PRICING OF VARIANCE AND VOLATILITY SWAPS WITH SEMI-MARKOV VOLATILITIES

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ABSTRACT. In this paper, we price variance (Theorem 1) and volatility (Theorem 2) swaps for stochastic volatilities driven by semi-Markov processes. We also discuss some extensions of the obtained results such as local semi-Markov volatility, Dupire formula for the local semi-Markov volatility and residual risk associated with the swap pricing.

1 Introduction We consider a semi-Markov modulated market consisting of a riskless asset or bond, $B$, and a risky asset or stock, $S$, whose dynamics depend on a semi-Markov process $x$. Using the martingale characterization of semi-Markov processes, we note the incompleteness of semi-Markov modulated markets and find the minimal martingale measure. We price variance (Theorem 1) and volatility (Theorem 2) swaps for stochastic volatilities driven by the semi-Markov processes. We also discuss some extensions of the obtained results such as local semi-Markov volatility, Dupire formula for the local semi-Markov volatility and residual risk associated with the swap pricing. We mention that stochastic volatility driven by semi-Markov process is a natural generalization of Markov and other regime-switching models discussed below (see, for example, Section 2.2, (iv) and (v), and references mentioned there).

The paper is organized as follows. The literature review and necessarily notions is presented in Section 2. Martingale characterization of semi-Markov processes is considered in Section 3. Minimal martingale measure for stock price with semi-Markov volatility is constructed in Section 4. Section 5 contains pricing of variance swap for stochastic volatility driven by a semi-Markov process. We present a closed form solution in this case. Example of variance swap for stochastic volatility

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driven by two-state continuous-time Markov chain is presented in Section 6. Section 7 is devoted to the pricing of volatility swap for stochastic volatility driven by a semi-Markov process. In Section 8, we consider some extensions of the obtained results as a plan for a future work: local/current semi-Markov volatility, Dupire formula for a semi-Markov local volatility, risk-minimizing strategies and residual risk associated with the swaps pricing.

2 Literature review and necessary notions

2.1 Types of volatilities Volatility $\sigma$ is the standard deviation of the change in value of a financial instrument with a specific horizon. It is often used to quantify the risk of the instrument over that time period. The higher volatility, the riskier the security.

Historical volatility is the volatility of a financial instrument based on historical returns. It’s a standard deviation (uses historical (daily, weekly, monthly, quarterly, yearly)) price data to empirically measure the volatility of a market or instrument in the past.

The annualized volatility $\sigma$ is the standard deviation of the instrument’s logarithmic returns in a year:

$$\sigma := \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (R_i - \bar{R})^2},$$

where $R_i := \ln \frac{S_{t_i}}{S_{t_{i-1}}}$, $\bar{R} := \frac{1}{n} \sum_{i=1}^{n} \ln \frac{S_{t_i}}{S_{t_{i-1}}}$, $S_{t_i}$ is an asset price at time $t_i, i = 1, 2, \ldots, n$.

Implied volatility is related to historical volatility, however the two are distinct. Historical volatility is a direct measure of the movement of the underlier’s price (realized volatility) over recent history. Implied volatility, in contrast, is set by the market price of the derivative contract itself, and not the underlier. Therefore, different derivative contracts on the same underlier have different implied volatilities. Most derivative markets exhibit persistent patterns of volatilities varying by strike. The pattern displays different characteristics for different markets. In some markets, those patterns form a smile curve. In others, such as equity index options markets, they form more of a skewed curve. This has motivated the name ‘volatility skew’. For markets where the graph is downward sloping, such as for equity options, the term volatility skew is often used. For other markets, such as FX options or equity index options, where the typical graph turns up at either end, the more familiar
term volatility smile is used. In practice, either the term volatility smile or volatility skew may be used to refer to the general phenomenon of volatilities varying by strike.

The models by Black and Scholes [2] (continuous-time (B,S)-security market) and Cox et al. [8] (discrete-time (B,S)-security market (binomial tree)) are unable to explain the negative skewness and leptokurtic (fat tail) commonly observed in the stock markets. The famous implied-volatility smile would not exist under their assumptions.

Given the prices of call or put options across all strikes and maturities, we may deduce the volatility which produces those prices via the full Black-Scholes equation (see [2, 11, 15]). This function has come to be known as local volatility. Local volatility-function of the spot price $S_t$ and time $t$: $\sigma(S_t, t)$ (see Dupire [15] formulae for local volatility).

Level-dependent volatility (e.g., CEV or Firm Model)-function of the spot price alone. To have a smile across strike price, we need $\sigma$ to depend on $S: \sigma \equiv \sigma(S_t)$. In this case, the volatility and stock price changes are now perfectly correlated.

2.2 Models for volatilities In the early 1970’s, Black and Scholes [2] made a major breakthrough by deriving pricing formulas for vanilla options written on the stock. The Black-Scholes model assumes that the volatility term is a constant. This assumption is not always satisfied by real-life options as the probability distribution of an equity has a fatter left tail and thinner right tail than the lognormal distribution (see [26]), and the assumption of constant volatility $\sigma$ in financial model (such as the original Black-Scholes model) is incompatible with derivatives prices observed in the market. Stochastic volatility models are used in the field of quantitative finance to evaluate derivative securities, such as options, swaps. By assuming that the volatility of the underlying price is a stochastic process rather than a constant, it becomes possible to more accurately model derivatives.

The above issues have been addressed and studied in several ways, such as:

(i) Volatility is assumed to be a deterministic function of the time: $\sigma \equiv \sigma(t)$ (see [42]); Merton [32] extended the term structure of volatility to $\sigma := \sigma_1$ (deterministic function of time), with the implied volatility for an option of maturity $T$ given by $\sigma_T^2 = \frac{1}{T} \int_0^T \sigma_u^2 \, du$;

(ii) Volatility is assumed to be a function of the time and the current level of the stock price $S(t)$: $\sigma \equiv \sigma(t, S(t))$ (see [26]); the dynamics of the stock price satisfies the following stochastic differential
equation:

\[ dS(t) = \mu S(t) dt + \sigma(t, S(t)) S(t) dW_1(t), \]

where \( W_1(t) \) is a standard Wiener process;

(iii) The time variation of the volatility involves an additional source of randomness, besides \( W_1(t) \), represented by \( W_2(t) \), and is given by

\[ d\sigma(t) = a(t, \sigma(t)) dt + b(t, \sigma(t)) dW_2(t), \]

where \( W_2(t) \) and \( W_1(t) \) (the initial Wiener process that governs the price process) may be correlated (see \([5, 24, 27]\));

(iv) The volatility depends on a random parameter \( x \) such as \( \sigma(t) \equiv \sigma(x(t)) \), where \( x(t) \) is some random process (see \([16, 21, 37]\)). Cox and Ross \([8]\) valued options for alternative stochastic processes. Harrison and Pliska \([23]\) introduced and studied arbitrage and completeness of Brownian market. Föllmer and Sondermann \([17]\) introduced and studied locally minimizing risk strategies. Hamilton \([22]\) introduced Markov switching into the econometric mainstream. Föllmer and Schweizer \([18]\) studied hedging under incomplete information using the minimal martingale measure. Di Masi et al. \([13]\) obtained option pricing formula for stochastic volatility driven by a Markov chain in continuous time. Swishchuk \([36]\) obtained an option pricing formula for a model with stochastic volatility driven by a semi-Markov process. Gray \([20]\) combined GARCH effects with Markov switching. Griego and Swishchuk \([21]\) obtained the Black-Scholes formula for a market in a Markov random environment;

(v) Another approach is connected with stochastic volatility, namely, uncertain volatility scenario (see \([5]\)). This approach is based on the uncertain volatility model developed in Avellaneda et al. \([1]\), where a concrete volatility surface is selected among a candidate set of volatility surfaces. This approach addresses the sensitivity question by computing an upper bound for the value of the portfolio under arbitrary candidate volatility, and this is achieved by choosing the local volatility \( \sigma(t, S(t)) \) among two extreme values \( \sigma_{\text{min}} \) and \( \sigma_{\text{max}} \) such that the value of the portfolio is maximized locally;

(vi) The volatility \( \sigma(t, S_t) \) depends on \( S_t := S(t + \theta) \) for \( \theta \in [-\tau, 0] \), namely, stochastic volatility with delay (see \([30]\)).

In the approach (i), the volatility coefficient is independent of the current level of the underlying stochastic process \( S(t) \). This is a deterministic volatility model, and the special case where \( \sigma \) is a constant reduces
to the well-known Black-Scholes model that suggests changes in stock prices are lognormal distributed. But the empirical test by Bollerslev [3] seems to indicate otherwise. One explanation for this problem of a lognormal model is the possibility that the variance of \( \log(S(t)/S(t-1)) \) changes randomly. This motivated the work of Chesney and Scott [7], where the prices are analyzed for European options using the modified Black-Scholes model of foreign currency options and a random variance model. In their works the results of Hull and White [27], Scott [33] and Wiggins [41] were used in order to incorporate randomly changing variance rates.

In the approach (ii), several ways have been developed to derive the corresponding Black-Scholes formula: one can obtain the formula by using stochastic calculus and, in particular, the Ito’s formula (see [34], for example).

A generalized volatility coefficient of the form \( \sigma(t, S(t)) \) is said to be level-dependent. Because volatility and asset price are perfectly correlated, we have only one source of randomness given by \( W_1(t) \). A time and level-dependent volatility coefficient makes the arithmetic more challenging and usually precludes the existence of a closed-form solution. However, the arbitrage argument based on portfolio replication and a completeness of the market remain unchanged.

The situation becomes different if the volatility is influenced by a second “non-tradable” source of randomness. This is addressed in the approach (iii), (iv) and (v) we usually obtain a stochastic volatility model, which is general enough to include the deterministic model as a special case. The concept of stochastic volatility was introduced by Hull and White [27], and subsequent development includes the works of Wiggins [41], Johnson and Shanno [29], Scott [33], Stein and Stein [35] and Heston [24]. We also refer to Frey [19] for an excellent survey on level-dependent and stochastic volatility models (see also [12]).

Hobson and Rogers [25] suggested a new class of nonconstant volatility models, which can be extended to include the aforementioned level-dependent model and share many characteristics with the stochastic volatility model. The volatility is non-constant and can be regarded as an endogenous factor in the sense that it is defined in terms of the past behaviour of the stock price. This is done in such a way that the price and volatility form a multi-dimensional Markov process.

The Generalized Auto-Regression Conditional Heteroskedacity (GARCH) model (see [3]) is another popular model for estimating stochastic volatility. It assumes that the randomness of the variance process varies with the variance, as opposed to the square root of the variance
as in the Heston model. The standard GARCH(1,1) model has the following form for the variance differential:

$$d\sigma_t = \kappa(\theta - \sigma_t)\,dt + \gamma \sigma_t\,dB_t.$$ 

The GARCH model has been extended via numerous variants, including the NGARCH, LGARCH, EGARCH, GJR-GARCH, etc. (see, e.g., [14]).

2.3 Variance and volatility swaps Volatility swaps are forward contracts on future realized stock volatility, variance swaps are similar contract on variance, the square of the future volatility, both these instruments provide an easy way for investors to gain exposure to the future level of volatility.

A stock’s volatility is the simplest measure of its risk less or uncertainty. Formally, the volatility $\sigma_R$ is the annualized standard deviation of the stock’s returns during the period of interest, where the subscript $R$ denotes the observed or “realized” volatility.

The easy way to trade volatility is to use volatility swaps, sometimes called realized volatility forward contracts, because they provide pure exposure to volatility(and only to volatility).

Demeterfi et al. [10] explained the properties and the theory of both variance and volatility swaps. They derived an analytical formula for theoretical fair value in the presence of realistic volatility skews, and pointed out that volatility swaps can be replicated by dynamically trading the more straightforward variance swap.

Javaheri et al. [28] discussed the valuation and hedging of a GARCH(1,1) stochastic volatility model. They used a general and exible PDE approach to determine the first two moments of the realized variance in a continuous or discrete context. Then they approximate the expected realized volatility via a convexity adjustment.

Brockhaus and Long [4] provided an analytical approximation for the valuation of volatility swaps and analyzed other options with volatility exposure.

Working paper by Théoret et al. [40] presented an analytical solution for pricing of volatility swaps, proposed by Javaheri et al. [28]. They priced the volatility swaps within framework of GARCH(1,1) stochastic volatility model and applied the analytical solution to price a swap on volatility of the S&P500 Canada Index (5-year historical period: 1997–2002).

Elliott and Swishchuk [16] studied option pricing formulae and pricing swaps for Markov-modulated Brownian with jumps. Variance and
volatility swaps for financial markets with stochastic volatility that follow Heston model have been studied in [38] and variance swaps for financial markets with stochastic volatilities with delay have been studied in [39].

3 Martingale characterization of semi-Markov processes

3.1 Markov renewal and semi-Markov processes

Let \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) be a probability space with filtration \(\mathcal{F}_t\), \((X, \mathcal{X})\) be a measurable space and \(Q(x, B, t) = P(x, B)G_x(t), x \in X, B \in \mathcal{X}, t \in R_+\), be a semi-Markov kernel. Let us consider a \((X \times R_+, \mathcal{X} \otimes B_+)-valued\) stochastic process \((x_n, \tau_n; n \geq 0)\) with \(\tau_0 \leq \tau_1 \leq \cdots \leq \tau_n \leq \tau_{n+1} \leq \cdots\) (see [31, 37]).

**Definition 1.** A Markov renewal process is a two component Markov chain, \((x_n, \tau_n; n \geq 0)\), homogeneous with respect to the second component with transition probabilities

\[
P(x_{n+1} \in B, \tau_{n+1} - \tau_n \leq t | \mathcal{F}_n) = Q(x_n, B, t) = P(x, B)G_x(t) \quad a.s.
\]

Let us define the counting process of jumps \(\nu(t)\) by \(\nu(t) = \sup\{n \geq 0 : \tau_n \leq t\}\), that gives the number of jumps of the Markov renewal process in the time interval \((0, t]\).

**Definition 2.** A stochastic process \(x(t), t \geq 0\), defined by the following relation \(x(t) = x_{\nu(t)}\) is called a semi-Markov process, associated to the Markov renewal process \((x_n, \tau_n; n \geq 0)\).

**Remark 1.** Markov jump processes are special cases of semi-Markov processes with semi-Markov kernel \(Q(x, B, t) = P(x, B)[1 - e^{-\lambda(x)t}]\).

**Definition 3.** The following auxiliary process \(\gamma(t) = t - \tau_{\nu(t)}\) is called the backward recurrence time-the time period since the last renewal epoch until \(t\) (or the current life in the terminology of reliability theory, or age random variable).

**Remark 2.** The current life is a Markov process with generator \(Q(f(t) = f'(t) + \lambda(t)[f(0) - f(t)]\), where \(\lambda(t) = -\frac{\mathcal{F}'(t)}{\mathcal{F}(t)}, \mathcal{F}(t) = 1 - F(t), \) Domain \((Q) = C^1(R)\).
Remark 3. If we expand the state space of the semi-Markov process to include a component that records the amount of time already spent in the current state, then this additional information in the state description makes the semi-Markov process Markovian. For example, the following process \((x(t), \gamma(t))\) is a Markov process.

Definition 4. The compensating operator \(Q\) of the Markov renewal process is defined by the following relation

\[
Qf(x_0, \tau_0) = q(x_0)E[f(x_1, \tau_1) - f(x_0, \tau_0)|\mathcal{F}_0],
\]

where \(q(x) = 1/m(x)\), \(m(x) = \int_0^{+\infty} \mathcal{G}_x(t) \, dt\), \(\mathcal{F}_t = \sigma\{x(s), \tau_\nu(s); 0 \leq s \leq t\}\).

Proposition 1. The compensating operator of the Markov renewal process can be defined by the relation

\[
Qf(x, t) = q(x) \left[ \int_0^{+\infty} G_x(ds) \int_X P(x, dy) f(y, t + s) - f(x, t) \right].
\]

This statement follows directly from Definition 4.

Proposition 2. Let \((x_n, \tau_n)\) be the Markov renewal process, \(Q\) be the compensating operator,

\[
m_n := f(x_n, \tau_n) - \sum_{i=1}^n (\tau_i - \tau_{i-1})Qf(x_{i-1}, \tau_{i-1}),
\]

and \(\mathcal{F}_n = \sigma\{x_k, \tau_k; k \leq n\}\). Then the process \(m_n\) is a \(\mathcal{F}_n\)-martingale for any function \(f\) such that \(E_x[f(x_1, \tau_1)] < +\infty\).

Let \(y(t)\) be a Markov process with infinitesimal generator \(Q\).

Proposition 3. The process

\[
m(t) := f(y(t)) - f(y) - \int_0^t Qf(y(s)) \, ds
\]

is an \(\mathcal{F}_t\) -martingale (see [31]).

This statement follows from Dynkin formula.
Proposition 4. The quadratic variation of the martingale $m(t)$ is the process

$$
\langle m(t) \rangle = \int_0^t [Qf^2(y(s)) - 2f(y(s))Qf(y(s))] \, ds.
$$

(See [16].)

3.2 Jump measure for semi-Markov process The jump measure for $x(t)$ is defined in the following way (see [31]):

$$
\mu([0,t] \times A) = \sum_{n \geq 0} 1(x_n \in A, \tau_n \leq t), \quad A \in \mathcal{X}, \ t \geq 0.
$$

It is known (see [31]) that predictable projection (compensator) for $x(t)$ has the form

$$
\nu(dt, dy) = \sum_{n \geq 0} 1(\tau_n < t \leq \tau_{n+1}) \frac{P(x_n, dy)g_{x_n}(t)}{G_{x_n}(t)} \, dt,
$$

where $G_x(t) = 1 - G_x(t)$, $g_x(t) = dG_x(t)/dt, \ \forall x \in X$.

3.3 Martingale characterization of semi-Markov processes Let $x_t$ be a semi-Markov process with semi-Markov kernel $Q(x, B, t) = P(x, B)G_x(t)$ and $\gamma(t)$ be the current life.

Lemma 1. The process

$$
m_t^f := f(x_t, \gamma(t)) - \int_0^t Qf(x_s, \gamma(s)) \, ds
$$

is a martingale with respect to the filtration $F^*_t := \sigma\{x_s, \tau_{s(a)}; 0 \leq s \leq t\}$, where $Q$ is the infinitesimal operator of Markov process $(x_t, \gamma(t))$:

$$
Qf(x, t) = \frac{df(x, t)}{dt} + g_x(t) \int_X P(x, dy)[f(y, t) - f(x, t)].
$$

This statement follows from Proposition 3 and the fact that $(x(t), \gamma(t))$ is a Markov process (see Remark 3).

Let us calculate the quadratic variation of the martingale $m_t^f$.

Lemma 2. Let $Q$ be such that if $f \in \text{Domain}(Q)$, then $f^2 \in \text{Domain}(Q)$. The quadratic variation $\langle m_t^f \rangle$ of the martingale $m_t^f$ in (1) is equal to

$$
\langle m_t^f \rangle = \int_0^t [Qf^2(x_s, \gamma(s)) - 2f(x_s, \gamma(s))Qf(x_s, \gamma(s))] \, ds.
$$
This statement follows from Proposition 4 and the fact that \((x(t), \gamma(t))\) is a Markov process (see Remark 3).

**Lemma 3.** Let the following condition (Novikov’s condition) be satisfied

\[
E^P \exp \left\{ \frac{1}{2} \int_0^t [Qf^2(x_s, \gamma(s)) - 2f(x_s, \gamma(s))Qf(x_s, \gamma(s))] ds \right\} < +\infty,
\]
\[\forall f^2 \in \text{Domain}(Q).\]

Then \(E^P e^f_t = 1\), where

\[e^f_t := e^{m^f_t - \frac{1}{2}m^f_t},\]

and \(e^f_t\) in above is a \(P\)-martingale (Doléans-Dade martingale).

4 Minimal risk-neutral (martingale) measure for stock price with semi-Markov stochastic volatility

4.1 Current life stochastic volatility driven by semi-Markov process (current life semi-Markov volatility) Let \(x_t\) be a semi-Markov process in measurable phase space \((X, X)\).

We suppose that the stock price \(S_t\) satisfies the following stochastic differential equation

\[dS_t = S_t(r dt + \sigma(x_t, \gamma(t)) dw_t)\]

with the volatility \(\sigma := \sigma(x_t, \gamma(t))\) depending on the process \(x_t\), which is independent on standard Wiener process \(w_t\), and the current life \(\gamma(t) = t - \tau_v(t), \mu \in \mathbb{R}\). We call the volatility \(\sigma(x_t, \gamma(t))\) the current life semi-Markov volatility.

We note that process \((x_t, \gamma(t))\) is a Markov process on \((X, \mathbb{R}+)\) with infinitesimal operator

\[Qf(x, t) = \frac{df(x, t)}{dt} + \frac{g_x(t)}{G_2(t)} \int_X P(x, dy)[f(y, t) - f(x, t)].\]

4.2 Minimal martingale measure We consider the following \((B, S)\)-security market \((B\) stands for ‘Bond’ and \(S\) stands for ‘Stock’). Let the stock price \(S_t\) satisfies the following equation

\[dS_t = S_t(\mu dt + \sigma(x_t, \gamma(t)) dw_t),\]
where $\mu \in R$ is the appreciation rate and $\sigma(x_t, \gamma(t))$ is the semi-Markov volatility, and the bond price $B(t)$ is

$$B(t) = B_0 e^{rt},$$

where $r > 0$ is the risk-free rate of return (interest rate).

As long as we have two sources of randomness, Brownian motion $w(t)$ and semi-Markov process $x_t$, the above $(B,S)$-security market (5)–(6) is incomplete (see [16, Theorem 1]) and there are many risk-neutral (or martingale) measures. We are going to construct the minimal martingale measure (see [18]). With respect to this construction (see [16, Lemma 4]) the following measure $P^*$ is the minimal martingale measure.

Using Girsanov’s Theorem (see [39]) we obtain the following result concerning the minimal martingale measure in the above market.

**Lemma 4.** Under the assumption $\int_0^T \left( \frac{r - \mu}{\sigma(x_t, \gamma(t))} \right)^2 dt < +\infty$, a.s., the following holds ($T < +\infty$):

1) There is a probability measure $P^*$ equivalent to $P$ such that

$$\frac{dP^*}{dP} = \exp \left\{ \int_0^T \frac{r - \mu}{\sigma(x_t, \gamma(t))} \, dw(t) - \frac{1}{2} \int_0^T \left( \frac{r - \mu}{\sigma(x_t, \gamma(t))} \right)^2 \, dt \right\}$$

is its Radon-Nikodym density.

2) The discounted stock price $Z(t) = \frac{S_t}{B(t)}$ is a positive local martingale with respect to $P^*$ and is given by

$$Z(t) = \exp \left\{ -\frac{1}{2} \int_0^t \sigma^2(x_s, \gamma(s)) \, ds + \int_0^t \sigma(x_s, \gamma(s)) \, dw^*(s) \right\},$$

where $w^*(t) = \int_0^t \frac{r - \mu}{\sigma(x_s, \gamma(s))} \, ds + w(t)$ is a standard Brownian motion with respect to $P^*$.

**Remark 4.** Measure $P^*$ is called the minimal martingale measure.

**Remark 5.** We note that under risk-neutral measure $P^*$ the stock price $S_t$ satisfies the following equation:

$$dS_t = S_t \left( r \, dt + \sigma(x_t, \gamma(t)) \, dw^*(t) \right)$$

and discounted process $Z(t)$ has the presentation

$$dZ(t) = Z(t) \sigma(x_s, \gamma(s)) \, dw^*(t).$$
Remark 6. A sufficient condition (Novikov’s condition) for the right-hand side of (7) to be a martingale is

\[ E \exp \left\{ \frac{1}{2} \int_0^T \left( \frac{r - \mu}{\sigma(x_t, \gamma(t))} \right)^2 dt \right\} < +\infty. \]

5 Pricing of variance swaps for stochastic volatility driven by a semi-Markov process  

A variance swap is a forward contract on annualized variance, the square of the realized volatility. Its payoff at expiration is equal to

\[ N(\sigma_R^2(x) - K_{\text{var}}), \]

where \( \sigma_R^2(x) \) is the realized stock variance (quoted in annual terms) over the life of the contract,

\[ \sigma_R^2(x) := \frac{1}{T} \int_0^T \sigma^2(x_s, \gamma(s)) ds, \]

\( K_{\text{var}} \) is the delivery price for variance, and \( N \) is the notional amount of the swap in dollars per annualized volatility point squared. The holder of variance swap at expiration receives \( N \) dollars for every point by which the stock’s realized variance \( \sigma_R^2(x) \) has exceeded the variance delivery price \( K_{\text{var}} \) (see [6]).

Pricing a variance forward contract or swap is no different from valuing any other derivative security. The value of a forward contract \( F \) on future realized variance with strike price \( K_{\text{var}} \) is the expected present value of the future payoff in the risk-neutral world:

\[ P_{\text{var}}(x) = E\{e^{-rT}(\sigma_R^2(x) - K_{\text{var}})\}, \]

where \( r \) is the risk-free discount rate corresponding to the expiration date \( T \), and \( E \) denotes the expectation with respect to the minimal martingale measure \( P \) (from now on we use simpler notation \( P \) instead of \( P^* \)).

Let us show how we can calculate \( EV(x) \), where \( V(x) := \sigma_R^2(x) \). For that we need to calculate \( E\sigma^2(x_t, \gamma(t)) \).

We note (see Section 2, Lemma 1) that for \( \sigma(x) \in \text{Domain}(Q) \) the following process

\[ m_t^\sigma := \sigma(x_t, \gamma(t)) - \sigma(x, 0) - \int_0^t Q\sigma(x_s, \gamma(s)) ds \]
is a zero-mean martingale with respect to $\mathcal{F}_t := \sigma\{x_s, \tau_{\nu(s)}; 0 \leq s \leq t\}$ and $Q$ is the infinitesimal operator defined in (4).

The quadratic variation of the martingale $m^\tau_t$ by Lemma 2 is equal to

$$\langle m^\tau_t \rangle = \int_0^t [Q\sigma^2(x_s, \gamma(t)) - 2\sigma(x_s, \gamma(t))Q\sigma(x_s, \gamma(t))] ds,$$

$\sigma^2(x, t) \in \text{Domain}(Q)$.

Since $\sigma(x_s, \gamma(s))$ satisfies the following stochastic differential equation

$$d\sigma(x_1, \gamma(t)) = Q\sigma(x_1, \gamma(t)) dt + dm^\tau_t$$

then we obtain from Itô formula (see [39]) that $\sigma^2(x_t, \gamma(t))$ satisfies the following stochastic differential equation

$$d\sigma^2(x_t, \gamma(t)) = 2\sigma(x_t, \gamma(t)) dm^\tau_t$$

$$+ 2\sigma(x_t, \gamma(t))Q\sigma(x_t, \gamma(t)) dt + d\langle m^\tau_t \rangle,$$

where $\langle m^\tau_t \rangle$ is defined in (11).

Substituting (11) into (12) and taking the expectation of both parts of (12) we obtain that

$$E\sigma^2(x_t, \gamma(t)) = \sigma^2(x, 0) + \int_0^t QE\sigma^2(x_s, \gamma(s)) ds,$$

and solving this equation we have

$$E\sigma^2(x_t, \gamma(t)) = e^{tQ}\sigma^2(x, 0).$$

Finally, we obtain

$$EV(x) = \frac{1}{T} \int_0^T e^{tQ}\sigma^2(x, 0) dt.$$

In this way, we have obtained the following result that follows from (10)–(13).

**Theorem 1.** The value of a variance swap for semi-Markov stochastic volatility $\sigma(x_t, \gamma(t))$ equals to

$$\mathcal{P}_{\text{var}}(x) = e^{-rT} \left( \frac{1}{T} \int_0^T e^{tQ}\sigma^2(x, 0) dt - K_{\text{var}} \right),$$

where $Q$ is defined in (4).
5.1 Closed form solution for the variance swap

Rather than using (14) to compute variance swap we can use the following identity

\[ e^{tQ}\sigma^2(x,0) = \lim_{n \to +\infty} \left( I + \frac{Q}{n} \right)^n \sigma^2(x,0), \]

where \( I \) is an identity operator.

In this way, we can approximate \( e^{tQ}\sigma^2(x,0) \) by raising \((I + \frac{Q}{n})\) to the \( n \)th power. For example, in the case of \( n = 1 \) we have \( e^{tQ} \approx I + Qt \), and we get the following closed form solution for variance swap in (14):

\[ P_{\text{var}}(x) = e^{-rT} \left( \frac{1}{T} \int_0^T e^{tQ}\sigma^2(x,0) \, dt - K_{\text{var}} \right) \]

\[ \approx e^{-rT} \left( \frac{1}{T} \int_0^T (I + Qt)\sigma^2(x,0) \, dt - K_{\text{var}} \right) \]

\[ = e^{-rT} \left( \sigma^2(x,0) - K_{\text{var}} + \frac{T}{2} Q\sigma^2(x,0) \right). \]

As we can see, the value of variance swap is the sum of the discounted value of the spread between initial variance \( \sigma^2(x,0) \) at state \( x \) and strike price \( K \), and discounted value associated with the semi-Markov switchings.

In the same way, we can get any order of approximation (or closed form solution) for the value of variance swap in (14).

6 Example of variance swap for stochastic volatility driven by two-state continuous-time Markov chain

Let \( Q \) be a generator of two-state continuous time Markov chain

\[ Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \]

and

\[ P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix} \]

be a Markov transition function. Thus,

\[ P(t) = e^{tQ}. \]

In this case, the variance takes two values: \( \sigma^2(1) \) and \( \sigma^2(2) \).
From formula (14) it follows that the value of variance swap in this case is equal to

$$P(i) = e^{-rT} \left( \frac{1}{T} \int_0^T [p_{i1}(s)\sigma^2(1) + p_{i2}(s)\sigma^2(2)] \, ds - K_{var} \right)$$

for $i = 1, 2$.

Thus, the value of variance swap depends on the initial state of Markov chain.

**Remark 7.** We note, that we can approximate the value of variance swap in (15) using the approximation of Section 5.1. In this way, we have the following approximation, for example, when $n = 1$:

$$P(i) = e^{-rT} \left( (\sigma^2(i) - K) + \frac{T}{2} (q_{i1}\sigma^2(1) + q_{i2}\sigma^2(2)) \right),$$

where $(q_{ij})_{i,j=1}^2$ are entries of infinitesimal matrix $Q$ and defined above.

We also note, that if Markov chain is stationary with ergodic distribution $(p_1, p_2)$, then the value of variance swap is equal to

$$P = p_1 P(1) + p_2 P(2),$$

where $P(i), i = 1, 2$, are defined in (15) (see also [16]).

### 7 Pricing of volatility swaps for stochastic volatility driven by a semi-Markov process

#### 7.1 Volatility swap

Volatility swaps are forward contracts on future realized stock volatility.

A stock’s volatility is the simplest measure of its risk less or uncertainty. Formally, the volatility $\sigma_R(S)$ is the annualized standard deviation of the stock’s returns during the period of interest, where the subscript $R$ denotes the observed or “realized” volatility for the stock $S$.

The easy way to trade volatility is to use volatility swaps, sometimes called realized volatility forward contracts, because they provide pure exposure to volatility (and only to volatility) (see [16]).

A stock volatility swap is a forward contract on the annualized volatility. Its payoff at expiration is equal to

$$N(\sigma_R(S) - K_{vol}),$$
where $\sigma_R(S)$ is the realized stock volatility (quoted in annual terms) over the life of contract,

$$
(16) \quad \sigma_R(S) := \sqrt{\frac{1}{T} \int_0^T \sigma_s^2 \, ds},
$$

$\sigma_t$ is a stochastic stock volatility, $K_{vol}$ is the annualized volatility delivery price, and $N$ is the notional amount of the swap in dollar per annualized volatility point. The holder of a volatility swap at expiration receives $N$ dollars for every point by which the stock’s realized volatility $\sigma_R$ has exceeded the volatility delivery price $K_{vol}$. The holder is swapping a fixed volatility $K_{vol}$ for the actual (floating) future volatility $\sigma_R$. We note that usually $N = \alpha I$, where $\alpha$ is a converting parameter such as 1 per volatility-square, and $I$ is a long-short index (+1 for long and −1 for short).

Pricing a variance forward contract or swap is no different from valuing any other derivative security. The value of a forward contract $F$ on future realized variance with strike price $K_{var}$ is the expected present value of the future payoff in the risk-neutral world:

$$
(17) \quad P_{vol}(x) = E \{ e^{-rT} (\sigma_R(x) - K_{vol}) \},
$$

where $r$ is the risk-free discount rate corresponding to the expiration date $T$, and $E$ denotes the expectation with respect to the minimal martingale measure $P$ (from now on we use simpler notation $P$ instead of $P^*$).

Thus, for calculating variance swaps we need to know only $E \{ \sigma^2_R(S) \}$, namely, mean value of the underlying variance.

To calculate volatility swaps we need more. From Brockhaus-Long [4] approximation (which is used the second order Taylor expansion for function $\sqrt{x}$) we have (see also [28, p. 16])

$$
(18) \quad E \{ \sqrt{\sigma_R^2(S)} \} \approx \sqrt{E \{ V \}} - \frac{Var \{ V \}}{8E \{ V \}^{3/2}},
$$

where $V := \sigma^2_R(S)$ and $\frac{Var \{ V \}}{8E \{ V \}^{3/2}}$ is the convexity adjustment.

Thus, to calculate volatility swaps we need both $E \{ V \}$ and $Var \{ V \}$.

### 7.2 Pricing of volatility swap

As we can see from (18), to calculate volatility swaps we need both $E \{ V \}$ and $Var \{ V \}$. 
We have already calculated $E\{\sigma^2_f(S)\} = E[V]$ (see (13)). Let us calculate $\text{Var}(V) := E[V]^2 - (E[V])^2$. In this way, we need $E[V]^2 = E\sigma^4_R(S)$. Taking into account the expression for $V = \sigma^2_R(S)$ we have

\begin{equation}
E[V]^2 = \frac{1}{T^2} \int_0^T \int_0^T E[\sigma^2(x_t, \gamma(t))\sigma^2(x_s, \gamma(s)) \, dt \, ds].
\end{equation}

In this way, the variance of $V$, $\text{Var}(V)$, is

\begin{equation}
\text{Var}(V) = E[V]^2 - (E[V])^2
= \frac{1}{T^2} \int_0^T \int_0^T E\sigma^2(x_t, \gamma(t))\sigma^2(x_s, \gamma(s)) \, dt \, ds
- \left(\frac{1}{T} \int_0^T e^{tQ}\sigma^2(x, 0) \, dt\right)^2.
\end{equation}

Taking into account (16)–(20), we obtain

\begin{equation}
\mathcal{P}_{\text{vol}}(x) = e^{-rT}[E\sigma_R(S) - K_{\text{vol}}]
= e^{-rT}[\sqrt{E} \int_0^T \sigma^2(x_s, \gamma(s)) \, dt - K_{\text{vol}}]
\approx e^{-rT}[\sqrt{E}V - \frac{\text{Var}(V)}{8(EV)^{3/2}} - K_{\text{vol}}]
= e^{-rT}\left\{ \sqrt{\frac{1}{T}} \int_0^T e^{tQ}\sigma^2(x, 0) \, dt
- \left[ \frac{1}{T^2} \int_0^T \int_0^T E\sigma^2(x_t, \gamma(t))\sigma^2(x_s, \gamma(s)) \, dt \, ds
- \left(\frac{1}{T} \int_0^T e^{tQ}\sigma^2(x, 0) \, dt\right)^2 \right]\right/ \left(8\left(\frac{1}{T} \int_0^T e^{tQ}\sigma^2(x, 0) \, dt\right)^{3/2}\right) - K_{\text{vol}} \right\}.
\end{equation}

Summarizing (17)–(21), we have the following.

**Theorem 2.** The value of volatility swap for semi-Markov stochastic
volatility $\sigma(x, \gamma(t))$ equals to

\[
\mathcal{P}_{vol}(x) \approx e^{-rT} \left\{ \sqrt{\frac{1}{T} \int_0^T e^{tQ} \sigma^2(x, 0) \, dt} \right. \\
- \left. \left[ \frac{1}{T^2} \int_0^T \int_0^T E \sigma^2(x, \gamma(t)) \sigma^2(x_s, \gamma(s)) \, dt \, ds \right] \\
- \left( \frac{1}{T} \int_0^T e^{tQ} \sigma(x, 0) dt \right)^2 \right\} \left( 8 \frac{1}{T} \int_0^T e^{tQ} \sigma^2(x, 0) dt \right)^{3/2} - K_{vol} \right\}.
\]

8 Discussions of some extensions

8.1 Local current stochastic volatility driven by a semi-Markov process (local current semi-Markov volatility) We suppose that the stock price $S_t$ satisfies the following stochastic differential equation

\[
dS_t = S_t(r \, dt + \sigma_{loc}(S_t, x_t, \gamma(t)) \, dW_t)
\]

with the volatility $\sigma := \sigma_{loc}(S_t, x_t, \gamma(t))$ depending on the process $x_t$, which is independent on standard Wiener process $W_t$, stock price $S_t$, and the current life $\gamma(t) = t - \tau(v(t))$.

Remark 8. We note that process $(S_t, x_t, \gamma_t)$ is a Markov process on $(\mathbb{R}_+, X, \mathbb{R}_+)$ with infinitesimal operator

\[
Qf(s, x, t) = \frac{\partial f(s, x, t)}{\partial t} + \frac{g_x(t)}{G_x(t)} \int_X P(x, dy)[f(s, y, t) - f(s, x, t)] \\
+ rS \frac{\partial f(s, x, t)}{\partial s} + \frac{1}{2} \sigma^2(s, x, 0) S^2 \frac{\partial^2 f(s, x, t)}{\partial s^2}.
\]

Using the same procedure as in section 4 with infinitesimal operator $Q$ in (23) we get the following result.

**Theorem 3.** The value of a variance swap for Markov stochastic volatility $\sigma(x, t)$ equals to

\[
P(x) = e^{-rT} \left( \int_0^T e^{tQ} \sigma^2(s, x, 0) \, dt - K_{var} \right),
\]
where $Q$ is defined in (23).

8.2 Local stochastic volatility driven by a semi-Markov process (local semi-Markov volatility) We suppose that the stock price $S_t$ satisfies the following stochastic differential equation

$$dS_t = S_t(r dt + \sigma(S_t, x_t, t) dw_t)$$

with the volatility $\sigma := \sigma(S_t, x_t, t)$ depending on the process $x_t$, which is independent on standard Wiener process $w_t$, stock price $S_t$ and current time $t$.

Suppose that $\sigma(S, x, t)$ is differentiable function by $t$ with bounded derivative. Then we can reduce the problem to the previous one by the following expansion:

$$\sigma(S_t, x_t, t) = \sigma(S_t, x_t, \gamma(t)) + \tau_{\nu(t)} \frac{d\sigma(S_t, x_t, t)}{dt} + o(\tau_{\nu(t)}).$$

The error of estimation will be

$$E[\sigma(S_t, x_t, t) - \sigma(S_t, x_t, \gamma(t))]^2 \leq E[\tau_{\nu(t)}]^2 \times C,$$

where $C = \max_{0 \leq t \leq T} E[\frac{d\sigma(S_t, x_t, t)}{dt}]^2$.

8.3 Dupire formula for semi-Markov local volatility Unlike the implied volatility produced by applying the Black-Scholes formulae to market prices, the local volatility is the volatility implied by the market prices and the one factor Black-Scholes. In 1994, Dupire showed (see [15]) that if the spot price follows a risk-neutral random walk of the form

$$\frac{dS}{S} = (r - D) dt + \sigma(t, S) dW$$

and if no-arbitrage market prices for European vanilla options are available for all strikes $K$ and expires $T$, then $\sigma(t, S)$ can be extracted analytically from these option prices. If $C$ denotes the price of a European call with strike $K$ and expiry $T$, we obtain Dupire’s famous equation:

$$\frac{\partial C}{\partial T} = \sigma_{local}(K, T) \frac{K^2 \partial^2 C}{2 \partial K^2} - (r - D)K \frac{\partial C}{\partial K} - DC,$$

where $r$ and $D$ are interest and dividend rates, respectively. After rearranging this equation, we obtain the direct expression to calculate the local volatility (Dupire formulae):

$$\sigma_{local}(K, T) = \sqrt{\frac{\frac{\partial C}{\partial T} + (r - D)K \frac{\partial C}{\partial K} + DC}{K^2 \frac{\partial^2 C}{\partial K^2}}}.$$
We suppose that the stock price $S_t$ satisfies the following stochastic differential equation

$$dS_t = S_t((r - D) dt + \sigma_{loc}(S_t, x_t, t) dw_t)$$

with the volatility $\sigma := \sigma(S_t, x_t, \gamma(t))$ depending on the process $x_t$, which is independent on standard Wiener process $w_t$, stock price $S_t$, and the current life $\gamma(t) = t - \tau_x(t)$. Here, $D$ is the dividend rate.

Let $C$ denotes the price of a European call with strike $K$ and expiry $T$ for our $(B, S)$-security market with semi-Markov volatility. Then $C$ satisfies the following equation:

$$\frac{\partial C}{\partial T} + (r - D)K \frac{\partial C}{\partial K} - \sigma_{loc}(K, x, T) \frac{\partial^2 C}{\partial K^2} + D C + QC = 0,$$

where $Q$ is defined by

$$Qf(x, t) = \left( g_x(t) \int_X P(x, dy) [f(y, t) - f(x, t)] \right).$$

Then the semi-Markov local volatility can be calculated using the following expression:

$$\sigma_{loc}(K, x, T) = \sqrt{\frac{QC}{2\sigma^2} + \frac{(r - D)K \frac{\partial C}{\partial K} + DC - QC}{\sigma^2}}.$$

where $Q$ is defined in (25). Of course, if $\sigma$ does not depend on $x$ (and $C$ does not depend on $x$ as well), then $QC = 0$ in (26) and (26) coincides with (24).

8.4 Risk-minimizing strategies (or portfolios) and residual risk

Let $\pi(t) := (\alpha(t), \beta(t))$ is a portfolio or strategy (predictable processes), $V_t(\pi) = \alpha(t) + \beta(t)S_t$ is the value process, where $S_t$ satisfies

$$dS_t = S_t\sigma(S_t, x_t, \gamma(t)) dw^*(t).$$

We suppose that $r = 0$ just for simplicity. Let’s define the cost process

$$C_t(\pi) = V_t(\pi) - \int_0^t \beta(u) dS_u.$$

The residual risk is defined by the formula (see [18])

$$R_t(\pi) = E[(C_T(\pi) - C_t(\pi))^2 | \mathcal{F}_t].$$
Portfolio \( \pi(t) \) is said to be \( H \) admissible if \( V_T(\pi) = H \). Also \( \pi(t) \) is self-financing if the cost process is a martingale. The \( H \)-admissible self-financing portfolio \( \pi^* \) is called a risk-minimizing if, for any \( H \) admissible \( \pi \) for any \( t \)

\[
R_t(\pi^*) \leq R_t(\pi).
\]

**Proposition 5.** (see [36]) The risk-minimizing \( H \)-admissible strategy \( \pi^* = (\alpha^*, \beta^*) \) is given by the following formula:

\[
\beta^*(t) = u_s(S_t, x, t)
\]

and

\[
\alpha^*(t) = V_t(\pi^*) - \alpha^*(t)S_t,
\]

where

\[V_t(\pi^*) = E^* f(S_T) + \int_0^t u_s(S_r, x(r), r) dS_r + \int_0^t \int_X \varphi(r, y)(\mu - \nu) (dr, dy),\]

\[\varphi(r, y) := u(S_r, y, r) - u(S_r, x(r), r),\]

and function \( u(z, x, t) \) satisfies the following Cauchy problem:

\[
\begin{cases}
   u_t(z, x, t) + \frac{1}{2} \sigma^2(z, x, t) z^2 u_{zz}(z, x, t) + Qu(z, x, t) = 0 \\
   u(z, xT) = f(z),
\end{cases}
\]

with operator \( Q \) as

\[
Q u(z, x, t) = \frac{g_x(t)}{G_x(t)} \int_X P(x, dy) [u(z, y, t) - u(z, x, t)],
\]

and measures \( \mu \) and \( \nu \) have defined in Section 2.2.

**Proposition 6.** (see [36]) The residual risk process has the form

\[
R_t(\pi^*) = E \left( \int_t^T [Qu^2(S_r, x(r), r) - 2u(S_r, x(r), r)Qu(S_r, x(r), r)] ds | F_t \right).
\]

In particular, the total residual risk on the interval \([0, T]\) at the moment \( t = 0 \) is equal to

\[
R_0(\pi^*) = E \left( \int_0^T [Qu^2(S_r, x(r), r) - 2u(S_r, x(r), r)Qu(S_r, x(r), r)] ds \right).
\]
Remark 9. We note that process

\begin{equation}
\tag{31}
m_t := u(S_t, x(t), \gamma(t)) - u(z, x, 0) - \int_0^t \left[ Q + \frac{d}{dr} \right] u(S_r, x(r), \gamma(r)) \, dr
\end{equation}

is an $\mathcal{F}_t$-martingale, where $\mathcal{F}_t = \sigma \{ x(s), w(s); 0 \leq s \leq t \}$.

Proposition 7. The total residual risk on the interval $[0, T]$ at time $t = 0$ is the risk-neutral expectation of quadratic variation (or characteristic) $\langle m_t \rangle$ of the martingale $m_t$ in (31):

\begin{equation}
\tag{32}
R_0(\pi^*) = E^* \langle m_t \rangle.
\end{equation}

Proof. Follows from the definition of the residual risk, (30) and the following relationship:

\begin{equation}
\langle m_t \rangle = \int_0^T \left[ \left( Q + \frac{d}{dr} \right) u^2(S_r, x(r), r) - 2u(S_r, x(r), r) \right. \\
\left. \left( Q + \frac{d}{dr} \right) u(S_r, x(r), r) \right] \, ds \\
= \int_0^T \left[ Qu^2(S_r, x(r), r) - 2u(S_r, x(r), r)Qu(S_r, x(r), r) \right] \, ds.
\end{equation}

Hence,

\[ R_0(\pi^*) = E^* \langle m_t \rangle, \]

and (32) follows.

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