ABSTRACT. This article addresses the motion of a projectile that is launched from the top of a tower and lands on a given surface in space. The goal is to determine explicit and manageable formulas for the direction of launch in space that allows the projectile to travel as far as possible. While the classical critical point method fails to produce a solution formula even in the special case of shooting to a slanted plane, here we develop a different general approach that leads to remarkably simple equations and solution formulas. Our method is in the spirit of Lagrange multipliers and tailored to motion in three dimensions. Essentially, the optimization problem is reduced to the solution of three equations of geometric type. One of these conditions just means that the Jacobian determinant for the position function of the projectile has to vanish at an optimal solution. It turns out that the theory works well even in the more general setting of launch from a moving vehicle. Applications are given to the case of motion without air resistance and also to the case when the retarding force is proportional to the velocity of the projectile. In particular, surprisingly simple explicit solution formulas are derived for the case of shooting to a slanted plane.

1 Introduction and motivation

As an introductory example, we consider the following classical model for the motion of a projectile in space. Given a tower of height \( h > 0 \), we suppose that the projectile is launched at time \( t = 0 \) with initial position \((0, 0, h)\), muzzle speed \( s > 0 \), angle of inclination \( \theta \) with respect to the plane \( z = h \), and polar angle \( \varphi \) with respect to the \( xy \)-plane. Ignoring, for now, air resistance and other possible forces, we assume that the motion of the projectile is governed only by constant acceleration of magnitude \( g > 0 \) due to gravity in the direction of the negative \( z \)-axis. The position vector \((x(t), y(t), z(t))\) of the projectile at time \( t \geq 0 \) is then provided by the standard parametric
equations
\[ x(t) = s \cos \theta \cos \varphi \cdot t, \quad y(t) = s \cos \theta \sin \varphi \cdot t, \quad z(t) = h + s \sin \theta \cdot t - \frac{1}{2}gt^2, \]
valid for all \( t \geq 0 \) for which the projectile stays in the air until it hits a surface represented here by the graph of a function \( z = f(x, y) \) or, more generally, a level surface of the form \( \psi(x, y, z) = 0 \). To emphasize the dependence on the two launch angles \( \theta \) and \( \varphi \), we write \( \mathbf{r}(t, \theta, \varphi) = (x(t, \theta, \varphi), y(t, \theta, \varphi), z(t, \theta, \varphi)) \) for the position of the projectile at time \( t \) when launched with \( \theta \) and \( \varphi \).

Now suppose that the task is to find angles \( \theta \) and \( \varphi \) that allow the projectile to travel as far as possible in a given direction that, after a suitable choice of the coordinate system, may be taken to be the direction of the positive \( x \)-axis. Thus we are interested in the maximization of \( x(t, \theta, \varphi) \) subject to the constraint \( \mathbf{r}(t, \theta, \varphi) = 0 \). Of course, when \( \psi(x, y, z) = z \), and also in other simple cases, it is obvious that the optimal launch occurs for \( \varphi = 0 \), which reduces the setting to motion in the \( xz \)-plane and the task to finding only the optimal angle of elevation \( \theta \). However, in general, both \( \theta \) and \( \varphi \) need to be determined.

A typical example is the case of shooting to a slanted plane with an equation of the form \( z = x + y \) for given real constants \( \beta \) and \( \gamma \). A natural approach to the corresponding optimization problem is to first solve the impact condition
\[ z(t, \theta, \varphi) = \beta x(t, \theta, \varphi) + \gamma y(t, \theta, \varphi) \]
for \( t > 0 \) in terms of \( \theta \) and \( \varphi \). An elementary solution of this quadratic equation results in an explicit formula for the unique impact time \( t = t(\theta, \varphi) > 0 \) that also depends on the five parameters \( g, h, s, \beta \) and \( \gamma \). We are thus led to the problem of maximizing
\[ d(\theta, \varphi) = x(t(\theta, \varphi), \theta, \varphi). \]
Unfortunately, the critical point equations \( d_\theta(\theta, \varphi) = 0 \) and \( d_\varphi(\theta, \varphi) = 0 \) for this unconstrained optimization problem are much too hard for a paper-and-pencil approach and even for computer algebra systems: it turns out that both Mathematica and Maple run out of memory when trying to solve this pair of equations.

In fact, as already observed in \([5, 6, 7]\), the critical point approach is unsatisfactory even for the two-dimensional version of this optimization problem where there is no dependence on the polar angle \( \varphi \). More
precisely, if the goal is to maximize $x(t, \theta, 0)$ subject to the constraint $z(t, \theta, 0) = \beta x(t, \theta, 0)$, then computer algebra systems do provide solutions to the corresponding critical point equation in this one-variable special case. For example, after suitable simplification, version 6.0 of Mathematica yields eight solutions of the critical point equation for $\theta$, namely,

$$\pm \arccos \left( \pm \sqrt{\frac{2gh^2 + 3ghs^2 + (1 + \beta^2) s^4 \pm \sqrt{\beta^2 s^6 (2gh + (1 + \beta^2) s^2)}}{2(g^2h^2 + 2ghs^2 + (1 + \beta^2) s^4)}} \right)$$

together with the disclaimer that some solutions may not have been found. However, such a solution formula for the optimal angle $\theta$ is extremely complicated, and the unspecified $\pm$ signs make it almost impossible to understand how each of the parameters $g, h, s$, and $\beta$ affects $\theta$ when all the other parameters are kept constant. For instance, one might expect $\theta$ to be a decreasing function of $h$, but this is hard to infer from (1).

In this article, we develop a different approach to optimization problems of this type for general projectile motion in space. Our method is in the spirit of Lagrange multipliers and may be viewed as an extension of the two-dimensional theory from our recent article [5]. For classical accounts, we refer to [1, 2, 9].

In order to be able to handle a variety of examples, we consider an arbitrary function $\rho(x, y, z)$ that plays the role of a generalized distance function. Typical choices for $\rho(x, y, z)$ are $x, z, x^2 + y^2, x^2 + y^2 + z^2$ and $x^2 + y^2 + (z - h)^2$, as well as the square roots of the last three expressions.

We are then interested in the optimization of $\rho(r(t, \theta, \varphi))$ subject to the constraint $\psi(r(t, \theta, \varphi)) = 0$, where $r(t, \theta, \varphi)$ represents the motion of an arbitrary projectile, thus including, for example, the possibility of air resistance. From the Lagrange multiplier theorem we derive, in Theorem 2, two necessary conditions for the solutions of this optimization problem. One of these conditions simply means that the Jacobian determinant for the position function $r$ has to vanish at an optimal solution $(t, \theta, \varphi)$. Not only for motion without air resistance, but also for the case when air resistance is proportional to the velocity of the projectile, the Jacobian condition leads to a remarkably simple equation with a striking geometric interpretation. The other condition involves certain cross and dot products for the gradients of $\rho, \psi$, and $r$ and is again of geometric type.
Our applications will illustrate that, in many cases, these two conditions allow us to express both the optimal polar angle \( \varphi \) and the impact time \( t \) for optimal launch in terms of the optimal angle of elevation \( \theta \). The optimization problem is therefore reduced to the solution of an impact equation for the one remaining variable \( \theta \). In fact, our approach extends to a more general setting where the projectile is launched from a moving vehicle such as a car or a hot air balloon.

In particular, we obtain, in Theorem 3, amazingly simple explicit solution formulas for the above-mentioned case of launching to a slanted plane, where the critical point approach fails miserably. This will also shed new light on the solution formula (1) for the two-dimensional case. Moreover, our approach leads to a surprisingly short expression for the maximal distance that the projectile is able to travel.

2 General principles

Although the emphasis of this article is on optimization in three dimensions, we begin with a brief discussion of the conceptually much simpler case of two dimensions. This case will serve as a motivation for our approach in three dimensions and is, of course, of independent interest.

In the following result, we consider open subsets \( \Gamma \) and \( \Omega \) of \( \mathbb{R}^2 \) and continuously differentiable real-valued functions \( \rho \) and \( \psi \) on \( \Omega \). Also, let \( r : \Gamma \to \Omega \) be a transformation from \( \Gamma \) into \( \Omega \) with continuously differentiable component functions \( x \) and \( y \) on \( \Gamma \), so that \( r(t, \theta) = (x(t, \theta), y(t, \theta)) \) for all \((t, \theta) \in \Gamma\).

In this general setting, we are interested in the optimization of \( \rho(r(t, \theta)) \) over all points \((t, \theta) \in \Gamma\) that satisfy the condition \( \psi(r(t, \theta)) = 0 \). Such an optimization problem may be addressed by the method of Lagrange multipliers. Here our goal is to rewrite the classical Lagrange equations for the compositions \( \rho \circ r \) and \( \psi \circ r \) directly in terms of certain Jacobian determinants for the single functions \( \rho, \psi, \) and \( r \).

**Theorem 1.** Suppose that the point \((t, \theta) \in \Gamma\) provides a local extremum of the composition \( \rho \circ r \) subject to the constraint \( \psi(r(t, \theta)) = 0 \). Then

\[
\begin{bmatrix}
\rho_x(r(t, \theta)) & \rho_y(r(t, \theta)) \\
\psi_x(r(t, \theta)) & \psi_y(r(t, \theta))
\end{bmatrix}
\begin{bmatrix}
x_t(t, \theta) \\
y_t(t, \theta)
\end{bmatrix}
= 0 \quad \text{or} \quad
\begin{bmatrix}
x(t, \theta) \\
y(t, \theta)
\end{bmatrix}
= 0.
\]

**Proof.** An application of the Lagrange multiplier theorem to the compositions \( u = \rho \circ r \) and \( v = \psi \circ r \) on \( \Gamma \) shows that either \( \nabla v(t, \theta) \) is the zero vector or that \( \nabla u(t, \theta) = \lambda \nabla v(t, \theta) \) for some \( \lambda \in \mathbb{R} \). This means
that the list of vectors \((\nabla u(t, \theta), \nabla v(t, \theta))\) is linearly dependent and thus satisfies
\[
\begin{bmatrix}
u_x(t, \theta) & \nu_y(t, \theta) \\
v_z(t, \theta) & \nu_y(t, \theta)
\end{bmatrix} = 0.
\]
On the other hand, the multivariate chain rule guarantees that
\[
\begin{bmatrix}
u_x(t, \theta) & \nu_y(t, \theta) \\
v_z(t, \theta) & \nu_y(t, \theta)
\end{bmatrix} = \begin{bmatrix}
\rho_x(r(t, \theta)) & \rho_y(r(t, \theta)) \\
\psi_x(r(t, \theta)) & \psi_y(r(t, \theta))
\end{bmatrix} \begin{bmatrix}
x_x(t, \theta) & x_y(t, \theta) \\
y_x(t, \theta) & y_y(t, \theta)
\end{bmatrix}.
\]
Since the determinant of the product of two matrices is equal to the product of their determinants, we conclude that the determinant of at least one of the last two matrices is zero, as desired.

Of particular interest is the case when the constraint function \(\psi\) is induced by the graph of a function of one variable, in the sense that \(\psi(x, y) = y - f(x)\) for some continuously differentiable function \(f\). In this case, for arbitrary \((t, \theta)\) for which \(y(t, \theta) = f(x(t, \theta))\), we have
\[
\begin{bmatrix}
\rho_x(r(t, \theta)) & \rho_y(r(t, \theta)) \\
\psi_x(r(t, \theta)) & \psi_y(r(t, \theta))
\end{bmatrix} = \rho_x(r(t, \theta)) + \rho_y(r(t, \theta))f'(x(t, \theta)) = \sigma'(x(t, \theta)),
\]
where the function \(\sigma\) is given by \(\sigma(x) = \rho(x, f(x))\). Thus, in the setting of Theorem 1, we conclude that either \(x(t, \theta)\) is a critical point of \(\sigma\) or that
\[
x_x(t, \theta)y_y(t, \theta) = x_y(t, \theta)y_x(t, \theta).
\]
This special case of Theorem 1 was recently obtained in Theorem 1 of [5] with a somewhat different approach. The result turned out to be quite useful for the optimization theory of projectile motion in two dimensions. In particular, Section 4.2 of [5] provides an example to illustrate that it is essential to include the critical points of \(\sigma\) in this context. This shows that, in general, the first alternative in the conclusion of Theorem 1 cannot be excluded.

To establish a certain counterpart of the preceding result in three dimensions, we now consider a pair of open subsets \(\Gamma\) and \(\Omega\) of \(\mathbb{R}^3\). Also, let \(\rho\) and \(\psi\) denote continuously differentiable real-valued functions on \(\Omega\), and let \(r : \Gamma \to \Omega\) be a transformation from \(\Gamma\) into \(\Omega\) with continuously differentiable component functions \(x\), \(y\), and \(z\) on \(\Gamma\), so that
\[
r(t, \theta, \varphi) = \langle x(t, \theta, \varphi), y(t, \theta, \varphi), z(t, \theta, \varphi) \rangle
\]
for all \((t, \theta, \varphi) \in \Gamma\). As usual,

\[
J = \begin{vmatrix} x_t & x_\theta & x_\varphi \\ y_t & y_\theta & y_\varphi \\ z_t & z_\theta & z_\varphi \end{vmatrix}
\]

stands for the Jacobian determinant of the transformation \(r\). In this setting, we obtain the following main general result of this article.

**Theorem 2.** Suppose that the point \((t, \theta, \varphi) \in \Gamma\) provides a local extremum of the composition \(\rho \circ r\) subject to the constraint \(\psi \circ r(t, \theta, \varphi) = 0\). Then the identities

\[
\begin{align*}
(r_t \times r_\theta) \cdot [\nabla \rho \times \nabla \psi] \circ r &= 0, \\
(r_\theta \times r_\varphi) \cdot [\nabla \rho \times \nabla \psi] \circ r &= 0, \\
(r_\varphi \times r_t) \cdot [\nabla \rho \times \nabla \psi] \circ r &= 0,
\end{align*}
\]

hold at the point \((t, \theta, \varphi)\). Moreover, we have either

\[
(\nabla \rho \times \nabla \psi) \circ r(t, \theta, \varphi) = 0 \quad \text{or} \quad J(t, \theta, \varphi) = 0.
\]

Finally, if \((r_\theta \times r_\varphi)\) \((t, \theta, \varphi)\) is nonzero, then the three equations in (2) are satisfied at the point \((t, \theta, \varphi)\) precisely when either

\[
(\nabla \rho \times \nabla \psi) \circ r(t, \theta, \varphi) = 0
\]

or the two identities

\[
J = 0 \quad \text{and} \quad (r_\theta \times r_\varphi) \cdot [\nabla \rho \times \nabla \psi] \circ r = 0
\]

hold at the point \((t, \theta, \varphi)\). A similar characterization obtains when either \((r_t \times r_\theta)\) \((t, \theta, \varphi)\) or \((r_\varphi \times r_t)\) \((t, \theta, \varphi)\) is nonzero.

The proof of this theorem will be given in the next section. As illustrated by Theorems 3 and 5 below and also by the additional examples at the end of Section 4, it turns out that, in many important cases, both cross products \(r_\theta \times r_\varphi\) and \(\nabla \rho \times \nabla \psi\) are nonzero at all points of interest. To maximize \(\rho \circ r(t, \theta, \varphi)\) subject to the constraint \(\psi \circ r(t, \theta, \varphi) = 0\), it then suffices, by Theorem 2, to solve the three equations

\[
\begin{align*}
J(t, \theta, \varphi) &= 0, \\
(r_\theta \times r_\varphi) \circ r(t, \theta, \varphi) \cdot (\nabla \rho \times \nabla \psi) \circ r(t, \theta, \varphi) &= 0, \\
\psi \circ r(t, \theta, \varphi) &= 0,
\end{align*}
\]
for \((t, \theta, \varphi) \in \Gamma\). In our applications to optimization problems for the motion of projectiles, we will see that the first and second of these equations are amazingly simple, not only for motion without air resistance, but also for the case when air resistance is proportional to the velocity of the projectile. This will allow us to eliminate both \(t\) and \(\varphi\) and thus to reduce the three-dimensional constrained optimization problem to the solution of just one impact equation for the single variable \(\theta\).

As in the case of two dimensions, of special interest is the case of a constraint function of the form 
\[
\psi(x, y, z) = z - f(x, y)
\]
for some continuously differentiable function \(f\). If \((t, \theta, \varphi)\) is a point that satisfies both 
\[
z(t, \theta, \varphi) = f(x(t, \theta, \varphi), y(t, \theta, \varphi))\]
and \((\nabla \rho \times \nabla \psi)(x(t, \theta, \varphi)) = 0\), then it is easily seen that \((x(t, \theta, \varphi), y(t, \theta, \varphi))\) is a critical point of the function \(\sigma\) given by 
\[
\sigma(x, y) = \rho(x, y, f(x, y)).
\]
Thus, in the setting of Theorem 2, we obtain that either \((x(t, \theta, \varphi), y(t, \theta, \varphi))\) is a critical point of \(\sigma\) or that \(J(t, \theta, \varphi) = 0\).

3 Proof of the main result

We first collect a few tools from linear algebra. Throughout this discussion, let \(p\) and \(q\) be two column vectors in \(\mathbb{R}^3\), and let \(A\) denote a real \(3 \times 3\) matrix with row vectors \(a, b,\) and \(c\). We will repeatedly use the fact that the cross product and the dot product in \(\mathbb{R}^3\) are related by the formula
\[
(a \times b) \cdot (p \times q) = \begin{vmatrix} a \cdot p & b \cdot p \\ a \cdot q & b \cdot q \end{vmatrix},
\]
which may be verified by a routine computation. Moreover, the list \((p, q)\) is linearly dependent precisely when \(p \times q\) is the zero vector. Here we are interested in certain geometric conditions on \(A, p,\) and \(q\) that characterize the linear dependence of the list \((Ap, Aq)\).

**Lemma 1.** The list \((Ap, Aq)\) is linearly dependent precisely when the following conditions are fulfilled:

\[
\begin{align*}
(a \times b) \cdot (p \times q) &= 0, \\
(b \times c) \cdot (p \times q) &= 0, \\
(c \times a) \cdot (p \times q) &= 0.
\end{align*}
\]

**Proof.** Suppose first that \((Ap, Aq)\) is linearly dependent. If both \(Ap\) and \(Aq\) are nonzero, then there exists a nonzero scalar \(\lambda \in \mathbb{R}\) for which
Aq = λAp and therefore,
\[ a \cdot q = \lambda a \cdot p, \quad b \cdot q = \lambda b \cdot p \quad \text{and} \quad c \cdot q = \lambda c \cdot p. \]

Pairwise cross-multiplication of these three equations followed by cancellation of \( \lambda \) yields the system:

\[
\begin{aligned}
(a \cdot p)(b \cdot q) &= (a \cdot q)(b \cdot p), \\
(b \cdot p)(c \cdot q) &= (b \cdot q)(c \cdot p), \\
(c \cdot p)(a \cdot q) &= (c \cdot q)(a \cdot p).
\end{aligned}
\]

Evidently, the preceding identities remain valid if one of the vectors \( Ap \) or \( Aq \) is the zero vector, since this happens precisely when \( a \cdot p = b \cdot p = c \cdot p = 0 \) or \( a \cdot q = b \cdot q = c \cdot q = 0 \). Consequently, in either case, formula (4) allows us to conclude from (6) that the three identities in (5) are all satisfied.

Conversely, suppose that (5) holds. Then, again by (4), we infer that (6) is fulfilled. To show that \((Ap, Aq)\) is linearly dependent, it suffices to consider the case when \( Ap \) is nonzero. Without loss of generality, we may assume that \( a \cdot p \neq 0 \). With the choice \( \lambda = (a \cdot q) / (a \cdot p) \) we conclude from (6) that \( Aq = \lambda Ap \), as desired.

If the list \((p, q)\) happens to be linearly dependent or if the span of the list \((a, b, c)\) is at most one dimensional, then it is easily seen that all the conditions of Lemma 1 are fulfilled. Hence, the result is of real interest only when \((p, q)\) is linearly independent and the dimension of the span of \((a, b, c)\) is at least two. In the latter case, we obtain a useful characterization that involves the determinant of \( A \):

**Lemma 2.** If the list \((Ap, Aq)\) is linearly dependent, then either
\[
p \times q = 0 \quad \text{or} \quad |A| = 0.
\]

Moreover, if \( a \times b \) is not the zero vector, then the list \((Ap, Aq)\) is linearly dependent if and only if either \( p \times q = 0 \) or we have both
\[ |A| = 0 \quad \text{and} \quad (a \times b) \cdot (p \times q) = 0. \]

**Proof.** Suppose first that the list \((Ap, Aq)\) is linearly dependent. If \( p \times q \) is nonzero, then the list \((p, q)\) is linearly independent, and it follows that
A fails to be invertible. Thus $|A| = 0$, and the identity $(a \times b) \cdot (p \times q) = 0$ is immediate from Lemma 1.

Conversely, if $p \times q = 0$, then the list $(p, q)$ is linearly dependent, and therefore so is $(Ap, Aq)$. On the other hand, if $|A| = 0$, then the list $(a, b, c)$ is linearly dependent. But the list $(a, b)$ is linearly independent provided that $a \times b$ is nonzero. Consequently, in this case, we obtain the representation $c = \lambda a + \mu b$ for suitable $\lambda, \mu \in \mathbb{R}$. Hence, the identity $(a \times b) \cdot (p \times q) = 0$ implies that

$$(b \times c) \cdot (p \times q) = \lambda(b \times a) \cdot (p \times q) + \mu(b \times b) \cdot (p \times q) = 0$$

and similarly $(c \times a) \cdot (p \times q) = 0$. Again by Lemma 1, it follows that the list $(Ap, Aq)$ is linearly dependent. \hfill \square

Obviously, if either $(b, c)$ or $(c, a)$ is linearly independent, then Lemma 2 remains valid once its last identity is replaced by $(b \times c) \cdot (p \times q) = 0$ or $(c \times a) \cdot (p \times q) = 0$, respectively. To compare the determinant condition $|A| = 0$ with the system of equations in (5), we note that $|A| = 0$ precisely when

$$(a \times b) \cdot c = (b \times c) \cdot a = (c \times a) \cdot b = 0.$$ 

This observation is immediate from the definition of the determinant.

**Proof of Theorem 2.** Consider the compositions $u = \rho \circ r$ and $v = \psi \circ r$ on $\Gamma$. Since $(t, \theta, \varphi)$ provides a local extremum for $u$ subject to the constraint $v(t, \theta, \varphi) = 0$, we conclude from the Lagrange multiplier theorem that either $\nabla v(t, \theta, \varphi)$ is the zero vector or that $\nabla u(t, \theta, \varphi) = \lambda \nabla v(t, \theta, \varphi)$ for some $\lambda \in \mathbb{R}$. This means precisely that the list $(\nabla u(t, \theta, \varphi), \nabla v(t, \theta, \varphi))$ is linearly dependent. We now introduce the two column vectors

$$p = \nabla \rho (r(t, \theta, \varphi)) \quad \text{and} \quad q = \nabla \psi (r(t, \theta, \varphi))$$

in $\mathbb{R}^3$ as well as the matrix

$$A = \begin{bmatrix} x_t(t, \theta, \varphi) & y_t(t, \theta, \varphi) & z_t(t, \theta, \varphi) \\ x_\theta(t, \theta, \varphi) & y_\theta(t, \theta, \varphi) & z_\theta(t, \theta, \varphi) \\ x_\varphi(t, \theta, \varphi) & y_\varphi(t, \theta, \varphi) & z_\varphi(t, \theta, \varphi) \end{bmatrix} = \begin{bmatrix} r_t(t, \theta, \varphi) \\ r_\theta(t, \theta, \varphi) \\ r_\varphi(t, \theta, \varphi) \end{bmatrix}.$$ 

Since the multivariate chain rule ensures that $Ap = \nabla u(t, \theta, \varphi)$ and $Aq = \nabla v(t, \theta, \varphi)$, we infer that the list $(Ap, Aq)$ is linearly dependent. The assertions are then immediate from Lemmas 1 and 2. \hfill \square
Projectile motion without air resistance

Throughout this section, we consider the motion of a projectile with parametric equations

\[
\begin{align*}
    x(t, \theta, \varphi) &= (u + s \cos \theta \cos \varphi)t, \\
    y(t, \theta, \varphi) &= (v + s \cos \theta \sin \varphi)t, \\
    z(t, \theta, \varphi) &= h + (w + s \sin \theta)t - \frac{1}{2}gt^2,
\end{align*}
\]

where \( g, h, s > 0 \) and \( u, v, w \in \mathbb{R} \) are given constants. An obvious interpretation of the additional parameters \( u, v \) and \( w \) is that here the projectile is launched at time \( t = 0 \) from a vehicle that moves with velocity vector \( (u, v, w) \) and is at the location \( (0, 0, h) \) at time \( t = 0 \).

To apply Theorem 2 to this case, we need to compute the corresponding Jacobian determinant \( J \) and the cross products involving \( r_t \), \( r_\theta \), and \( r_\varphi \) at an arbitrary point \((t, \theta, \varphi)\) in \( \mathbb{R}^3 \). All this can easily be done with the aid of a computer algebra system, but computations using only paper and pencil also work nicely in this setting.

First, the Jacobian determinant for (7) is given by

\[
J(t, \theta, \varphi) = -s^2t^2 \cos \theta(s + u \cos \theta \cos \varphi + v \cos \theta \sin \varphi + w \sin \theta - gt \sin \theta).
\]

With the notation

\[
\delta(\theta, \varphi) = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, \sin \theta)
\]

for the unit vector in \( \mathbb{R}^3 \) that provides the direction given by the angles \( \theta \) and \( \varphi \), we obtain the following characterization of the Jacobian condition.

**Proposition 3.** Let \( t > 0 \), \( -\pi/2 < \theta < \pi/2 \), and \( -\pi \leq \varphi \leq \pi \). Then, in the setting of (7), the identity \( J(t, \theta, \varphi) = 0 \) holds if and only if

\[
tg \sin \theta = s + u \cos \theta \cos \varphi + v \cos \theta \sin \varphi + w \sin \theta,
\]

and this happens precisely when the vectors \( \delta(\theta, \varphi) \) and \( r_t(t, \theta, \varphi) \) are perpendicular. In particular, if \( u = v = w = 0 \), then \( J(t, \theta, \varphi) = 0 \) exactly when \( 0 < \theta < \pi/2 \) and

\[
t = \frac{s}{g \sin \theta},
\]

and this happens if and only if the tangent lines to the trajectory of the projectile at the positions \( (0, 0, h) \) and \( r(t, \theta, \varphi) \) are perpendicular.
Proof. Since (7) entails that \( \mathbf{r}_t(t, \theta, \varphi) = s \mathbf{δ}(\theta, \varphi) + \langle u, v, w - gt \rangle \), we obtain

\[ \mathbf{r}_t(t, \theta, \varphi) \cdot \mathbf{δ}(\theta, \varphi) = s + u \cos \theta \cos \varphi + v \cos \theta \sin \varphi + w \sin \theta - tg \sin \theta. \]

Hence, the result is immediate from the preceding formula for \( J(t, \theta, \varphi) \).

Thus, in applications of Theorem 2 to projectile motion governed by (7), the Jacobian condition allows us to eliminate \( t \) by a simple formula that, quite remarkably, is independent of both \( \rho \) and \( \psi \). The only exception occurs when

\[ \mathbf{r}_t(0, \theta, \varphi) \cdot \mathbf{δ}(\theta, \varphi) = s + u \cos \theta \cos \varphi + v \cos \theta \sin \varphi + w \sin \theta = 0. \]

Note that, by the Cauchy-Schwarz inequality, this possibility is precluded provided that \( |\langle u, v, w \rangle| < s \). Indeed, if \( \mathbf{r}_t(0, \theta, \varphi) \cdot \mathbf{δ}(\theta, \varphi) = 0 \), then \( \langle u, v, w \rangle \cdot \mathbf{δ}(\theta, \varphi) = -s \) and therefore, \( s = |\langle u, v, w \rangle \cdot \mathbf{δ}(\theta, \varphi)| \leq |\langle u, v, w \rangle| \).

For classical projectile motion in two dimensions, it was already observed earlier, for instance in [3], that, in the case of optimal launch to a horizontal line, the tangent lines to the trajectory of the projectile at the points of launch and impact have to be perpendicular. A more general result of this type was recently obtained in Theorem 4 of [6]. The archetypical case is, of course, given by shooting from \( h = 0 \) to ground level, where it has been known at least since Galileo that the optimal angle of elevation is \( \theta = \pi/4 \) and that the two tangent lines for this launch are perpendicular.

It turns out that the vector \( \mathbf{δ}(\theta, \varphi) \) continues to play a central role in this context. In fact, another simple computation shows that

\[ (\mathbf{r}_\theta \times \mathbf{r}_\varphi)(t, \theta, \varphi) = -s^2 t^2 \cos \theta \mathbf{δ}(\theta, \varphi). \]

In particular, \( (\mathbf{r}_\theta \times \mathbf{r}_\varphi)(t, \theta, \varphi) \) is nonzero whenever \( t > 0 \) and \( -\pi/2 < \theta < \pi/2 \). For the other two cross products, we obtain more complicated results, namely,

\[ (\mathbf{r}_t \times \mathbf{r}_\theta)(t, \theta, \varphi) = st \langle v \cos \theta + (s + (w - gt) \sin \theta) \sin \varphi, \]

\[ -u \cos \varphi - (s + (w - gt) \sin \theta) \cos \varphi, \]

\[ \sin \theta \langle v \cos \varphi - u \sin \varphi \rangle, \]
and similarly

\[(r_x \times r_y) (t, \theta, \varphi) = st \cos \theta (w - gt + s \sin \theta), \]
\[\sin \varphi (w - gt + s \sin \theta), \]
\[-s \cos \theta - u \cos \varphi - v \sin \varphi].\]

However, under the condition \(J(t, \theta, \varphi) = 0\), these formulas may be simplified to

\[(r_t \times r_y) (t, \theta, \varphi) = st (v \cos \varphi - u \sin \varphi) \delta(\theta, \varphi)\]

and

\[(r_x \times r_z) (t, \theta, \varphi) = -st \cot \theta (s \cos \theta + u \cos \varphi + v \sin \varphi) \delta(\theta, \varphi).\]

Thus, in the case \(J(t, \theta, \varphi) = 0\), the three cross products are all multiples of \(\delta(\theta, \varphi)\), but the best choice for our purpose is \(r_x \times r_y\). Incidentally, if \(u = v = 0\), then \(r_t \times r_y\) vanishes at all points \((t, \theta, \varphi)\) for which \(J(t, \theta, \varphi) = 0\) and thus is of no use to us.

In the following, we consider the natural domain for our optimization problems, namely the set \(D\) consisting of all \((t, \theta, \varphi)\) for which \(t \geq 0\), \(-\pi/2 \leq \theta \leq \pi/2\), and \(-\pi \leq \varphi \leq \pi\). Also, as one might expect from the dimensionless analysis in the classical theory of projectile motion, the quantity

\[\kappa = 1 + \frac{2gh}{s^2}\]

will be useful; see Section 1.8 of [1] for details.

**Theorem 3.** Consider the projectile motion given by (7) for the case \(u = v = w = 0\). Then, for arbitrary real constants \(\beta\) and \(\gamma\), there exists exactly one point \((t, \theta, \varphi)\) in \(D\) that maximizes \(x(t, \theta, \varphi)\) subject to the constraint

\[z(t, \theta, \varphi) = \beta x(t, \theta, \varphi) + \gamma y(t, \theta, \varphi),\]

namely the point given by

\[\theta = \arccot \sqrt{\kappa + 2\beta^2 + 2\gamma^2 - 2\beta \sqrt{\kappa + \beta^2 + \gamma^2}},\]
\[t = \frac{s}{g \sin \theta} = \frac{s}{g} \sqrt{1 + \kappa + 2\beta^2 + 2\gamma^2 - 2\beta \sqrt{\kappa + \beta^2 + \gamma^2}},\]
\[\varphi = \arcsin(-\gamma \tan \theta) = \arctan \left(\frac{\gamma}{\beta - \sqrt{\kappa + \beta^2 + \gamma^2}}\right).\]
Moreover, the maximal displacement is

\[ x(t, \theta, \varphi) = \frac{s^2}{g} (-\beta + \sqrt{\kappa + \beta^2 + \gamma^2}). \]

**Proof.** For given \( \theta \in [-\pi/2, \pi/2] \) and \( \varphi \in [-\pi, \pi] \), an elementary solution of the relevant quadratic equation shows that there exists exactly one time \( t = t(\theta, \varphi) > 0 \) for which the impact \( z(t, \theta, \varphi) = \beta x(t, \theta, \varphi) + \gamma y(t, \theta, \varphi) \) occurs. Moreover, it is easily verified that the distance function \( d \) given by \( d(\theta, \varphi) = x(t(\theta, \varphi), \theta, \varphi) \) is continuous on the compact rectangle \([-\pi/2, \pi/2] \times [-\pi, \pi]\). Thus, by the extreme value theorem, our optimization problem certainly has a solution. Hence, given an arbitrary point \((t, \theta, \varphi) \in D\) that maximizes \( x(t, \theta, \varphi) \) subject to the constraint under consideration, it remains to show that all the stipulated identities are fulfilled.

We apply Theorem 2 to the sets \( \Gamma = \Omega = \mathbb{R}^3 \) and the functions \( \rho \) and \( \psi \) given by \( \rho(x, y, z) = x \) and \( \psi(x, y, z) = z - \beta x - \gamma y \). Then \( \nabla \rho \times \nabla \psi = (0, -1, -\gamma) \) and

\[ (r_\theta \times r_\varphi) (t, \theta, \varphi) = -s^2 t^2 \cos \theta \delta(\theta, \varphi). \]

Since clearly \( t > 0 \) and \( -\pi/2 < \theta < \pi/2 \), we conclude that both cross products are nonzero. Thus Theorem 2 ensures that \((t, \theta, \varphi)\) satisfies the three conditions listed in (3). By Proposition 3, the Jacobian condition \( J(t, \theta, \varphi) = 0 \) yields \( \theta > 0 \) and \( t = s/(g \sin \theta) = (s/g) \csc \theta \). Moreover, the cross product condition

\[ (r_\theta \times r_\varphi) (t, \theta, \varphi) \cdot (\nabla \rho \times \nabla \psi) (r(t, \theta, \varphi)) = 0 \]

implies that \( \delta(\theta, \varphi) \cdot (0, -1, -\gamma) = 0 \) and therefore, \( \sin \varphi = -\gamma \tan \theta \).

Also, since \( x(t, \theta, \varphi) = st \cos \theta \cos \varphi > 0 \), we obtain \( \cos \varphi > 0 \) and hence,

\[ \cos \varphi = \sqrt{1 - \sin^2 \varphi} = \sqrt{1 - \gamma^2 \tan^2 \theta}. \]

Elimination of \( t \) and \( \varphi \) in the impact condition \( \psi (r(t, \theta, \varphi)) = 0 \) now leads to

\[ h + \frac{s^2}{g} - \frac{s^2}{2g} \csc^2 \theta = \frac{s^2}{g} \cot \theta (\beta \cos \varphi + \gamma \sin \varphi) \]

and consequently,

\[ \frac{2gh}{s^2} + 2 - \csc^2 \theta = 2 \beta \cot \theta \sqrt{1 - \gamma^2 \tan^2 \theta - 2\gamma^2}. \]
It may be tempting to employ a computer algebra system to solve the preceding equation for $\theta$, but, perhaps somewhat surprisingly, neither Mathematica nor Maple is helpful here, even when assisted by a suitable substitution. Nevertheless, the desired formula for $\theta$ may be found in a few steps. Indeed, using the definition of $\kappa$ and the fact that $\csc^2 \theta = 1 + \cot^2 \theta$, we may rewrite the last identity in the form

$$\kappa + 2\gamma^2 - \cot^2 \theta = 2\beta \sqrt{\cot^2 \theta - \gamma^2}.$$ 

With the notation $c = \kappa + 2\gamma^2$, we conclude that $\mu = \cot^2 \theta$ satisfies the equation

$$c - \mu = 2\beta \sqrt{\mu - \gamma^2},$$

hence $c^2 - 2c\mu + \mu^2 = 4\beta^2 (\mu - \gamma^2)$, and therefore,

$$\mu^2 - 2(c + 2\beta^2)\mu + c^2 + 4\beta^2\gamma^2 = 0.$$ 

The solutions of this quadratic equation for $\mu$ are

$$\mu \pm = c + 2\beta^2 \pm \sqrt{(c + 2\beta^2)^2 - c^2 - 4\beta^2\gamma^2}$$

$$= c + 2\beta^2 \pm 2|\beta| \sqrt{c + \beta^2 - \gamma^2}.$$ 

A glance at these two formulas for $\mu \pm$ reveals that both solutions are real and, in fact, non-negative. Moreover, because

$$c + \beta^2 - \gamma^2 = \kappa + \beta^2 + \gamma^2 > \beta^2,$$

the last formula for $\mu \pm$ shows that $\mu_- < c < \mu_+$ whenever $\beta \neq 0$. Consequently, only one of these solutions satisfies the original equation (8). Indeed, if $\beta > 0$, then (8) ensures that $\mu \leq c$ and thus $\mu = \mu_-$, while the case $\beta < 0$ leads to $c \leq \mu$ and therefore $\mu = \mu_+$. We conclude that the identities

$$\mu = c + 2\beta^2 - 2\beta \sqrt{c + \beta^2 - \gamma^2} = \kappa + 2\beta^2 + 2\gamma^2 - 2\beta \sqrt{\kappa + \beta^2 + \gamma^2}$$

hold for arbitrary real $\beta$, which establishes the claim concerning $\theta$. Incidentally, we note that the delicate issue of the $\pm$ sign in this context is
probably the reason why computer algebra systems fail to produce the preceding formula for $\mu$. Moreover, the identity $\csc^2 \theta = 1 + \cot^2 \theta$ yields

$$\begin{align*}
t &= \frac{s}{g} \csc \theta = \frac{s}{g} \sqrt{1 + \cot^2 \theta} \\
&= \frac{s}{g} \sqrt{1 + \kappa + 2\beta^2 + 2\gamma^2 - 2\beta\sqrt{\kappa + \beta^2 + \gamma^2}},
\end{align*}$$

as desired. Finally, because

$$\cot^2 \theta - \gamma^2 = \kappa + \beta^2 + \gamma^2 - 2\beta\sqrt{\kappa + \beta^2 + \gamma^2} + \beta^2$$

$$= \left(-\beta + \sqrt{\kappa + \beta^2 + \gamma^2}\right)^2,$$

we arrive at

$$\tan \varphi = \frac{\sin \varphi}{\cos \varphi} = \frac{-\gamma \tan \theta}{\sqrt{1 - \gamma^2 \tan^2 \theta}} = \frac{-\gamma}{\sqrt{\cot^2 \theta - \gamma^2}},$$

and similarly

$$x(t, \theta, \varphi) = st \cos \theta \cos \varphi = \frac{s^2}{g} \cot \theta \sqrt{1 - \gamma^2 \tan^2 \theta}$$

$$= \frac{s^2}{g} \left(-\beta + \sqrt{\kappa + \beta^2 + \gamma^2}\right).$$

This establishes all the remaining claims.  

For the data $g = 9.81 \text{ m/sec}^2$, $h = 50 \text{ m}$, $s = 80 \text{ m/sec}$, $\beta = 0.2$, and $\gamma = 0.1$, the preceding result leads to the optimal angles $\theta_{\text{opt}} = 0.84$ and $\varphi_{\text{opt}} = -0.11$, while the maximal distance is 585.16 m. These results are confirmed by the following graphics for the distance as a function of the two variables $\theta$ and $\varphi$.  

It is interesting to note that, in the special case $\gamma = 0$, the formula for the optimal angle of elevation in Theorem 3 assumes the simple form

$$\theta = \arccot \left(-\beta + \sqrt{\kappa + \beta^2}\right).$$

In particular, we see that the solution formula (1) from the critical point approach to the two-dimensional counterpart of our optimization problem admits a stunning simplification. The preceding formula for $\theta$ was previously obtained in Section 4.1 of $[5]$, also based on Lagrange multipliers, while an alternative elementary proof was recently developed in Theorem 2 of $[6]$.

In the context of Theorem 3, it is natural to wonder what happens when $\mathbf{h} = (u, v, w)$ is not the zero vector. After all, also in this case, Proposition 3 allows us to eliminate $t$, while the condition on the cross products leads, exactly as in the proof of Theorem 3, to the simple formula $\sin \varphi = -\gamma \tan \theta$. Thus, as above, Theorem 2 reduces this optimization problem to the solution of a certain impact equation for $\theta$, but it turns out that, in general, this equation provides a formidable challenge. In fact, computer algebra systems run out of memory when attempting to solve this equation.

In the following result, we restrict ourselves to a certain manageable special case. In particular, here we assume that $h = 0$, which requires a small adjustment of the natural domain for this optimization problem. Given $s, v, w,$ and $\gamma$, let $D_s$ consist of all $(t, \theta, \varphi) \in D$ that satisfy the condition $w + s \sin \theta \geq \gamma (v + s \cos \theta \sin \varphi)$.

**Theorem 4.** Consider the projectile motion given by (7) for the case $h = 0, u = 0,$ and $v, w \in \mathbb{R}$. Then, for arbitrary real $\gamma$ for which $w + s >$
\(\gamma v\), there exists exactly one point \((t, \theta, \varphi) \in D_*\) that maximizes \(x(t, \theta, \varphi)\) subject to the constraint
\[
z(t, \theta, \varphi) = \gamma y(t, \theta, \varphi),
\]

namely the point given by
\[
\theta = \arcsin \left( \frac{\gamma v - w + \sqrt{(\gamma v - w)^2 + 8s^2(1 + \gamma^2)}}{4s(1 + \gamma^2)} \right),
\]
\[
\varphi = \arcsin(-\gamma \tan \theta) \quad \text{and} \quad t = \frac{s + v \cos \theta \sin \varphi + w \sin \theta}{g \sin \theta}.
\]

**Proof.** If the angles \(\theta \in [-\pi/2, \pi/2]\) and \(\varphi \in [-\pi, \pi]\) satisfy the condition
\[
w + s \sin \theta \geq \gamma (v + s \cos \theta \sin \varphi),
\]
then either the quadratic equation \(z(t, \theta, \varphi) = \gamma y(t, \theta, \varphi)\) has \(t = 0\) as a double root or there exists a unique time \(t = t(\theta, \varphi) > 0\) for which \(z(t, \theta, \varphi) = \gamma y(t, \theta, \varphi)\). Moreover, the inequality \(w + s > \gamma v\) guarantees that \(D_*\) is non-empty. As in the derivation of Theorem 3, a routine application of the extreme value theorem now ensures that the optimization problem at hand has a solution, so it remains to be seen that any solution \((t, \theta, \varphi)\) is of the desired form. This will follow from Theorem 2. Clearly, the condition \(w + s > \gamma v\) implies that \(t > 0\) and \(\theta \neq \pm \pi/2\). Moreover, we have \(\theta \geq 0\), since otherwise the angles \(|\theta|\) and \(\varphi\) would be admissible and satisfy
\[
z(t, |\theta|, \varphi) > z(t, \theta, \varphi) \quad \text{for all} \ \tau > 0,
\]

hence \(t(|\theta|, \varphi) > t(\theta, \varphi)\), and therefore \(x(t(|\theta|, \varphi), |\theta|, \varphi) > x(t, \theta, \varphi)\), which is impossible. Also, the first two conditions of \((3)\) lead to \(t y \sin \theta = s + v \cos \theta \sin \varphi + w \sin \theta\) and \(\sin \varphi = -\gamma \tan \theta\). Hence the impact condition \(z(t, \theta, \varphi) = \gamma y(t, \theta, \varphi)\) may be rewritten in the form
\[
2s(1 + \gamma^2) \sin^2 \theta + (w - \gamma v) \sin \theta - s = 0,
\]
as the patient reader will easily check. The solution of this quadratic equation for \(\sin \theta \geq 0\) is straightforward and leads to the desired conclusion. \(\square\)
We close this section with a few additional examples to illustrate the significance and power of the cross product condition

\[(r_\theta \times r_\varphi) (t, \theta, \varphi) \cdot (\nabla \rho \times \nabla \psi) (r(t, \theta, \varphi)) = 0,\]

here in the setting of (7) and for the case \( t > 0 \) and \( \theta \neq \pm \pi/2 \).

First, if \( \rho(x, y, z) = x^2 + y^2 \) and \( \psi(x, y, z) = z \), then \( \nabla \rho \times \nabla \psi = 2 \langle y, -x, 0 \rangle \), and (9) turns into the condition \( u \sin \varphi = v \cos \varphi \).

Also, if the goal is to shoot as high as possible to a certain cylindrical wall, then \( \rho(x, y, z) = z; \psi(x, y, z) = x^2 + y^2 - 1 \), and \( \nabla \rho \times \nabla \psi = 2 \langle -\gamma y, \beta x, 0 \rangle \), so that condition (9) assumes the form

\[\beta \sin \varphi (u + s \cos \theta \cos \varphi) = \gamma \cos \varphi (v + s \cos \theta \sin \varphi).\]

In the special case \( \beta = \gamma > 0 \), this simplifies again to \( u \sin \varphi = v \cos \varphi \).

Finally, for the choice \( \rho(x, y, z) = x^2 + y^2 + z^2 - 1 \), we obtain \( \nabla \rho \times \nabla \psi = 4z \langle y, -x, 0 \rangle \). Evidently, \( \rho(x, y, z) \) will be minimal for \( x = y = 0 \) and maximal for \( z = 0 \), but the condition \( z(t, \theta, \varphi) = 0 \) may not be feasible. For \( z(t, \theta, \varphi) \neq 0 \), condition (9) leads to \( u \sin \varphi = v \cos \varphi \).

5 Projectile motion with linear air resistance As in [1, 4, 5, 8] for the two-dimensional setting, we now turn to the case of projectile motion in space when the retarding force is proportional to the velocity vector with a negative constant factor. This leads to the second-order initial value problem

\[
x''(t) = -\alpha x'(t), \quad x'(0) = u + s \cos \theta \cos \varphi, \quad x(0) = 0,
\]

\[
y''(t) = -\alpha y'(t), \quad y'(0) = v + s \cos \theta \sin \varphi, \quad y(0) = 0,
\]

\[
z''(t) = -g - \alpha z'(t), \quad z'(0) = w + s \sin \theta, \quad z(0) = h,
\]

where \( \alpha > 0 \) is the air resistance coefficient. It is not hard to verify that the solution of this uncoupled system of ordinary differential equations is given by

\[
\begin{aligned}
x(t, \theta, \varphi) &= \frac{u + s \cos \theta \cos \varphi}{\alpha} (1 - e^{-\alpha t}), \\
y(t, \theta, \varphi) &= \frac{v + s \cos \theta \sin \varphi}{\alpha} (1 - e^{-\alpha t}), \\
z(t, \theta, \varphi) &= h - \frac{g}{\alpha} t + \frac{g + \alpha (w + s \sin \theta)}{\alpha^2} (1 - e^{-\alpha t}).
\end{aligned}
\]
For the corresponding Jacobian determinant $J(t, \theta, \varphi)$, a slightly tedious computation leads to

$$
- \frac{(1 - e^{-\alpha t})^2 s^2 \cos \theta}{e^{\alpha t} \alpha^3} (as + (g - e^{-\alpha t} \alpha w) \sin \theta + \alpha \cos \theta (u \cos \varphi + v \sin \varphi)).
$$

Complicated as this formula may be, we obtain the following counterpart of Proposition 3.

**Proposition 4.** Let $t > 0$, $-\pi/2 < \theta < \pi/2$, and $-\pi \leq \varphi \leq \pi$ be given, and suppose that $\mathbf{r}_t(0, \theta, \varphi) \cdot \delta(\theta, \varphi) \neq 0$. Then, in the setting of (10), the identity $J(t, \theta, \varphi) = 0$ holds if and only if $\theta \neq 0$ and

$$
t = \frac{1}{\alpha} \ln \left( 1 + \frac{\alpha (s + u \cos \theta \cos \varphi + v \cos \theta \sin \varphi + w \sin \theta)}{g \sin \theta} \right),
$$

and this happens exactly when the vectors $\delta(\theta, \varphi)$ and $\mathbf{r}_t(t, \theta, \varphi)$ are perpendicular. In particular, if $u = v = w = 0$, then $J(t, \theta, \varphi) = 0$ precisely when $0 < \theta < \pi/2$ and

$$
t = \frac{1}{\alpha} \ln \left( 1 + \frac{\alpha s}{g \sin \theta} \right),
$$

and this happens if and only if the tangent lines to the trajectory of the projectile at the positions $(0, 0, h)$ and $\mathbf{r}(t, \theta, \varphi)$ are perpendicular.

**Proof.** From (10) we conclude that

$$
\mathbf{r}_t(t, \theta, \varphi) = (s \delta(\theta, \varphi) + (u, v, w)) e^{-\alpha t} + \left( 0, 0, \frac{g}{\alpha} (e^{-\alpha t} - 1) \right).
$$

Taking the preceding formula for $J(t, \theta, \varphi)$ into account, we arrive at

$$
\left( - \frac{(1 - e^{-\alpha t})^2 s^2 \cos \theta}{\alpha^2} \right) \mathbf{r}_t(t, \theta, \varphi) \cdot \delta(\theta, \varphi) = J(t, \theta, \varphi).
$$

Thus, $J(t, \theta, \varphi) = 0$ if and only if $\delta(\theta, \varphi)$ and $\mathbf{r}_t(t, \theta, \varphi)$ are perpendicular. Moreover, another glance at the formula for $J(t, \theta, \varphi)$ reveals that $J(t, \theta, \varphi) = 0$ precisely when

$$
g \sin \theta (e^{\alpha t} - 1) = \alpha (s + u \cos \theta \cos \varphi + v \cos \theta \sin \varphi + w \sin \theta).$$
Since the condition \( r_t(0, \theta, \varphi) \cdot \delta(\theta, \varphi) \neq 0 \) ensures that the right-hand side of the preceding identity is nonzero, we infer that the condition \( J(t, \theta, \varphi) = 0 \) entails that \( \theta \) is nonzero. All the assertions are now immediate.

We mention in passing that the formulas of the preceding result converge to the corresponding expressions in Proposition 3 as \( \alpha \) approaches 0. This may be verified by a straightforward application of l’Hospital’s rule.

Moreover, it is not hard to see that

\[
(r_\theta \times r_\varphi)(t, \theta, \varphi) = \frac{(1 - e^{-\alpha t})^2 s^2 \cos \theta}{\alpha^2} \delta(\theta, \varphi).
\]

In particular, \((r_\theta \times r_\varphi)(t, \theta, \varphi)\) is nonzero whenever \( t > 0 \) and \( \theta \neq \pm \pi/2 \). As one might expect from the case of motion without air resistance, the formulas for \( r_t \times r_\theta \) and \( r_\varphi \times r_t \) are much more involved, but it turns out that at any point \((t, \theta, \varphi)\) for which \( J(t, \theta, \varphi) = 0 \) all three cross products are multiples of \( \delta(\theta, \varphi) \). We leave the verification to the interested reader.

Thus, somewhat surprisingly, it follows that the cross product condition (9) for the case of linear air resistance leads to the same formula as in the case without air resistance and therefore yields the same relationship between the optimal angles \( \theta \) and \( \varphi \). In particular, we obtain the following partial counterpart of Theorem 3.

**Theorem 5.** Consider the projectile motion given by (10) for the case \( u = v = w = 0 \), let \( \beta \) and \( \gamma \) be real constants, and suppose that \((t, \theta, \varphi)\) maximizes \( x(t, \theta, \varphi) \) subject to the constraint

\[
z(t, \theta, \varphi) = \beta x(t, \theta, \varphi) + \gamma y(t, \theta, \varphi).
\]

Then \( 0 < \theta < \pi/2 \), and we have

\[
t = \frac{1}{\alpha} \ln \left( 1 + \frac{\alpha s}{g \sin \theta} \right) \quad \text{and} \quad \varphi = \arcsin(-\gamma \tan \theta).
\]

Moreover, \( \theta \) satisfies the equation

\[
h = \frac{g}{\alpha^2} \ln \left( 1 + \frac{\alpha s}{g \sin \theta} \right)
\]

\[
+ \frac{sg + \alpha s^2 \left( (1 + \gamma^2) \sin \theta - \beta \sqrt{1 - (1 + \gamma^2) \sin^2 \theta} \right)}{\alpha (\alpha s + g \sin \theta)} = 0.
\]
Proof. This is another consequence of Theorem 2. Indeed, the assertions for \( t \) and \( \theta \) result from Proposition 4 and the cross product condition (9), while (11) is nothing but the impact condition evaluated at the indicated choice of \( t \) and \( \varphi \).

In the special case \( \beta = \gamma = 0 \), an explicit solution formula for (11) was provided in Theorem 5 of [5]. This formula involves the Lambert \( W \) function which, as explained in [8], is a powerful tool in this context. We do not know a solution formula for the general case, but, for specific data, it is not difficult to see that (11) has a unique solution which can then be determined numerically.

For instance, if \( g = 9.81 \text{ m/sec}^2 \), \( h = 50 \text{ m} \), \( s = 80 \text{ m/sec} \), \( \alpha = 0.2 \text{ sec}^{-1} \), \( \beta = 0.2 \), and \( \gamma = 0.1 \), then the left-hand side of (11) defines a strictly increasing function of the parameter \( p = \sin \theta \), and the \texttt{FindRoot} command of \textit{Mathematica} leads to the unique root \( p = 0.528 \). Thus Theorem 5 yields the optimal solution \( \theta = \arcsin p = 0.557 \), \( \varphi = -0.062 \), \( t = 7.04 \text{ sec} \), and \( x(t, \theta, \varphi) = 256 \text{ m} \).

The preceding graphics displays a few typical trajectories for this example as well as the optimal trajectory in the vertical plane induced by the polar angle \( \varphi \). Evidently, the tangent lines to this curve at the points of launch and impact are perpendicular, as predicted by Proposition 4.

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