniform grid spacing $\Delta \tilde{x} = 0$, meaning that equation (3.1.12c) would not appear in (3.1.10).

The finite-difference discretization generates a linear system of $N_i \cdot N_j \cdot N_k$ simultaneous algebraic equations

\begin{equation}
A \phi = b
\end{equation}

where $N_i, N_j, N_k$ represent the number of grid points in each of the respective $(x, y, z)$ coordinate directions. One can easily see that the size of the linear system increases as a cubic which demonstrates the difficulty of modelling in 3-D. The matrix $A$, however, has at most seven entries in any row allowing the use of sparse matrix techniques. The sparse matrix solver used was Mat1 which utilizes the Yale sparse matrix storage technique. Mat1 is an iterative solver that implements an Orthomin acceleration technique as well as an incomplete factorization to approximate the solution of a linear system of equations [11].
Analytic solutions of (2.5) [22] often represent the buried source as a
delta function which is impossible to represent exactly using numerical
techniques. To verify the numerical results, the asymptotic behavior of
the computed potential was examined for the case of a surface electrode
on a homogeneous halfspace. Assuming a form of the solution like

\[ \phi = A/r^b, \]

(3.1.14)

and taking logarithms yields

\[ \ln(\phi) = \ln(A) + b \ln(r). \]

(3.1.15)

If (3.1.2) is true, one expects the value of \( b \) to be constant and equal
to \(-1\). This relationship can be seen in Figure 4 for the Robbin
and Dirichlet boundary conditions. The value of \( b \) is calculated by
a linear regression fit of (3.1.15) using the value of \( \phi \) at all grid points.
The length scale \( L_x \) represents the outer edge of the computational
domain where the boundary conditions are applied. The model was
run with equal lengths in all the coordinate directions implying that
\( L_y = L_x \) and \( L_z = L_x \). The value of \( b \) computed for Dirichlet
boundary conditions improves as the size of the computational domain
is increased. The value for Robbin boundary conditions, however, is
exactly \(-1\), independent of the domain size.

3.2. Solution of inverse problem. The cost function is an eight-
dimensional multimodal scalar function. One possible technique to
find the global minimum would be to calculate the cost for all possible
configurations and choose the smallest value calculated. This type of
brute force solution is possible theoretically, but impractical given finite
computing time and resources.

Many algorithms exist for the minimization of functions [3]. For exam-
ple, a gradient method can be used to seek out a minimum. In our
case, gradient methods are unsuitable since no analytic information
about the derivatives of the cost function is available. Also, cross-
sections of the cost function verify its multimodal nature (Figure 5).
While some dimensions show \( C \) to be strongly unimodal other dimen-
sions contain several local minima. A derivative method could become
trapped in a local minimum where no better local solution exists. Fur-
thermore, calculating the cost function is computationally expensive
FIGURE 4. Plot of $b$ versus $L_z$ for the numerical solution of a surface electrode and a conductively homogeneous earth for both Dirichlet and Robin boundary conditions. The variable $b$ is defined as the slope of the linear regression of $\ln(\phi) = \ln(A) + b \ln(r)$ where $\phi$ is the potential and $A$ is a constant. The variable $r$ is the distance from the source electrode and $L_z$ is the size of the finite domain.

Since each evaluation requires completely solving the potential field equations of (2.5). Hence, a minimum number of function evaluations is desired in order to locate the global minimum. A suitable heuristic technique for this type of optimization is simulated annealing. It is worth emphasizing that this is a heuristic method so it does not guarantee the discovery of the global minimum but it performs well at finding a near global minimum.

Simulated annealing is based on the Metropolis algorithm [15] and draws analogies to the freezing of crystals. The procedure requires the following formulation:

1) Description of all possible system configurations
2) Generator of random changes to configuration
3) Cost Function (to maximize or minimize)
4) Control Parameter $T$ to step down (annealing schedule).

The current state of the system is given by the eight box parameters described in Figure 2. The random changes consist of perturbing one of these eight box parameters. The magnitude of this perturbation raises a difficult question. In a discrete system such as the Travelling Salesman Problem in [14], the random changes are straightforward. In the proposed problem, each of the parameters is real valued and hence can achieve a continuum of values. Minimization of functions with continuous variables has been discussed in [5, 6]. The method used in [6] involves choosing at random an $n$-dimensional vector from
the $n$-dimensional unit hypersphere, where $n$ is the dimensionality of the function to be minimized. This unit vector is then multiplied by a set of scaling factors for each dimension. The system state is then moved by the scaled vector. In the present example, choosing a new configuration need not be so complicated. A change of less than one grid spacing fails to change the finite difference linearization of (3.1.9) since the conductivity is stored only at discrete grid points. Accordingly, the potential field will remain unchanged leaving the cost function the same. An appropriate magnitude for any perturbation is then given by the grid spacing for each dimension. A smaller change would go unnoticed while a larger change would skip intermediate configurations. A perturbation for parameters in the $x$-direction would be $x_0 \pm \Delta x_0$ and $L_1 \pm \Delta x_0$. Similarly, the changes in the $y$ and $z$ directions are $y_0$, $L_2 \pm \Delta y_0$ and $z_0$, $L_3 \pm \Delta z_0$. A natural increment does not exist for the dip and strike angles. A perturbation of $\gamma$, $\theta \pm 5^\circ$ was found to give acceptable resolution as well as providing changes of reasonable magnitude. Thus, at each step an increase or decrease of one of the eight box parameters yields a maximum of 16 possible choices to achieve a new configuration.

The annealing process proceeds as follows:

a) start with an initial configuration
b) calculate cost function
c) generate random change to configuration
d) recalculate cost function

e) accept or reject step

f) adjust temperature if necessary

g) if annealing is complete stop, else go to c).

The temperature parameter $T$ is assigned an initial value greater than the average change in the cost function for any particular random configuration change. This is commonly referred to as the melting temperature. This initial temperature should be high enough so that the algorithm can choose any state in the solution domain, similar to a molecule being allowed to move throughout a liquid. Randomly selected configuration changes are performed at a particular temperature level until some maximum number are attempted or a preset limit of successful reconfigurations is achieved. Then the temperature, $T$, is reduced by 20% and the process is repeated until no successful changes occur. The temperature at the $k$th level is then

$$(3.2.1) \quad T_k = (0.80)^k T_0,$$

where $T_0$ is the initial temperature. The lowering of temperature is known as the cooling or annealing schedule. The ideal cooling schedule is difficult to predict. Cooling too quickly may result in quenching. This refers to the cost function achieving a local rather than global minimum. Cooling too slowly wastes computational effort. Convergence results for the simulated annealing algorithm as well as ideal cooling schedules can be seen in [12]. As the temperature is lowered, the system becomes less like a fluid and more like a solid. By performing the cooling sufficiently slowly, the system can cool into its crystalline state (global minimum cost). When no more changes are accepted, it is assumed that the minimum cost is reached and the reconstruction of the high conductance box is complete.

The annealing process continues until one of the following two stopping conditions is met: 1) the minimum value of the cost function is found, or 2) continuous perturbations of the system produce no acceptances and hence no new configurations are generated.

The first criteria assumes that the theoretical minimum is known which is not always the case. Of course, with the synthetic noise free data the minimum cost that can be achieved is $C = 0$ (Figure 8a). This fact determines exactly when to stop. With the introduction of noisy
data, however, the global minimum is some unknown value greater than zero (Figure 8b). The second criteria is usually gauged in terms of the ratio of acceptances to perturbation attempts. A suitable stopping condition exists if this ratio is less than 10%. Fortunately, the current model possesses a finite number of state changes (16 in total). This property simplifies the second stopping criteria. Extensive searching at the low temperature levels of annealing is not necessary since there are only a finite number of changes to the configuration possible. If all 16 changes are tried and rejected, then a minimum is assumed and further searching is pointless. It is true that this minimum may be local, but a careful choice of the annealing schedule will guarantee that this is a global minimum.

Simulated annealing accepts a step unconditionally if $\Delta C < 0$. If $\Delta C > 0$ then this step is accepted based on the Boltzmann's
FIGURE 7. Surface plots of the magnitude of the electric field $|E|$ for a) noiseless synthetic data and b) with 40% uniformly distributed random noise added.

probability,

$$Pr(\Delta C) \sim \exp \left(-\frac{\Delta C}{T}\right).$$

This algorithm is not greedy, since there is always a finite chance of
FIGURE 8. Schematic of the cost function for a) noiseless synthetic data and b) noisy data. The situation in b) foreshadows the theoretical breakdown of the method. If the global minimum cost does not correspond to the correct solution, then the inversion will fail.
accepting a higher cost value in order to jump out of a local minimum \cite{14}.

Another acceptance scheme different from the Boltzmann's probability was considered for comparison. Threshold accepting (TA), as proposed by \cite{9}, is similar to simulated annealing; however, the acceptance of a random change that increases the cost function is slightly different. In TA a random change with $\Delta C > 0$ is accepted if $\Delta C < T$, where $T$ is the control parameter. This prevents accepting new configurations which are far worse. In simulated annealing there is always a finite probability that a very bad change is accepted. The advantages of threshold accepting are seen in the low temperature fine tuning of the solution. Here, computational effort is saved since poor changes, which would have to be undone later, are not accepted.

4. Validation. Testing of the reconstruction algorithm was done using synthetically generated data. The synthetic data set is created in the following manner. A set of eight box parameters (Table 1) was chosen to run through the forward solver. The resulting solution, $\phi$, was used to find the electric field measurements on the surface. These values were regarded as the experimentally measured values which the algorithm was attempting to recover. There are two pieces of information one can exploit from this synthetic data: 1) the generating conductivity structure is known and 2) the global minimum is guaranteed to be zero. The latter fact is a result of the cost function being nonnegative and the linear system in (3.1.9) having a unique solution.

**TABLE 1.** Box parameters for synthetically generated data set.

<table>
<thead>
<tr>
<th>$(x_0, y_0, z_0)$</th>
<th>$(L_1, L_2, L_3)$</th>
<th>Dip and Strike</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-0.375, 0.25, 1.0)$</td>
<td>$(1.75, 1.0, 0.5)$</td>
<td>$\gamma = 30^\circ$, $\theta = -30^\circ$</td>
</tr>
</tbody>
</table>

Reconstruction of the original configuration proceeded by choosing an initial guess box and then applying the Metropolis algorithm. The initial guess was chosen to be a unit cube with zero dip and strike angles (see Figure 6). Solving for the guess box and calculating the cost function yields an initial cost of $C = 90.23$. The simulated annealing algorithm is then allowed to perturb this initial configuration and
accept or reject new guesses accordingly. The entire annealing process requires 438 iterations (Table 2) to successfully reconstruct the correct box. Of these attempted perturbations, 233 changes were accepted.

As with any real world experiment, errors, whether measurement or otherwise, will be present. The merit of an inversion scheme relies upon its ability to deal with random noise in the data. Uniformly distributed random noise was added to the synthetically generated data set to test the inversion scheme’s robustness. To add $N\%$ noise to a measurement $M$, the formula used is

\[(4.1) \quad M_{\text{noisy}} = \left(1 \pm \frac{N}{100} \cdot U[0, 1]\right) \cdot M,\]

where $U[0, 1]$ represents a uniformly distributed random number between 0 and 1. The effect of 40% random noise added to the surface signal can be seen in Figure 7. Inversion results with the noisy data can be seen in Table 2. Note that the known minimum cost is no longer 0. The known minimum value is calculated by substituting the synthetic data and the synthetic data with noise in the cost formula (2.11). This new minimum cost should represent the cost when the reconstruction is complete. The inversion worked with up to 40% random noise added. The solution effort, however, increases dramatically with the addition of noise. The number of iterations required by the simulated annealing search increases by an order of magnitude from 0% noise to 35% noise. This can be attributed to the blurring of the cost functions global minimum when noise is added as sketched in Figure 8b. In fact, the actual minimum achieved in the 35% noise case is lower than the minimum cost value expected. This phenomenon represents the theoretical breakdown of the inversion scheme. If the cost value for the true answer is not the global minimum, then the simulated annealing search will be unable to locate the conductivity structure which generated the data in the first place. For 50% random noise the annealing search failed to converge.

5. Further results. The reconstruction method was further tested using synthetically generated data that represents a real life ore body. The dimensions for this test configuration were extracted from a vein type body found on the Cavendish Geophysical test range [23]. The test range consists of an extensively surveyed hard rock site. It
TABLE 2. Performance of annealing reconstruction with uniformly distributed random noise added to the surface field data.

<table>
<thead>
<tr>
<th>(%)</th>
<th>Known</th>
<th>Final</th>
<th>Accepted</th>
<th>Attempted</th>
<th>((x_0, y_0, z_0))</th>
<th>((L_1, L_2, L_3))</th>
<th>((\gamma, \theta))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>233</td>
<td>438</td>
<td>((-0.375, 0.25, 1.0))</td>
<td>((1.75, 1.0, 0.5))</td>
<td>((30^\circ, -30^\circ))</td>
</tr>
<tr>
<td>10</td>
<td>2.55</td>
<td>2.55</td>
<td>253</td>
<td>671</td>
<td>((-0.375, 0.25, 1.0))</td>
<td>((1.75, 1.0, 0.5))</td>
<td>((30^\circ, -30^\circ))</td>
</tr>
<tr>
<td>30</td>
<td>22.87</td>
<td>22.86</td>
<td>314</td>
<td>1111</td>
<td>((-0.5, 0.25, 1.0))</td>
<td>((2.0, 1.0, 0.5))</td>
<td>((30^\circ, -35^\circ))</td>
</tr>
<tr>
<td>35</td>
<td>40.2</td>
<td>40.1</td>
<td>1893</td>
<td>3472</td>
<td>((-0.5, 0.25, 0.9325))</td>
<td>((1.5, 1.0, 0.625))</td>
<td>((30^\circ, -30^\circ))</td>
</tr>
<tr>
<td>40</td>
<td>50.4</td>
<td>50.4</td>
<td>3345</td>
<td>9004</td>
<td>((-0.375, 0.25, 1.0))</td>
<td>((1.75, 1.0, 0.5))</td>
<td>((30^\circ, -30^\circ))</td>
</tr>
<tr>
<td>50</td>
<td>82.7</td>
<td></td>
<td>(\infty)</td>
<td>(\infty)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
was established in 1967 and is used for the testing and evaluation of electromagnetic prospecting techniques. The tests conducted include airborne and ground based EM surveys as well as a comprehensive diamond drilling program. The drilling program involved removing core samples which were classified as to type of rock and minerals present. Metals found included Cu, Pb, Ni and Sn in sulphide type deposits. The dimensions used were that of the Zone A body which lies 3m below the surface and is the shape of a rectangular slab 200m by 20m by 3m. These dimensions were nondimensionalized using the drill hole depth \( L = 8.9 \text{m} \). Borehole resistivity measurements conducted in [16] measured the conductivity of the ore body and the surrounding ground. This ratio was approximately 10,000 to 1. The sharp conductivity contrast, however, proved difficult for the numerical solver; hence, a conductivity ratio of 100 to 1 was used.

**TABLE 3.** Box parameters for the synthetically generated data set modelled after the Zone A ore body found at the Cavendish Geophysical Test Range. The hole depth \( L = 8.9 \text{m} \) was used to nondimensionalize all the distance measurements.

<table>
<thead>
<tr>
<th>((x_0, y_0, z_0))</th>
<th>((L_1, L_2, L_3))</th>
<th>Dip and Strike</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0.25, 1.08, 1.50))</td>
<td>((0.38, 15.18, 2.5))</td>
<td>(\gamma = -20^\circ, \theta = 20^\circ)</td>
</tr>
</tbody>
</table>

The initial box parameters for the Cavendish Zone A body can be seen in Table 3 while the details of the recovery can be seen in Table 4. The inversion achieved a lowest cost corresponding to \( C = 33.1 \) starting from an initial cost value of \( C = 1003.0 \). The nonzero lowest cost is not a shortcoming of the minimization algorithm, but rather a limitation of the coarseness of the finite difference mesh. A large grid spacing in the \( y \)-direction, \( \Delta y = 0.5 \), was required in order to fit this dimension within the domain. This coarseness makes exact recovery impossible. Most importantly, the dip and strike angles are in the correct direction and are close in magnitude.

The Cavendish Zone A ore body dimensions were also used to test inversion in the presence of a thin conductive layer near the surface. This layer is known as conductive overburden. The higher conductivity found in an overburden layer is a result of moisture and dissolved salts lying in the soil above the bedrock. The effect of this layer is to partially shield the electrical response of the buried ore body and thereby reduce the magnitude of the electric fields measured on the surface. The
overburden was given a conductivity ratio relative to the background of \( \sigma_{\text{over}} / \sigma_{\text{lo}} = 10 \) and a scaled thickness \( L/4 \). This thickness corresponds to a layer approximately 2m thick. The results of the inversion with this conductive layer can be seen in Table 5 as well as results with a conductive layer twice as thick (i.e., \( L/2 \)).

The conductive overburden does not seriously affect the length and width recovery of the inversion algorithm, but it does seriously affect the resolving power in the z-direction. Determination of the width and length of the conductive anomaly relies on the location and magnitude of the peaks of the surface fields. The recovery of the depth information depends on the complete structure of the surface fields over a large area on the surface. The smoothing of the signal on the surface is more pronounced with a highly conductive layer near the surface as is demonstrated by comparing Figure 9a) and Figure 9b). This
TABLE 4. Recovery of ore body for dimensions from the Cavendish Geophysical Test Range.

<table>
<thead>
<tr>
<th>Known</th>
<th>Final</th>
<th>Accepted</th>
<th>Attempted</th>
<th>((x_0, y_0, z_0))</th>
<th>((L_1, L_2, L_3))</th>
<th>((\gamma, \theta))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\theta)</td>
<td>33.1</td>
<td>258</td>
<td>597</td>
<td>((0.5,0,0,2.0))</td>
<td>((0.5,15.0,1.75))</td>
<td>((-30^\circ, 20^\circ))</td>
</tr>
</tbody>
</table>

TABLE 5. Recovery of ore body for dimensions from the Cavendish Geophysical Test Range with a thin layer of conductive overburden lying on the surface. The layer of overburden was chosen to have a conductivity ratio \(\sigma_{over}/\sigma_{lo} = 10\).

<table>
<thead>
<tr>
<th>Layer</th>
<th>Minimum Cost</th>
<th>Re-configurations</th>
<th>Reconstructed Box</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Thickness</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Known</td>
<td>Final</td>
<td>Accepted</td>
<td>Attempts</td>
</tr>
<tr>
<td>(L/4)</td>
<td>0</td>
<td>25.67</td>
<td>245</td>
</tr>
<tr>
<td>(L/2)</td>
<td>0</td>
<td>16.54</td>
<td>325</td>
</tr>
</tbody>
</table>
smoothing makes the recovery of the depth information difficult. Most importantly, the dip and strike angles are recovered exactly in the case of a layer of thickness $L/4$. Hence, the proposed inversion schemes work reasonably well in the presence of a conductive overburden.

6. Discussion. A full three-dimensional model of the mise-à-la-massee prospecting method has been developed. A finite difference approximation was used to solve numerically the electrostatic potential when given the conductivity field. By making regularizing assumptions pertaining to the conductivity substructure, the inverse problem is converted to a search for the global minimum of an eight-dimensional scalar cost function. This idea proves successful with a simple synthetic noise free data set.

A more difficult reconstruction occurs when uniformly generated
random noise is added to the surface field data. Here the simulated annealing minimization requires more iterations to find the global minimum as the amount of noise is increased. The algorithm is successful with up to 40% random noise. The theoretical breakdown of the method begins to occur beyond 40% noise. This means that the global minimum of the cost function corresponds to a conductivity structure which is not necessarily the generating conductivity structure. Hints of this breakdown can be seen in the 35% noise case, but these false minimums are still close to the correct answer. With 50% random noise the search for a global minimum failed outright.

A realistic ore body reconstruction is successfully completed using dimensions from the Cavendish Geophysical Test Range. Further extensions of the model included inversion in the presence of a conductive overburden.

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REFERENCES


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