A DISCRETE TIME DELAYED NEURAL NETWORK WITH POTENTIAL FOR ASSOCIATIVE MEMORY REVISITED

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ABSTRACT. We consider a nonlinear difference equation with a time delay which represents the simplest possible discretely updated neural network. We show that provided that the integral time delay is large enough, this self feedback system exhibits a huge number of asymptotically stable periodic orbits. Therefore, such a network has potential for associative memory and pattern recognition.

1 Introduction

Having the ability to store and retrieve information in a robust manner is of interest for many practical purposes. It is well known that the coexistence of a large number of stable periodic patterns or stable equilibria is the basis for this mechanism in neural networks. Temporal delays, which occur in implementations, are unavoidable due to factors such as the distances between neurons, and the finite switching speeds of amplifiers. In the work of Wu and Zhang [10], inspired by Walther [7], it was shown that a simple discrete time delayed neural network can have coexistence of a huge number of stable periodic orbits provided that the integral time delay is big enough. The limitations of their results come from the fact that the network does not have enough trainable parameters with which information can be encoded. Wang and Zou [8] generalized these ideas by considering a discrete time network consisting of two layers with bidirectional connection topology, with time delays corresponding to each neuron. The contribution of Wang and Zou is that the network has the same desired dynamical properties as that of Wu and Zhang’s, but for a much greater number of trainable parameters. A common feature of both of the latter networks is that the signal transmission functions involved are perturbations of the sign function. That is, a step function with two
distinct values. The contribution of the present work is to demonstrate that modifying the signal transmission function in the network originally considered by Wu and Zhang, can lead to a much greater capacity for storing information. Using the same techniques as Wu and Zhang, we consider a class of functions which are perturbations of a step function taking on three values. We show that the corresponding delayed neural network has coexistence of a huge number of periodic patterns and that this number is much bigger than that obtained by Wu and Zhang, provided that the time delay is big enough. We conclude with a discussion of how these techniques maybe possible to extend to perturbations of step functions taking on any odd number of values bigger than 1, and a comparison with the numbers obtained in this study with those of Wu and Zhang [10].

2 Preliminaries Consider the following nonlinear difference equation with a fixed time delay $k \geq 2$ of integer value:

$$x_n = \beta x_{n-1} + \alpha f(x_{n-k}),$$

where $\alpha > 0$ and $\beta \in (0, 1)$ are model parameters, $n \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$ belongs to the class of functions satisfying the following conditions

$$\begin{align*}
  f(x) &= 0 & \text{if } |x| \leq \epsilon \\
  |f(x) - f(y)| &\leq L|x - y| & \text{if } x, y > \epsilon \text{ or } x, y < -\epsilon \\
  |f(x) - 1| &\leq \mu & \text{if } x > \epsilon \\
  |f(x) + 1| &\leq \mu & \text{if } x < -\epsilon
\end{align*}$$

for some constants $L, \mu \geq 0$ and $\epsilon > 0$. We shall call this class of functions $C(L, \mu, \epsilon)$. Note that the function

$$f(x) = \begin{cases} 
  0 & \text{if } |x| \leq \epsilon \\
  1 & \text{if } x > \epsilon \\
  -1 & \text{if } x < -\epsilon
\end{cases}$$

is a step function belonging to this class that takes on the values/symbols $-1$, $1$ and $0$. We shall see that these symbols play a vital role. The class $C(L, \mu, \epsilon)$ permits, hopefully, engineering implementation and applications, for a wide range of signal (input-output) functions.

Equation (1) represents the simplest self-feedback system representing a single neuron with instantaneous decay rate $\beta$, nonlinear feedback
with delay \( k \), the synaptic weight \( \alpha \), and a typical signal transmission function. By a solution of equation (1), we mean any sequence of real numbers \( \{x_n\}_{n=-k}^\infty \) which satisfies (1) for each \( n \geq 0 \). Given any sequence \( (x_{-k}, x_{-k+1}, \ldots, x_{-1}) \) in \( \mathbb{R}^k \), there is clearly a unique solution of (1) coinciding with the given sequence at \( n = -k, \ldots, -1 \). Therefore, it is natural to view equation (1) as a discrete semidynamical system on \( \mathbb{R}^k \), which will be made clear below. We will show that under certain constraints, (1) exhibits a large number of stable periodic orbits provided that the time delay \( k \) is large enough, and we will show that for each \( p \mid k \) one can easily count the number of stable \( p \)-periodic orbits.

A delay difference equation is not an infinite dimensional system. A simple change of variables shows that such a scalar delay difference equation gives rise to a discrete semidynamical system on higher dimensions. Namely, given a solution \( \{x_n\}_{n=-k}^\infty \) of equation (1) if we define \( \{w_n\}_{n=0}^\infty \) by \( w_n = (x_{n-k}, \ldots, x_{n-1}) \), then

\[
(4) \quad w_{n+1} = F(w_n) \quad \text{for each} \quad n \geq 0 \quad \text{where} \quad F : \mathbb{R}^k \to \mathbb{R}^k
\]

is given component-wisely by (4)

\[
(5) \quad F_j(w) = \begin{cases} w_{j+1} & \text{if } 1 \leq j \leq k-1 \\ \beta w_k + \alpha f(w_1) & \text{if } j = k. \end{cases}
\]

Thus, \( \{x_n\}_{n=-k}^\infty \) is a solution of (1) if and only if \( \{w_n\}_{n=0}^\infty \) satisfies (4) for \( n \geq 0 \).

Observe that the step function given by (3) takes on the three values \(-1, 1 \) and \( 0 \). Consider the collection \( \Sigma = \{-1, 0, 1\}^k \) consisting of sequences of length \( k \) taking on the values \(-1, 0 \) and \( 1 \). Let \( \pi \) be the clockwise cyclic permutation on \( \Sigma \). That is \( \pi(\sigma_n) = \sigma_{n+1} \) for \( n < k \) and \( \pi(\sigma_k) = \sigma_1 \). Note that \( \pi \) has as many periodic points as we like provided that \( k \) is big enough. In fact, we will see below that we can give a formula for the periodic points of the permutation \( \pi \). The idea in what is to follow is to construct appropriate subsets in \( \mathbb{R}^k \) so that the semiflow given by \( F \), restricted to the union of these sets, is topologically semiconjugate to that given by \( \pi \) on \( \Sigma \) when we think of \( \Sigma \) as a topological space in the discrete topology.

3 Notations and hypotheses  Here, we formulate constraints on the parameters and introduce some convenient notations.
Recall equation (1) and the definition of the class of functions $C(L, \mu, \epsilon)$. We assume that $\alpha > 0$, $0 < \beta < 1$, and $0 \leq \mu < 1$.

Further, let $b := \alpha(1 + \mu)/(1 - \beta)$. We assume that

$$b \leq \frac{3\epsilon - \alpha(1 - \mu)}{\beta} \quad \text{and} \quad \alpha > \frac{2\epsilon}{1 - \mu}.$$  \hspace{1cm} (6)

This implies $b < \epsilon/\beta$.

Finally, we assume that

$$0 \leq L < \frac{1 - \beta}{\alpha}.$$  \hspace{1cm} (7)

Let $\Omega := \{w \in \mathbb{R}^k \mid \|w\| < \epsilon/\beta\}$, where $\| \cdot \|$ denotes the maximum norm on $\mathbb{R}^k$. Let $\varphi$ be the step function taking on the values $-1$, $0$ and $1$ given by equation (3).

Let $\Sigma := \{-1, 0, 1\}^k$, that is, all sequences of length $k$ with values in $-1$, $0$ and $1$. Given $\sigma$ in $\Sigma$ we define the rectangle $\Omega_\sigma = \{w \in \Omega \mid \varphi(w_j) = \sigma_j \text{ for } j = 1, \ldots, k\}$.

$$r := \min \left\{ \alpha(1 - \mu) - 2\epsilon, \frac{\epsilon}{\beta} - b, \frac{\epsilon - \beta b}{1 + \beta} \right\} > 0.$$  \hspace{1cm} (8)

The following sets defined for each $0 < c < r$ will play an important role later,

$$\Omega_{\sigma,c} = \{w \in \Omega_\sigma \mid \forall j = 1, \ldots, k, \ |w_j| \leq c \text{ or } c \leq |w_j| \leq b + c\}.$$  \hspace{1cm} (9)

Such a set is a nonempty closed rectangle in $\mathbb{R}^k$.

Let $\pi : \Sigma \rightarrow \Sigma$ denote the clockwise cyclic permutation. For each $p \geq 1$ let $\Sigma_p$ be the collection of all $p$-periodic points of $\pi$. When we refer to the period of a permutation or an iteration, we always mean the minimum period from now on.

We are ready to state and prove some basic lemmas and propositions which will give us the existence, uniqueness, and stability theorems for periodic orbits of equation (1).

4 Technical lemmas We start with a simple combinatorial result which will help us count the number of (stable) periodic orbits of (1).

Lemma 1. Given $k \geq 2$ the following hold.
(i) $p \mid k$ if and only if $\Sigma_p \neq \emptyset$.

(ii) Let $N_p$ denote the cardinality of $\Sigma_p$. Then

$$N_p = \begin{cases} 3 & \text{if } p = 1 \\ 3p - 3 & \text{if } p \text{ is prime} \\ 3p - \sum_{q<p, q \mid p} N_q & \text{otherwise}. \end{cases}$$

(iii) $\Sigma = \bigcup_{p \mid k} \Sigma_p$.

Proof. This is fairly straightforward.

Lemma 2. $F(\Omega_\sigma) \subseteq \Omega_{\pi(\sigma)}$, and for each $0 < c < r$, we have that $F(\Omega_{\sigma,c}) \subseteq \Omega_{\pi(\sigma),c}$. In particular, for each positive integer $l$, $F^l(\Omega_\sigma) \subseteq \Omega_{\pi^l(\sigma)}$ and $F^l(\Omega_{\sigma,c}) \subseteq \Omega_{\pi^l(\sigma),c}$.

Proof. Given that $0 < c < r$, fix $w \in \Omega_{\sigma,c}$. Then by equation (5) to show that $F(w) \in \Omega_{\pi(\sigma),c}$ it suffices to show that

$$\varphi(F_k(w)) = \sigma_1,$$

and either $|F_k(w)| \leq \epsilon - c$ or $\epsilon + c \leq |F_k(w)| \leq b + c$.

To this end there are two cases to consider.

Case 1: $|w_1| \leq \epsilon - c$.

In this case, by equation (5), and using the definition of $r$, we have

$$|F_k(w)| = \beta|w_k| \leq \beta(b + c) \leq \beta b + \beta \frac{\epsilon - \beta b}{1 + \beta} = \beta b + \frac{\beta \epsilon}{1 + \beta} = \frac{(1 + \beta)\epsilon - (\epsilon - \beta b)}{1 + \beta} = \epsilon - \frac{\epsilon - \beta b}{1 + \beta} \leq \epsilon - c.$$  

Therefore, $\varphi(F_k(w)) = 0$ and $|F_k(w)| \leq \epsilon - c$.

Case 2: $\epsilon + c \leq |w_1| \leq b + c$.

Here, we have two subcases to consider.

The first subcase is $\epsilon + c \leq w_1 \leq b + c$. In this case, we have (using the definition of $b$)

$$F_k(w) = \beta w_k + \alpha f(w_1) \leq \beta w_k + \alpha (1 + \mu) \leq \beta(b + c) + \alpha(1 + \mu) = b + \beta c \leq b + c,$$
and, using the definition of \( r \),

\[
F_k(w) \geq \beta w_k + \alpha (1 - \mu) \geq \beta \left( -\frac{c}{\beta} \right) + \alpha (1 - \mu) = \epsilon + (\alpha (1 - \mu) - 2c) \geq \epsilon + c.
\]

So, \( \varphi(F_k(w)) = 1 \) and \( \epsilon + c \leq F_k(w) \leq b + c \).

The second subcase is \( -b - c \leq w_1 \leq -\epsilon - c \). In this case, using a similar estimation as that for the first subcase we can conclude that \( -b - c \leq F_k(w) \leq -\epsilon - c \) and that \( \varphi(F_k(w)) = -1 = \varphi(w_1) \).

This completes the proof of Lemma 2 for the set \( \Omega_{\sigma,c} \).

A similar (and easier) argument establishes the lemma for \( \Omega_{\sigma} \). The next lemma establishes some useful estimates for iterates of \( F \).

**Lemma 3.** Given \( \sigma \in \Sigma \) and \( w, w' \in \Omega_{\sigma} \), for each \( 1 \leq l \leq k \) and each \( (k - l) + 1 \leq j \leq k \),

\[
|F^j_l(w) - F^j_l(w')| \leq (\beta + \alpha L)\|w - w'\|.
\]

Consequently, \( \|F^j_l(w) - F^j_l(w')\| \leq \|w - w'\| \) and \( \|F^k_l(w) - F^k_l(w')\| \leq (\beta + \alpha L)\|w - w'\| \).

**Proof.** This lemma follows from a straightforward induction argument on the well ordered set of integers \( 1 \leq l \leq k \) using the definition of \( F \) and \( \Omega_{\sigma} \), and the fact that \( \beta + \alpha L < 1 \). \( \square \)

### 5 Existence, uniqueness, and stability of periodic orbits

Now we can state and prove the existence, uniqueness, and stability of \( p \)-periodic solutions of (1) for each non-negative divisor \( p \) of \( k \).

Recall that equation (4) reads \( w_{n+1} = F(w_n) \). We will denote solutions of (4) with initial data \( w_0 \) by \( w(n, w_0) \). Equivalently, \( w(n, w_0) \) is the \( n \)-th order iteration \( F^n(w_0) \).

**Proposition 1.** Fix \( 0 < c < r \). For each \( p | k \) and \( \sigma \in \Sigma_p \), there exists a \( p \)-periodic solution \( w(n, w^p_\sigma) \) of equation (4) having \( w^p_\sigma \in \Omega_{\sigma,c} \).

**Proof.** \( F^p(\Omega_{\sigma,c}) \subseteq \Omega_{\pi p(\sigma),c} = \Omega_{\sigma,c} \) by Lemma 2. Therefore, \( F^p \) is a continuous self map on \( \Omega_{\sigma,c} \), so Brouwer’s fixed point theorem gives us a fixed point, \( w^p_\sigma \in \Omega_{\sigma,c} \) of \( F^p \). Thus, letting \( w(n, w^p_\sigma) = F^n(w^p_\sigma) \) gives us the desired conclusion. \( \square \)

In particular, Proposition 1 gives us for each \( p | k \) and \( \sigma \in \Sigma_p \), a collection \( \{w(n, w^p_\sigma)\} \) of \( p \)-periodic solutions of (4), with initial data in \( \Omega_{\sigma} \), indexed by \( 0 < c < r \). The next result shows that for given \( p | k \) and \( \sigma \in \Sigma_p \) there can only be one such solution.
Proposition 2. Suppose that \( w(n, w^\sigma) \) and \( w(n, \tilde{w}^\sigma) \) are two periodic solutions of (4) with \( w^\sigma \) and \( \tilde{w}^\sigma \) lying in \( \Omega_\sigma \). Then \( w^\sigma = \tilde{w}^\sigma \), and hence, \( w(n, w^\sigma) = w(n, \tilde{w}^\sigma) \).

Proof. Let the period of \( w(n, w^\sigma) \) be \( p \) and the period of \( w(n, \tilde{w}^\sigma) \) be \( q \). For each \( n \in \mathbb{N} \), by Lemma 3 we see that

\[
\| w(n, w^\sigma) - w(n, \tilde{w}^\sigma) \| = \| F^{n+pqk}(w^\sigma) - F^{n+pqk}(\tilde{w}^\sigma) \|
\]

\[
\leq (\beta + \alpha L)^{pq} \| F^n(w^\sigma) - F^n(\tilde{w}^\sigma) \|
\]

\[
= (\beta + \alpha L)^{pq} \| w(n, w^\sigma) - w(n, \tilde{w}^\sigma) \|. (14)
\]

Since \( \beta + \alpha L < 1 \), we must have that \( w(n, w^\sigma) = w(n, \tilde{w}^\sigma) \) for all \( n \).

The next result gives the number of \( p \)-periodic solutions of equation (1) whose orbits lie in the region \( \Omega \).

Corollary. Given any \( p \geq 1 \), let \( \mathcal{S}_p \) denote the collection of \( p \)-periodic solutions of equation (4) whose orbits lie in the region \( \Omega \). The following are equivalent.

(i) \( \mathcal{S}_p \neq \emptyset \),
(ii) \( p \mid k \);
(iii) \( \mathcal{S}_p \) is in bijective correspondence with \( \Sigma_p \) and \( p \mid k \).

Proof. (i) \( \Rightarrow \) (ii).
Let \( w(n, w^0) \in \mathcal{S}_p \). Let \( \sigma = (\varphi(w^0_1), \ldots, \varphi(w^0_q)) \in \Sigma \). Then by Lemma 1, there is some \( q \mid k \) such that \( \sigma \in \Sigma_q \) and \( w^0 \in \Omega_\sigma \). Proposition 1 gives us some \( w^\sigma \in \Omega_\sigma \) such that \( w(n, w^\sigma) \) is a \( q \)-periodic solution of (4). Proposition 2 implies that these solutions must be equal and therefore \( p = q \Rightarrow p \mid k \).

(ii) \( \Rightarrow \) (iii).
This follows from Propositions 1 and 2.

(iii) \( \Rightarrow \) (i).
This follows from Lemma 1.

We now show that periodic solutions of (4) whose orbits lie in \( \Omega \) are asymptotically stable.

Proposition 3. Suppose that \( w(n, w^\sigma) \) is a periodic solution of (4) in \( \Omega \) with \( w^\sigma \in \Omega_\sigma \). Then there is some \( r_\sigma > 0 \) such that whenever \( \| w^0 - w^\sigma \| < r_\sigma \) then \( \lim_{n \to \infty} \| w(n, w^\sigma) - w(n, w^0) \| = 0 \).
Proof. By the Corollary, we know that the period $p$ of $w(n, w^\sigma)$ divides $k$. By Propositions 1 and 2, we can conclude that $w^\sigma \in \Omega_{\sigma, c}$ for each $0 < c < r$. So let

$$r_\sigma = \min_{j=1,\ldots,k} \left\{ \frac{\epsilon}{\beta} - |w^\sigma_j|, |\epsilon - |w^\sigma_j|| \right\}.$$ \hfill (15)

Note that $r_\sigma > 0$. To obtain the desired conclusion, for each $n \geq 0$, we write $n = qk + s$ with $q \geq 0$ and $0 \leq s < k$. If $w^0$ satisfies $\|w^0 - w^\sigma\| < r_\sigma$, then we have by Lemma 3 that

$$\|w(n, w^0) - w(n, w^\sigma)\| = \|F^{qk+s}(w^0) - F^{qk+s}(w^\sigma)\|$$

$$\leq \|F^{qk}(w^0) - F^{qk}(w^\sigma)\|$$

$$\leq (\beta + \alpha L)^q \|w^0 - w^\sigma\|.$$ \hfill (16)

Now, we write $(\beta + \alpha L)^q = (\beta + \alpha L)^{(n-s)/k} = (\beta + \alpha L)^{n/k} (\beta + \alpha L)^{-s/k}$. Since $(\beta + \alpha L)^{-s/k} < (\beta + \alpha L)^{-1}$, using $C := (\beta + \alpha L)^{-1}$, we get

$$\|w(n, w^\sigma) - w(n, w^0)\|$$

$$\leq C((\beta + \alpha L)^{1/k})^n \|w^0 - w^\sigma\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$ \hfill (17)

We end this section by describing the number of stable $p$-periodic orbits corresponding to solutions of (4) in $\Omega$.

**Proposition 4.** For each $p \mid k$, equation (4), or equivalently equation (1), has $N_p$ periodic solutions whose orbits lie in the region $\Omega$ all of which are asymptotically stable. Consequently, for $p \mid k$ equation (4) has $N_p/p$ asymptotically stable periodic orbits lying in $\Omega$ and the total number of stable periodic orbits of (4) lying in $\Omega$ is just $\sum_{p \mid k} N_p/p$.

Proof. The first part of the above proposition follows from Lemma 1, the Corollary, and Proposition 3. The second part follows from the first and a straightforward combinatorial argument. \hfill $\square$
6 Discussion  We have shown that equation (1) exhibits $\sum_{p=1}^{\infty} N_p/p$ asymptotically stable periodic orbits, so in particular, if $k$ is large enough, this number will be huge! (See Table 1 for some values of $N_p$ for different $p$). This type of behavior was already observed by Wu and Zhang [10] but only when the nonlinearity $f$ is a suitable generalization of the sign function (See Table 2 for the corresponding values by Wu and Zhang). The key difference is that in Wu and Zhang’s paper, states are characterized by two symbols, rather than 3. This leads to the question of whether these results can be generalized to the case where the nonlinearity is (some adequate generalization) of a finite valued step function. We believe the results obtained in this paper can be generalized to the case when the nonlinearity is a step function which takes on an odd number of values $2M + 1$ for $M \geq 1$. More precisely, when the nonlinear feedback has the form

$$f(x) = \begin{cases} 0 & \text{if } |x| \leq \epsilon \\ 1 & \text{if } \epsilon < x < \epsilon_1 \\ n & \text{if } \epsilon_{n-1} \leq x < \epsilon_n; \ 2 \leq n \leq M - 1 \\ M & \text{if } x \geq \epsilon_{M-1} \\ -1 & \text{if } -\epsilon_1 < x < -\epsilon \\ -n & \text{if } -\epsilon_n < x \leq -\epsilon_{n-1}; \ 2 \leq n \leq M - 1 \\ -M - 1 & \text{if } x \leq -\epsilon_{M-1}. \end{cases}$$

In this context, states are characterized by $2M + 1$ symbols. Note that what we have done in this paper corresponds to $M = 1$, so that $2M + 1 = 3$. The details are left for a future study.

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TABLE 1: Values of $N_p$ and $N_p/p$ for different $p$ computed using Lemma 1. (Note that these values are independent of $k$).
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TABLE 2: Values of $p$-periodic solutions $N^*_p$, and $p$-periodic orbits $N^*_p/p$, of equation (1) for different $p$ observed by Wu and Zhang [2004]. (These values are independent of $k$).

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