ANALYTICAL SOLUTION TO EDDY CURRENT TESTING OF CYLINDRICAL PROBLEMS WITH VARYING PROPERTIES

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ABSTRACT. An analytical solution to eddy current testing problems in cylindrical coordinates is obtained in the case the excitation coil of the probe is coaxially situated inside a cylindrical tube whose magnetic permeability, \( \mu(r) \), depends upon the radial coordinate, \( r \). In the particular cases \( \mu(r) = 1/r^2 \) and \( \mu(r) = 1/r \), this solution is expressible in terms of Bessel and Whittaker functions, respectively. These results can be used for the evaluation of different properties of cylindrical tubes.

1. Introduction. The eddy current method is widely used for quality control measurements of properties of products and materials of cylindrical shape, like the thickness of different coverings of tube walls, and the characteristics of materials, like the electrical conductivity (see [1, 6]).

In practice, the excitation coil of an eddy current probe is coaxially situated inside the tube so that the axial symmetry of the problem is taken into account by the mathematical modelling; however, it is usually assumed that the properties of the medium are constant.

Solutions to problems where the excitation coil is coaxially situated inside a multilayer tube are given in [9, 8, 4, 5, 2, 3]. If the medium has variable properties, the formulae obtained in these references can be used in the following way. The conducting medium is divided into a large number of layers and the properties of the medium are assumed to be constant within each layer. The obvious shortcoming of this approach is that one has to consider a sufficiently large number of layers in order to achieve the desired accuracy (for example, up to 50 layers are used in [14] for the solution of a similar problem for a plane multi-layer medium).
Therefore there is a need to construct a mathematical model for a medium with variable properties. This problem was solved in [12] for the case where the coil is situated above a conducting half-space whose magnetic permeability varies in the vertical direction; more precisely, \( \mu(z) = \mu_0 e^{-\alpha z} \), \( \alpha = \text{constant} \). A similar problem was solved in [11] for a double conductor line situated above a half-space made of a double-layer medium with variable properties. In [13], the electrical conductivity of the conducting half-space was taken to be \( \sigma = \sigma_0 r^n \), for \( n = 0, 2, 4, 6 \).

In the present paper we consider some analytical solutions of eddy current testing problems for a cylindrical medium with variable magnetic and electric properties. We assume that the excitation coil of the probe is coaxially situated inside a cylindrical tube of radius \( R \) whose magnetic permeability \( \mu \) varies inversely with a power of the radial coordinate \( r \): \( \mu(r) = r^\alpha \). In the particular cases \( \alpha = -2 \) and \( \alpha = -1 \), the solution can be expressed in terms of Bessel and Whittaker functions, respectively.

2. Governing equations. In the theory of eddy current testing, one usually neglects the displacement current and restricts oneself to the case of low frequencies. Thus, Maxwell's equations can be approximated by the following pair of equations:

\[
\begin{align*}
\text{(1)} & \quad \text{curl } H = \sigma E + I^e, \\
\text{(2)} & \quad \text{curl } E = -\frac{\partial B}{\partial t},
\end{align*}
\]

where \( E \) and \( H \) are the electric and magnetic field strengths, respectively, and \( B \) is the magnetic induction vector. \( I^e \) is the external current density and \( \sigma \) is the conductivity of the medium.

We assume that \( I^e \) is a sinusoidal function of time, so that \( E \) and \( H \) are also sinusoidal in time, that is,

\[
\begin{align*}
\text{(3)} & \quad H(r, \varphi, z, t) = \tilde{H}(r, \varphi, z)e^{j\omega t}, \\
\text{(4)} & \quad E(r, \varphi, z, t) = \tilde{E}(r, \varphi, z)e^{j\omega t},
\end{align*}
\]

where \((r, \varphi, z)\) is a system of cylindrical polar coordinates and \( j = \sqrt{-1} \).

Moreover, since the vectors \( B \) and \( H \) are related by the formula

\[
B = \mu_0 \mu(r) H,
\]
where $\mu(r)$ is the magnetic permeability of the medium and $\mu_0$ is the magnetic constant, equation (2) may be written in the form

$$\text{curl } \vec{E} = -j\omega \mu_0 \mu(r) \vec{H}. \tag{5}$$

The vector potential $\vec{A}$ is given by the relation

$$\text{curl } \vec{A} = \mu_0 \mu(r) \vec{H}. \tag{6}$$

It follows from (5) and (6) that $\text{curl } (\vec{E} + j\omega \vec{A}) = 0$ where $\vec{A}(r, \varphi, z, t) = \vec{A} e^{j\omega t}$. Thus, there exists a scalar function $\psi$ which satisfies the relation

$$\vec{E} + j\omega \vec{A} = -\nabla \psi. \tag{7}$$

Using (1), (3)–(6), we obtain

$$\text{curl} \left( \frac{1}{\mu_0 \mu(r)} \nabla \vec{A} \right) = \sigma (-j\omega \vec{A} - \nabla \psi) + \vec{I}^e. \tag{8}$$

If we assume that the vector potential $\vec{A}$ has only one nonzero component in the $\varphi$-direction and it is independent of $\varphi$, namely,

$$\vec{A}(r, \varphi, z) = A_\varphi(r, z) \vec{e}_\varphi, \tag{9}$$

where $\vec{e}_\varphi$ is the unit vector in the $\varphi$-direction, then equation (8) can be written in the form

$$\frac{\partial^2 A_\varphi}{\partial r^2} + \left( \frac{1}{r} - \frac{1}{\mu} \frac{d\mu}{dr} \right) \frac{\partial A_\varphi}{\partial r} - \left( \frac{1}{r^2} + \frac{1}{\tau} \frac{d\mu}{dr} + j\omega \sigma \mu_0 \mu \right) A_\varphi \tag{10}$$

$$+ \frac{\partial^2 A_\varphi}{\partial z^2} = -\mu_0 \mu I^e,$$

where $I^e$ is the $\varphi$-component of $\vec{I}^e = I^e \vec{e}_\varphi$, and $r$ and $z$ are dimensional variables.

3. Mathematical analysis. We consider an excitation coil with current, of radius $R$, coaxially situated inside an infinitely long cylindrical tube of inner radius $\rho_1$ and outer radius $\rho_2$. If we take the radius of the coil, $R$, as a measure of length $\rho$, then $r = \rho/R$ and $z$ are dimensionless coordinates. Thus, $r_1 = \rho_1/R$ and $r_2 = \rho_2/R$. 
FIGURE 1. Coil of radius $l$, tube of radii $r_1 < r_2$, and regions $R_0, R_1$ and $R_2$.

We need to solve equation (10) in each of the three regions $R_0, R_1$ and $R_2$ (shown in Figure 1):

$$R_0 = \{0 \leq r \leq r_1, 0 \leq \phi \leq 2\pi, -\infty < z < +\infty\},$$

$$R_1 = \{r_1 \leq r \leq r_2, 0 \leq \phi \leq 2\pi, -\infty < z < +\infty\},$$

$$R_2 = \{r \geq r_2, 0 \leq \phi \leq 2\pi, -\infty < z < +\infty\}.$$

Thus the mathematical formulation of the problem becomes

(11) \[ \frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r} \frac{\partial A_0}{\partial r} - \frac{1}{r^2} A_0 + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 IR^2 \delta(r - R) \delta(z), \]

(12) \[ \frac{\partial^2 A_1}{\partial r^2} + \left( \frac{1}{r} - \frac{1}{\mu \frac{d\mu}{dr}} \right) \frac{\partial A_1}{\partial r} - \left( \frac{1}{r^2} + \frac{1}{r\mu \frac{d\mu}{dr}} + R^2 j \omega \sigma \mu_0 \mu \right) A_1 \]

\[ + \frac{\partial^2 A_1}{\partial z^2} = 0, \]
\begin{align}
\frac{\partial^2 A_2}{\partial r^2} + \frac{1}{r} \frac{\partial A_2}{\partial r} - \frac{1}{r^2} A_2 + \frac{\partial^2 A_2}{\partial z^2} &= 0,
\end{align}

where the subscripts 0, 1 and 2 denote solutions in the regions $R_0$, $R_1$ and $R_2$, respectively.

The boundary conditions are

\begin{align}
A_0|_{r=r_1} &= A_1|_{r=r_1}, & \frac{\partial A_0}{\partial r}|_{r=r_1} &= \frac{1}{\mu_1} \frac{\partial A_1}{\partial r}|_{r=r_1}, \\
A_1|_{r=r_2} &= A_2|_{r=r_2}, & \frac{1}{\mu_2} \frac{\partial A_1}{\partial r}|_{r=r_2} &= \frac{\partial A_2}{\partial r}|_{r=r_2},
\end{align}

where $\mu_1 = \mu(r_1)$ and $\mu_2 = \mu(r_2)$.

Applying the Fourier cosine transform

\[ \tilde{A}_i(\lambda, r) = \int_0^\infty A_i(r, z) \cos \lambda z \, dz \]

to problem (11)--(15) we obtain

\begin{align}
\frac{d^2 \tilde{A}_0}{dr^2} + \frac{1}{r} \frac{d \tilde{A}_0}{dr} - \frac{1}{r^2} \tilde{A}_0 - \lambda^2 \tilde{A}_0 &= -\mu_0 \frac{IR^2}{2} \delta(r - R), \\
\frac{d^2 \tilde{A}_1}{dr^2} + \left( \frac{1}{r} - \frac{1}{\mu_2} \frac{d \mu}{dr} \right) \frac{d \tilde{A}_1}{dr} &+ \left( \frac{1}{r^2} + \frac{1}{r \mu_2} \frac{d \mu}{dr} + R^2 j \omega \mu_0 \mu \right) \tilde{A}_1 - \lambda^2 \tilde{A}_1 = 0, \\
\frac{d^2 \tilde{A}_2}{dr^2} + \frac{1}{r} \frac{d \tilde{A}_2}{dr} - \frac{1}{r^2} \tilde{A}_2 - \lambda^2 \tilde{A}_2 &= 0,
\end{align}

\begin{align}
\tilde{A}_0|_{r=r_1} &= \tilde{A}_1|_{r=r_1}, & \frac{d \tilde{A}_0}{dr}|_{r=r_1} &= \frac{1}{\mu_1} \frac{d \tilde{A}_1}{dr}|_{r=r_1}, \\
\tilde{A}_1|_{r=r_2} &= \tilde{A}_2|_{r=r_2}, & \frac{1}{\mu_2} \frac{d \tilde{A}_1}{dr}|_{r=r_2} &= \frac{d \tilde{A}_2}{dr}|_{r=r_2}.
\end{align}
While the solution to (16) and (18) can be expressed in terms of Bessel functions, the solution to (17) is, in general, more complicated. If \( \mu = \mu(r) \) is an analytic function of \( r \) and the origin, \( r = 0 \), is a regular singular point of (17), then the solution to (17) can be found by Frobenius's method. However, in some particular cases the solution can be found in closed form.

In the general case where \( \mu(r) = r^\alpha \), \( \alpha = \text{constant} \), equation (17) becomes

\[
\frac{d^2 \tilde{A}_1}{dr^2} + \frac{(1 - \alpha)}{r} \frac{d \tilde{A}_1}{dr} - \left( \frac{\alpha + 1}{r^2} + R^2 j \omega \sigma \mu_0 r^\alpha + \lambda^2 \right) \tilde{A}_1 = 0.
\]

We consider two particular cases. First, if \( \alpha = -2 \), equation (17) simplifies to

\[
\frac{d^2 \tilde{A}_1}{dr^2} + \frac{3}{r} \frac{d \tilde{A}_1}{dr} - \left( \frac{-1 + R^2 j \omega \sigma \mu_0}{r^2} + \lambda^2 \right) \tilde{A}_1 = 0,
\]

which, after some transformation, becomes a modified Bessel equation, whose solution (see [10, p. 146]) is

\[
\tilde{A}_1(\lambda, r) = C_1 \frac{I_p(\lambda r)}{r} + C_2 \frac{K_p(\lambda r)}{r},
\]

where \( p = R\sqrt{j \omega \sigma \mu_0} \) and \( I_p(y) \) and \( K_p(y) \) are the modified Bessel functions of order \( p \) of the first and second kinds, respectively.

Second, if \( \alpha = -1 \), then (21) reduces to

\[
\frac{d^2 \tilde{A}_1}{dr^2} + \frac{2}{r} \frac{d \tilde{A}_1}{dr} - \left( \frac{R^2 j \omega \sigma \mu_0}{r} + \lambda^2 \right) \tilde{A}_1 = 0,
\]

which is a particular form of the confluent hypergeometric equation

\[
\frac{d^2 y}{dx^2} + \left( a + \frac{b}{x} \right) \frac{dy}{dx} + \left( c + \frac{d}{x} + \frac{e}{x^2} \right) y = 0,
\]

with \( a = 0, b = 2, c = -\lambda^2, d = -j \omega \sigma \mu_0, e = 0 \) (see [7, p. 251]). If we use the substitution

\[
y = x^{-1} w(\kappa, \nu, \xi),
\]
where
\[ k = \frac{d}{2\lambda}, \quad \nu = \frac{1}{2}, \quad \xi = 2\lambda r, \]
and \( w(k, \nu, \xi) \) is a Whittaker function (see [7, p. 248]), then the solution to (24) can be expressed in terms of Whittaker functions.

4. Solution in the case \( \alpha = -2 \). We now treat in detail the case \( \alpha = -2 \). It is convenient to consider two subregions in the region \( R_0 \), namely, \( 0 < r < 1 \) and \( 1 < r < r_1 \). If we denote the solutions in these regions by \( \tilde{A}_{00}(\lambda, r) \) and \( \tilde{A}_{01}(\lambda, r) \), respectively, then the solution to (16) that is bounded at \( r = 0 \) can be written in the form
\[
(27) \quad \tilde{A}_{00}(\lambda, r) = C_3 I_1(\lambda r),
(28) \quad \tilde{A}_{01}(\lambda, r) = C_4 I_1(\lambda r) + C_5 K_1(\lambda r).
\]
The solution to (18) that is bounded at infinity is
\[
(29) \quad \tilde{A}(\lambda, r) = C_6 K_1(\lambda r).
\]
The function \( \tilde{A}_0 \) is continuous at the interface \( r = 1 \), that is,
\[
(30) \quad \tilde{A}_{00}|_{r=1} = \tilde{A}_{01}|_{r=1}.
\]
A second condition at \( r = 1 \) can be found by integrating (16) with respect to \( r \) from \( 1 - \epsilon \) to \( 1 + \epsilon \) and taking the limit as \( \epsilon \to +0 \):
\[
(31) \quad \left. \frac{d\tilde{A}_{01}}{dr} \right|_{r=1} - \left. \frac{d\tilde{A}_{00}}{dr} \right|_{r=1} = -\frac{\mu_0 I R^2}{2}.
\]
The values of the arbitrary constants \( C_1, C_2, \ldots, C_6 \), can be found from (19), (20), (30) and (31). These are
\[
(32) \quad C_3 = C_4 + \frac{\mu_0 I}{2} RK_1(\lambda),
(33) \quad C_4 = \frac{DI}{E},
(34) \quad C_5 = \frac{\mu_0 I}{2} RI_1(\lambda),
(35) \quad C_6 = \gamma C_2,
(36) \quad C_2 = \frac{\rho_1 I_1(\lambda \rho_1) C_4 + \rho_1 K_1(\lambda \rho_1) C_5}{\gamma I_p(\lambda \rho_1) + K_p(\lambda \rho_1)},
(37) \quad C_6 = C_1 \frac{I_p(\lambda \rho_2)}{\rho_2 K_1(\lambda \rho_2)} + C_2 \frac{K_p(\lambda \rho_2)}{\rho_2 K_1(\lambda \rho_2)},
\]
where

\[
\gamma = \{(1 + p - \mu_2) \lambda \rho_2 K_1(\lambda \rho_2) - \mu_2 \rho_2 K_0(\lambda \rho_2)\}K_p(\lambda \rho_2) + \lambda \rho_2 K_{p-1}(\lambda \rho_2) K_1(\lambda \rho_2) \}
\]

\[
\sqrt{\{(\mu_2 - 1 - p) K_1(\lambda \rho_2) + \mu_2 \rho_2 K_0(\lambda \rho_2)\} I_p(\lambda \rho_2) + \lambda \rho_2 I_{p-1}(\lambda \rho_2) K_1(\lambda \rho_2)} \}
\]

\[
D = \frac{\mu_0}{2} R I_1(\lambda) \{\mu_1 [\gamma I_p(\lambda \rho_1) + K_p(\lambda \rho_1)] \\
\times [\lambda \rho_1^2 K_0(\lambda \rho_1) + \rho_1 K_1(\lambda \rho_1)] + \rho_1 K_1(\lambda \rho_1)[\gamma \lambda \rho_1 I_{p-1}(\lambda \rho_1) - \gamma (1 + p) I_p(\lambda \rho_1) \\
- \lambda \rho_1 K_{p-1}(\lambda \rho_1) - (1 + p) K_p(\lambda \rho_1)] \}
\]

\[
E = \mu_1 [\gamma I_p(\lambda \rho_1) + K_p(\lambda \rho_1)][\lambda \rho_1^2 I_0(\lambda \rho_1) - \rho_1 I_1(\lambda \rho_1)] \\
- \rho_1 I_1(\lambda \rho_1)[\gamma \lambda \rho_1 I_{p-1}(\lambda \rho_1) - \gamma (1 + p) I_p(\lambda \rho_1) \\
- \lambda \rho_1 K_{p-1}(\lambda \rho_1) - (1 + p) K_p(\lambda \rho_1)].
\]

The induced change in impedance in the region \( R_0 \) due to the presence of the conducting tube is given by the formula

\[
Z_{\text{ind}} = \frac{j \omega}{I} \oint_A A_0(r, z) \, dl,
\]

where \( A_0(r, z) \) is the component of the vector potential in \( R_0 \) due to the presence of the conducting tube, and \( i \) is the contour of the coil.

Using the inverse Fourier cosine transform,

\[
A_i(r, z) = \frac{2}{\pi} \int_0^\infty \tilde{A}_i(\lambda, r) \cos \lambda z \, d\lambda,
\]

we obtain the induced change in impedance in region \( R_0 \) in the form

\[
Z_{\text{ind}} = 4 \omega R^2 Z_0,
\]

where

\[
Z_0 = j \int_0^\infty \frac{D}{E} I_1(\lambda) \, d\lambda.
\]
FIGURE 2. The change in impedance, $Z_0$, as a function of $\beta$, for $r_1 = 1.05$ and two values of $r_2$, namely, $r_2 = 1.2$ and $r_2 = 1.4$.

This formula is used to compute the induced change in impedance due to the presence of a conducting tube whose magnetic permeability varies as $\mu(r) = 1/r^2$ and $r_1 \leq r \leq r_2$.

5. Numerical results. Mathematica, version 2.1, was used on a Sun Microsystems Sparc 10 to evaluate integral (39) numerically because it computes the values of the modified Bessel functions $I_p(x)$ and $K_p(x)$ for the complex order $p$ and the real argument $x$, as well as the integral of a complex valued function. The central part of the Mathematica code is given in the appendix. It was found that the values of the integral from 0 to 20 did not change much over a larger interval.

The hodographs of the change in impedance, $Z_0$, are presented in Figure 2 for the fixed value of $r_1 = 1.05$ and two values of $r_2$, namely, $r_2 = 1.2$ and $r_2 = 1.4$. The arrows in this figure indicate the direction of increase of the dimensionless parameter $\beta = R \sqrt{\omega \sigma \mu_0}$, which is related to $p$ by the formula $p = \beta \sqrt{3}$. The end points of the curves for $r_2 = 1.2$ and $r_2 = 1.4$ correspond to $\beta = 7$ and $\beta = 6$, respectively. As can be seen from the figure, the growth of $\beta$ in these intervals leads to an increase of the modulus of $Z_0$. 
6. **Conclusion.** An analytical solution to eddy current testing problems in the case where the excitation coil of the probe is coaxially situated inside a circular conducting tube with variable magnetic properties is presented in this paper. Two particular cases are considered, namely, $\mu(r) = r^{-2}$ and $\mu(r) = r^{-1}$. The solution to the first case is given in detail, and it is shown how to construct an analytical solution for the second case. These solutions can be expressed in terms of integrals of Bessel and Whittaker functions.

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**Appendix**

This Appendix lists the central part of a Mathematica program for computing the modified Bessel functions $I_p(x)$ and $K_p(x)$ for complex order $p$ and real argument $x$, and the integral of a complex valued function.

```mathematica
r1=1.05
r2=1.2
b=1.
p=N[Sqrt[0+1 I]*b]
p1=p-1.
ml=1/r1^2
m2=1/r2^2
xr2[x-]:=x*r2
xr1[x-]:=x*r1
klr2[x-]:=N[BesselK[1,xr2[x]]]
k0r2[x-]:=N[BesselK[0,xr2[x]]]
kpr2[x-]:=N[BesselK[p,xr2[x]]]
kp1r2[x-]:=N[BesselK[p1,xr2[x]]]
ipr2[x-]:=N[BesselI[p,xr2[x]]]
ip1r2[x-]:=N[BesselI[p1,xr2[x]]]
ipr1[x-]:=N[BesselI[p,xr1[x]]]
kpr1[x-]:=N[BesselK[p,xr1[x]]]
```
\[\begin{align*}
k_{0r_1}[x_\cdot] &= N[BesselK[0,xr_1[x]]] \\
k_{1r_1}[x_\cdot] &= N[BesselK[1,xr_1[x]]] \\
I_{0r_1}[x_\cdot] &= N[BesselI[0,xr_1[x]]] \\
I_{1r_1}[x_\cdot] &= N[BesselI[0,xr_1[x]]] \\
\gamma_{0}[x_\cdot] &= kpr_2[x]*(1+p-m_2)k_{1r_2}[x] - x*m_2*r_2*k_{0r_2}[x] \\
&+ x*r_2^2*kpr_2[x]*k_{1r_2}[x] \\
\gamma_{1}[x_\cdot] &= ((m_2-1-p)k_{1r_2}[x] + x*m_2*r_2*k_{0r_2}[x])*ipr_2[x] \\
&+ x*r_2^2*ipr_2[x]*k_{1r_2}[x] \\
d[x_\cdot] &= N[BesselI[1,\dot{x}^2]] \\
&= (m_1*(\gamma_{0}[x_\cdot]*ipr_1[x]+kpr_1[x])* \\
&+ x*r_1^2*k_{0r_1}[x] + r_1*k_{1r_1}[x]) \\
&+ r_1*k_{1r_1}[x]*(\gamma_{1}[x_\cdot]*r_1*ipr_1[x]- \\
&- \gamma_{1}[x_\cdot]*(1+p)*ipr_1[x] - r_1*ipr_1[x]) \\
e[x_\cdot] &= m_1*(\gamma_{0}[x_\cdot]*ipr_1[x]+kpr_1[x]) \\
&+ x*r_1^2*i_{0r_1}[x] + r_1*i_{1r_1}[x]) \\
&+ - r_1*i_{1r_1}[x]*(\gamma_{1}[x_\cdot]*r_1*ipr_1[x]- \\
&- \gamma_{1}[x_\cdot]*(1+p)*ipr_1[x] - r_1*ipr_1[x]) \\
\text{NIntegrate}[d[x]/e[x],\{x,0,20\}] &
\end{align*}\]

REFERENCES


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