A REVIEW OF RECENT EXISTENCE AND BLOW-UP RESULTS FOR KINETIC MODELS OF CHEMOTAXIS

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ABSTRACT. We review recent global existence and blow-up results for some kinetic models of chemotaxis.

1 Introduction

The purpose of this paper is to review some recent global existence and blow-up results for kinetic models of chemotaxis with emphasis on the papers [5, 6, 7].

Chemotaxis, the directed motion of cells towards higher concentrations of chemoattractants, plays a very important role in many biological processes such as developmental biology (e.g., embryology, angiogenesis, pattern formation), immunology (inflammatory processes, wound healing) and self-organization of bacteria colonies. Experiments show that swimming bacteria such as E.coli undergo an erratic motion at the microscopic level and move in a series of ‘run’ and ‘tumble’. A ‘run’ is motion in a straight line, and typically, a run in a favourable direction will be long and a run in an unfavourable direction will be short. At the end of a run, the cell stops and ‘tumbles’, and this results in a re-orientation and a new run in a new direction. Since the mean time for tumbling is much shorter than the mean time for running, we can model chemotaxis as a velocity jump process.

At the macroscopic level chemotaxis is modeled by systems of parabolic equations, the most famous of which is the Keller-Segel system

\[
\begin{align*}
\frac{\partial n}{\partial t} &= D_1 \Delta n - \chi \nabla \cdot (n \nabla c), \\
\frac{\partial c}{\partial t} &= D_2 \Delta c + \alpha c - \beta n,
\end{align*}
\]

where \( n(t, x) \) is the density of the cells, \( c(t, x) \) is the concentration of the chemoattractant, \( \chi \) is a positive constant known as chemosensitivity, \( D_1 \) and \( D_2 \) are the diffusion coefficients for the cells and the chemical,
respectively, and $\epsilon$, $\alpha$ and $\beta$ are nonnegative constants. This system is often simplified to

\begin{align}
(1.2a) \quad & \partial_t n = \Delta n - \chi \nabla \cdot (n \nabla c), \\
(1.2b) \quad & -\Delta c = n.
\end{align}

For simplicity, we let $x$ vary in the whole space $\mathbb{R}^d$. The first term on the right hand side of (1.2a) represents the tendency of the cells to diffuse under their own Brownian motion and the second term their tendency to aggregate due to the presence of the chemoattractant. If $d = 1$, then diffusion is stronger than aggregation and solutions exist globally [17, 27]. If $d = 2$ then the two tendencies are evenly balanced. A typical result in this case is that we have global existence if the total mass of the cells is small, and blow-up in finite time if it is large. The critical value for the mass turns out to be $8\pi/\chi [3, 4, 13]$. The proof of blow-up for large mass relies on the simple fact that, for $M > 8\pi/\chi$, we have

\begin{equation}
\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 n(t, x) \, dx = 4M \left(1 - \frac{\chi}{8\pi} M\right) < 0.
\end{equation}

On the other hand, to prove global existence, one first shows an identity of the form

\begin{equation}
\frac{d}{dt} \int n^p \, dx = -c_1 \int |\nabla n^{p/2}|^2 \, dx + c_2 \int n^{p+1} \, dx,
\end{equation}

and then uses a Gagliardo-Nirenberg type inequality

\begin{equation}
\int n^{p+1} \, dx \leq c_3 M \int |\nabla n^{p/2}|^2 \, dx,
\end{equation}

to absorb $\int n^{p+1} \, dx$ into $\int |\nabla n^{p/2}|^2 \, dx$, provided that the mass $M$ is sufficiently small, thus gaining control of $L^p$ norms of $n$. This approach, which goes back to [24], fails to establish global existence under the optimal condition $M < 8\pi/\chi$. To achieve that, one uses the fact that the quantity

\begin{equation}
\mathcal{E}(t) = \int_{\mathbb{R}^2} n \log n \, dx + \frac{\chi}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(t, x) n(t, y) \log |x - y| \, dx \, dy
\end{equation}

is decreasing in time, since

\begin{equation}
\frac{d\mathcal{E}}{dt} = -\int_{\mathbb{R}^2} n |\nabla \log n - \chi \nabla S|^2 \, dx,
\end{equation}
together with a logarithmic Hardy-Littlewood-Sobolev inequality [2]

\begin{align}
(1.6) \quad \frac{M}{2} \int_{\mathbb{R}^2} n \log n \, dx + \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(x)n(y) \log |x - y| \, dx \, dy \geq C(M).
\end{align}

Combining these two, one obtains control of \( \int n \log n \, dx \), and this turns out to be sufficient for proving global existence; see [3, 13]. In dimensions \( d \geq 3 \) the critical space is \( L^{d/2} \). We refer the reader to [11].

At the mesoscopic level, chemotaxis is modeled by a kinetic equation for the density \( f(t, x, v) \) of the cells coupled to a parabolic or elliptic equation for the concentration \( S(t, x) \) of the chemoattractant. We focus our attention on the Othmer-Dunbar-Alt model [1, 25] :

\begin{align}
(1.7a) \quad \partial_t f + v \cdot \nabla_x f &= \int_V T[S](t, x, v, v')f(t, x, v') \, dv' \\
&\quad - \int_V T[S](t, x, v', v)f(t, x, v) \, dv', \\
(1.7b) \quad f(0, x, v) &= f_0(x, v), \\
(1.7c) \quad S - \Delta S = \rho := \int_V f(t, x, v) \, dv.
\end{align}

Here \( T[S](t, x, v, v') \geq 0 \) is the turning kernel, a measure of the frequency of turning from velocity \( v' \) to velocity \( v \) at position \( x \) and time \( t \). We will discuss examples of interesting turning kernels below. For the sake of simplicity, we take the velocity space \( V \) to be the unit ball.

The system (1.7) has been investigated in several papers. The ‘linear’ problem, that is, the problem with a given field \( S \), was studied in [18, 26]. A particularly interesting feature is the presence in the turning kernel of terms of the form \( S(t, x - \epsilon v') \) and \( S(t, x + \epsilon v) \). It introduces an asymmetry that turns out to be crucial in producing the drift term in the Keller-Segel model derived in the diffusion limit of equation (1.7); see [9, 18, 21, 22, 23, 28]. The presence of \( S(t, x - \epsilon v') \) means that the cells ‘measure’ the concentration of the chemical \( S \) at position \( x - \epsilon v' \) before changing their direction at position \( x \). This can be interpreted as an internal memory effect. On the other hand, the presence of \( S(t, x + \epsilon v) \) means that the cells ‘measure’ \( S \) at \( x + \epsilon v \). This can be achieved by the use of sensorial protrusions. For the sake of clarity, and since we are not going to discuss the drift-diffusion limit in this paper, we set \( \epsilon = 1 \).

The nonlinear IVP for (1.7) was first studied in [9]. The authors proved global existence of weak solutions in three dimensions under the
following pointwise hypothesis on the turning kernel
\[
0 \leq T[S](t, x, v, v') \leq C\left(1 + S(t, x + v) + S(t, x - v')\right)
\]
and for initial data \(0 \leq f_0 \in L^1 \cap L^\infty\). The proof uses strong dispersion estimates (see [16, 29]) and bootstraps higher \(L^p_{x,v}\)-norms of the solution \(f\), starting from the base case \(p = 1\) that corresponds to conservation of mass. The same method was used in [21] to prove global existence of weak solutions in two dimensions under the hypothesis
\[
0 \leq T[S](t, x, v, v') \leq C\left(1 + S(t, x + v) + S(t, x - v') + |\nabla S(t, x + v)| + |\nabla S(t, x - v')|\right),
\]
and in three dimensions under either of the hypotheses
\[
0 \leq T[S](t, x, v, v') \leq C\left(1 + S(t, x + v) + |\nabla S(t, x + v)|\right)
\]
or
\[
0 \leq T[S](t, x, v, v') \leq C\left(1 + S(t, x - v') + |\nabla S(t, x - v')|\right).
\]
As was pointed out in [21] terms involving \(S(t, x - v')\) or \(S(t, x + v)\) require the use of different dispersion estimates with different integrability exponents. When both terms are present it is difficult to find one set of exponents that makes both dispersion estimates work.

We refer the reader to [22, 23] for other results of this type that involve more general biologically relevant turning kernels.

2 Strichartz and dispersion estimates for the kinetic transport equation
Strichartz and dispersion estimates for the linear wave, Schrödinger, KdV, and other dispersive equations, are very useful in proving existence and regularity results for many nonlinear equations. We refer the reader to [30] for an introduction and an extensive bibliography.

Strichartz and dispersion estimates for the kinetic transport equation were proved in [8] and the first application to a nonlinear kinetic model was given in [5]. The dispersion estimate can be stated as follows.

**Proposition 2.1** (Dispersion estimate [8]). Let \(f_0 \in L^q(\mathbb{R}^d_x; L^p(\mathbb{R}^d_v))\) where \(1 \leq q \leq p \leq \infty\), and let \(f\) solve
\[
\partial_t f + v \cdot \nabla_x f = 0
\]
with initial data \( f(0, x, v) = f_0(x, v) \). Then

\[
(2.2) \quad \|f(t)\|_{L^p(\mathbb{R}_+^d; L^q(\mathbb{R}^d))} \leq \frac{1}{|t|^d\left(\frac{1}{q} - \frac{1}{p}\right)} \|f_0\|_{L^q(\mathbb{R}_+^d; L^p(\mathbb{R}^d))}.
\]

Since the solution of (2.1) with initial data \( f_0(x, v) \) is given by \( f(t, x, v) = f_0(x - tv, v) \), the dispersion estimate is simply the statement that for any function \( h \in L^q(\mathbb{R}_+^d; L^p(\mathbb{R}^d)) \), where \( 1 \leq q \leq p \leq \infty \), we have

\[
(2.3) \quad \|h(x - tv, v)\|_{L^p(\mathbb{R}_+^d; L^q(\mathbb{R}^d))} \leq \frac{1}{|t|^d\left(\frac{1}{q} - \frac{1}{p}\right)} \|h(x, v)\|_{L^q(\mathbb{R}_+^d; L^p(\mathbb{R}^d))}.
\]

Next we recall the Strichartz estimates of [8].

**Proposition 2.2** (Strichartz estimates [8]). Let \( d \geq 2 \) and let \( r, p, q, a \in [1, \infty] \) satisfy the conditions

\[
(2.4) \quad p \geq q, \quad \frac{2}{r} = d\left(\frac{1}{q} - \frac{1}{p}\right) < 1, \quad a = \frac{2pq}{p + q} \leq 2.
\]

If \( f(t, x, v) \) solves

\[
(2.5) \quad \partial_t f + v \cdot \nabla_x f = g, \quad f(0, x, v) = 0,
\]

then

\[
(2.6) \quad \|f\|_{L_t^r L_x^q L_v^r} \leq C \|g\|_{L_t^r L_x^q L_v^r}.
\]

If \( f(t, x, v) \) solves

\[
(2.7) \quad \partial_t f + v \cdot \nabla_x f = 0, \quad f(0, x, v) = f_0(x, v),
\]

then

\[
(2.8) \quad \|f\|_{L_t^r L_x^q L_v^r} \leq C \|f_0\|_{L_{x,v}^r}.
\]
3 Global existence for some kinetic models of chemotaxis

We can use the estimates of Section 2 to improve the global existence results of [9] and extend to three dimensions the results of [21]. Using the dispersion estimate of Proposition 2.1, we can prove the following.

**Theorem 3.1 ([5]).** Let \( d = 3 \) and suppose that the (continuous) turning kernel \( T[S] \) satisfies

\[
0 \leq T[S](t, x, v, v') \leq C \left( 1 + S(t, x + v) + S(t, x - v') + |\nabla S(t, x + v)| \right).
\]

Let \( q \in (1, 3/2) \). Then there exists an exponent \( p \in (3/2, 3) \) (depending on \( q \)) such that if the initial data \( 0 \leq f_0 \in L^1(\mathbb{R}^6) \) is such that the norm \( \|f_0(x - tv, v)\|_{L^p(\mathbb{R}^3_x; L^q(\mathbb{R}^3_v))} \) is finite for all \( t > 0 \), then the Cauchy problem (1.7) has a global weak solution \( f \) with \( f(t) \in L^1(\mathbb{R}^6) \cap L^p(\mathbb{R}^3_x; L^q(V)) \) for all \( t \geq 0 \).

The delocalization introduced by \( x + v \) and \( x - v' \) plays a fundamental role in the proof of this result.

Observe that hypothesis (3.1) does not allow putting together the two gradients \( \nabla S(t, x + v) \) and \( \nabla S(t, x - v') \). This is still an open problem in three dimensions for large data. However, if we add the assumption that the critical \( L^{3/2}_{x,v} \)-norm of the initial data is sufficiently small, then we can use the Strichartz estimates of Proposition (2.2) to prove global existence under a very general hypothesis on the turning kernel, see (3.2) and the even weaker (3.3).

**Theorem 3.2 ([5]).** Let \( d = 3 \). Consider nonnegative initial data \( f_0 \in L^1 \cap L^a \), where \( \frac{3}{2} \leq a \leq 2 \), and assume that \( \|f_0\|_{L^a(\mathbb{R}^6)} \) is sufficiently small. Assume that the (continuous) turning kernel \( T[S] \) satisfies the condition

\[
0 \leq T[S](t, x, v, v') \leq C \sum_{\pm} \left[ |S(t, x \pm v)| + |S(t, x \pm v')| + |\nabla S(t, x \pm v)| + |\nabla S(t, x \pm v')| \right].
\]

Then (1.7) has a global weak solution \( f \in L^a_t \left( [0, \infty); L^p(\mathbb{R}^3_x; L^q(V)) \right) \), where \( \frac{1}{p} = \frac{1}{a} - \frac{1}{b} \) and \( \frac{1}{q} = \frac{1}{a} + \frac{1}{b} \). This result also holds if hypothesis
(3.2) is replaced by the weaker: for all $p_1, p_2, p_3 \in [1, \infty]$ with $p_1 \geq \max(p_2, p_3)$, it holds

\[
\| T[S](t, x, v, v') \|_{L^p_x L^q_v L^r_v}^r \leq C(\| V_{p_1, p_2, p_3} \|_{L^p_v} + \| \nabla S(t, \cdot) \|_{L^p_v}).
\]

4 Models without delocalization effects; critical nonlinearities

Next, we would like to find what the critical nonlinearities are, that is, we would like to find turning kernels for which the dispersion and Strichartz estimates just fail to prove global existence. To this end we remove the delocalization that was so helpful in proving Theorem 3.1 and replace the right hand side in hypothesis (3.1) by $L^\infty_x$ norms of $S$ or $\nabla S$. In other words, we are now considering models without direct memory effects.

We start with dimension three. In view of the results of Theorem 3.1 it is natural to ask whether global existence still holds if we assume that

\[
0 \leq T[S](t, x, v, v') \leq C[1 + \| S(t, \cdot) \|^\alpha_{L^\infty(\mathbb{R}^3)}]
\]

for some positive exponent $\alpha$. In this case we can prove the following.

**Theorem 4.1** ([6]). Let $d = 3$ and suppose that the turning kernel satisfies (4.1). Let the initial data satisfy $0 \leq f_0 \in L^1 \cap L^\infty$. Then

(i) if $\alpha < 1$, then (1.7) has a global weak solution;

(ii) if $\alpha = 1$ and the critical $\| f_0 \|_{L^3_x}$ is sufficiently small, then (1.7) has a global weak solution.

The case of $\alpha = 1$ and large initial data is critical and remains open. However, the corresponding question in two dimensions has recently been answered in [7] as we now explain.

First of all, in two dimensions having a term like $\| S(t, \cdot) \|^\alpha_{L^\infty(\mathbb{R}^2)}$ presents no problems.

**Theorem 4.2** ([6]). Let $d = 2$ and suppose that the turning kernel satisfies (4.1) with any $\alpha > 0$. Let the initial data satisfy $0 \leq f_0 \in L^1 \cap L^\infty$. Then (1.7) has a global weak solution.

There is actually a lot of room in this case, as we still have global existence even if we assume exponential growth of the nonlinearity in the sense

\[
0 \leq T[S](t, x, v, v') \leq C(1 + \exp \| S(t, \cdot) \|^\alpha_{L^\infty(\mathbb{R}^2)})
\]
where \(0 \leq \alpha < 1\). The case \(\alpha = 1\) is critical and we can prove global existence if we assume in addition that the mass is small (the proof in [6] requires \(M < \pi\)). These results suggest that there is room for adding a gradient term to the turning kernel.

**Theorem 4.3 ([6]).** Let \(d = 2\) and suppose that the turning kernel satisfies

\[
0 \leq T[S](t, x, v, v') \leq C \left[ 1 + \|S(t, \cdot)\|_{L^\infty(R^d)}^\alpha + \|\nabla S(t, \cdot)\|_{L^\infty(R^d)}^\beta \right]
\]

with any \(\alpha > 0\) and \(0 \leq \beta < 1\). Let the initial data satisfy \(0 \leq f_0 \in L^1 \cap L^\infty\). Then (1.7) has a global weak solution.

A careful examination of the proofs in [6] shows that the critical quantity \(\|\nabla S(t, \cdot)\|_{L^\infty}\) in two dimensions behaves exactly as the critical quantity \(\|S(t, \cdot)\|_{L^\infty}\) in three dimensions. Although we still don’t know whether global existence for large data holds in this critical case in three dimensions, we do know that in two dimensions it doesn’t, and that there exist critical turning kernels for which the solution blows up in finite time. We discuss one case of this in the next section.

### 5 Blow-up for a kinetic model of chemotaxis

In two dimensions, consider the turning kernel

\[
T[S](t, x, v, v') = (v' \cdot \nabla S(t, x))_+.
\]

Recall that \(v'\) is the velocity before turning and \(v\) the velocity after. Therefore, the meaning of (5.1) is that the cells ‘always’ choose directions that form an angle smaller than \(\pi/2\) with \(\nabla S\). Notice also that this turning kernel is critical in the sense that it satisfies (4.2) with \(\beta = 1\). It is also critical with respect to the delocalization effects in the sense that we would be able to prove global existence in the case of \(\nabla S(t, x + \epsilon v)\) for any \(\epsilon > 0\).

**Theorem 5.1 ([7]).** Consider the system (1.7) in two dimensions with spherically symmetric initial data \(0 \leq f_0 \in L^1 \cap L^\infty\) and turning kernel given by (5.1). Then

(i) If the total mass \(M\) of the cells is sufficiently large and the second moment \(\iint |x|^2 f_0(x, v) \, dx \, dv\) is sufficiently small then the solution blows-up in finite time. The hypothesis on the second moment can be omitted if the equation \(S - \Delta S = \rho\) is replaced by the equation \(-\Delta S = \rho\).
(ii) If the mass $M$ is sufficiently small and $f_0(x, v) \leq C|x|^{-\gamma}$ for some $\gamma < 1$, then we have global existence of weak solutions.

We now sketch the blow-up argument in the simpler case that $S$ solves the equation $-\Delta S = \rho$. First, we calculate the first time derivative of the second moment of the solution. Unlike the case of the Keller-Segel equations where this derivative is constant (see (1.3)), here, it is given by

$$\frac{d}{dt} \iint \frac{1}{2} |x|^2 f(t, x, v) \, dx \, dv = \iint (x \cdot v) f(t, x, v) \, dx \, dv$$

Next, the second derivative is given by

$$\frac{d^2}{dt^2} \iint \frac{1}{2} |x|^2 f(t, x, v) \, dx \, dv = \iint |v|^2 f(t, x, v) \, dx \, dv$$

$$+ \ c \iint (x \cdot v) (v \cdot \nabla S(t, x)) \rho(t, x) \, dx \, dv$$

$$- \ c \iint (x \cdot v) |\nabla S(t, x)| f(t, x, v) \, dx \, dv$$

The first term on the right hand side can easily be estimated by the mass because the velocity is bounded:

$$\iint |v|^2 f(t, x, v) \, dx \, dv \leq CM.$$  

The second term on the right hand side can be calculated explicitly. We find:

$$\iint (x \cdot v) (v \cdot \nabla S(t, x)) \rho(t, x) \, dx \, dv = c \int (x \cdot \nabla S) \rho \, dx = -cM^2,$$

for some positive constant $c$. Finally, the third term on the right hand side can be expressed as the derivative of a positive finite quantity $P$,

$$\iint (x \cdot v) |\nabla S(t, x)| f(t, x, v) \, dx \, dv = \frac{dP}{dt}.$$  

Therefore, integrating once in time we find

$$\frac{d}{dt} \iint \frac{1}{2} |x|^2 f(t, x, v) \, dx \, dv \leq c + (cM - cM^2)t + c(P(0) - P(t))$$

$$\leq c + (cM - cM^2)t + P(0)$$
and the last quantity is negative if $M$ is sufficiently large.

The proof of part (ii) of Theorem 5.1 is based on a comparison argument. We refer the reader to [7] for the details. The critical value for the mass is not known. Recall that for the Keller-Segel system proving global existence under the optimal hypothesis $M < 8\pi/\chi$ relied on the identity (1.5) for $\mathcal{E}$ and no such identity is known in the kinetic case. Removing the assumption of spherical symmetry is also an open problem.

6 Models with internal variables Let $p(t, x, v, y_1, \ldots, y_m)$ be the density of cells, where $t$, $x$ and $v$ have the same meaning as above, and $y_1, \ldots, y_m$ are variables that describe the internal state of the cell ([12, 14]). We consider the model

$$\partial_t p + v \cdot \nabla_x p + \nabla_y \cdot (G(y, S)p) = \int_{v' \in V} \lambda[y]K(v, v')p(t, x, v', y)dv' - \lambda[y]p(t, x, v, y),$$

$$S - \Delta S = \rho,$$

where $S$ has the same meaning as in previous sections and $\rho = \iint p\, dv\, dy$. The transport along characteristics of the internal cellular dynamics is given by an equation of the form

$$\frac{dy}{dt} = G(y, S(t, x)), \quad y \in \mathbb{R}^m.$$

For example, the following equations are proposed in [14]:

$$\begin{align*}
\frac{dy_1}{dt} &= \frac{1}{\tau_e} (h(S) - (y_1 + y_2)) \quad \text{(excitation)}, \\
\frac{dy_2}{dt} &= \frac{1}{\tau_a} (h(S) - y_2) \quad \text{(adaptation)}.
\end{align*}$$

A simpler set of equations is proposed in [12]:

$$\begin{align*}
y_1 &= (h(S) - y_2)_+, \quad \text{(excitation)}, \\
\frac{dy_2}{dt} &= \frac{1}{\tau_a} (h(S) - y_2) \quad \text{(adaptation)}.
\end{align*}$$

Using the dispersion estimate of Section 2, we can prove the following.
Theorem 6.1. Let \( d = 3 \). Assume that the turning kernel has the form
\[
T = \lambda|y|K(v, v')
\]
where \( K \) is uniformly bounded, and \( \lambda \) grows at most linearly: \( \lambda|y| \leq C(1 + |y|) \). On the other hand, assume that \( G \) has a (sub)critical growth with respect to \( y \) and \( S \). Then there exists \( 0 \leq \alpha < 1 \) such that
\[
|G|(y, S) \leq C(1 + |y| + S^\alpha).
\]
Assume that we are given initial data \( p(0) \in L^1 \cap L^\infty \). Then there exists an exponent \( 1 < q < 3/2 \) such that the system (6.1) has global weak solutions with \( p(t) \in L^2_y L^1_v L^1_y \).

For more results and models involving kinetic equations see [10, 15, 19] and for surveys on kinetic and other aspects [20, 28, 29].

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