ANALYSIS AND NUMERICAL APPROXIMATION OF A FRICTIONAL UNILATERAL CONTACT PROBLEM WITH NORMAL COMPLIANCE

AREZKI TOUZALINE

ABSTRACT. We consider a static contact problem between a linear elastic body and an obstacle, the so-called foundation. The contact is frictional and is modelled with a normal compliance condition such that the penetration is restricted with unilateral constraint, and the associated version of Coulomb’s law of dry friction. We derive a variational formulation and prove its unique weak solvability if the friction coefficient is sufficiently small. Moreover, we prove the continuous dependence of the solution on the contact conditions. Also we study the finite element approximation of the problem and obtain an error estimate. Finally, we introduce an iterative method to solve the discrete problem.

1 Introduction

Contact mechanics is the branch of solid mechanics which typically involves two bodies instead of one and focuses its objective on their common interface rather than their interiors. Contact problems involving deformable bodies are quite frequent in industry as well as in daily life and play an important role in structural and mechanical systems. A first attempt to study contact problems within the framework of variational inequalities was made in [6]. The contact problem for elastic materials is one of the most used models in the theory of variational inequalities (see [6, 10]). In [10] we find a detailed analysis of the contact problem in elasticity with both mathematical and numerical studies. The analysis and the approximation by finite element methods of the unilateral contact models take an important place (see [7, 11]). These models present a great practical interest in mechanics (see [6]). The numerical studies of the Signorini contact problem were made in [1, 2, 3, 10]. In this work we study a static frictional contact problem for linear elastic materials. We assume that the contact is modelled with
a normal compliance condition similar to the one in [9] such that the penetration is restricted with unilateral constraint and the associated version of Coulomb’s law of dry friction. Under this condition the interpenetration of the body’s surface into the foundation is allowed and may be justified by considering the interpenetration and deformation of surface asperities. On the other hand, we want to point out the physical interest of the model studied here. Indeed, before the appearance of [9], it was well known that any restriction of the penetration was made in the compliance models. However, according again to [9], the method presented here considers a compliance model in which the compliance term doesn’t represent necessarily an important perturbation of the original problem without contact. This will help us to study the models, where a strictly limited penetration is performed with the limit procedure to the Signorini contact problem. The numerical study of the static contact problem with normal compliance was made in [8]. The aim of this paper is to extend this latest result to the case with finite penetration. We suppose that the displacement field is of class $H^2$ (the standard Sobolev space of degree 2) then we deduce an error estimate as $O(h^{3/4})$ where the notation $h > 0$ stands for the discretization parameter representing the greatest diameter of a triangle in the triangulation $T_h$. The paper is structured as follows. In Section 2 the mechanical problem (Problem $P_1$) is formulated, some notations are presented and the variational formulation is established. In Section 3 we give existence and uniqueness result if the friction coefficient is sufficiently small. In Section 4 we show a continuous dependence result. Finally, in Section 5 we study the finite element approximation of the displacement variational formulation. We establish the convergence of the finite element method and derive an error estimate under a regularity assumption on the solution. We solve the discrete problem by an iterative method which converges under certain assumptions.

2 Variational formulation

A linearly elastic body occupies a domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with a Lipschitz boundary $\Gamma$ that is partitioned into three measurable parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ where $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ are disjoint open sets such that $\text{meas}(\Gamma_1) > 0$. The body is subjected to volume forces of density $\phi_1$, prescribed zero displacements and tractions $\phi_2$ on the part $\Gamma_1$ and $\Gamma_2$, respectively. On $\Gamma_3$ the body is in frictional unilateral contact with a foundation.

Under these conditions, the classical formulation of the mechanical problem is the following.
Problem $P_1$. Find a displacement field $u: \Omega \rightarrow \mathbb{R}^d$ such that

\begin{align}
\sigma &= \mathcal{E}(u) \quad \text{in } \Omega, \\
\text{div } \sigma + \phi_1 &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma_1, \\
\sigma \nu &= \phi_2 \quad \text{on } \Gamma_2,
\end{align}

\begin{align}
\begin{cases}
\begin{aligned}
u &\leq g, \quad \sigma_\nu + p(\nu) \leq 0, \\
|\sigma_\tau| &\leq \mu p(\nu) \\
|\sigma_\tau| &< \mu p(\nu) \implies u_\tau = 0 \\
|\sigma_\tau| &\geq \mu p(\nu) \implies \exists \lambda \geq 0, \text{ s.t. } \sigma_\tau = -\lambda u_\tau
\end{aligned}
\end{cases} \quad \text{on } \Gamma_3.
\end{align}

In the study of Problem $P_1$ we shall adopt the following notations and hypotheses.

We denote by $S_d$ the space of second-order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$), while $\cdot$: and $|\cdot|$ will represent the inner product and Euclidean norm on $S_d$ and $\mathbb{R}^d$, respectively, i.e.,

\begin{align}
\begin{aligned}
u_i \nu_j &= (\nu, \nu)^{\frac{1}{2}}, \\
\sigma \cdot \tau &= \sigma_{ij} \tau_{ij}, \\
|\sigma| &= (\sigma \cdot \sigma)^{\frac{1}{2}},
\end{aligned}
\end{align}

For all $\sigma, \tau \in S_d$.

Here and below, the indices $i$ and $j$ run between 1 and $d$, and the summation convention over repeated indices is adopted.

We denote by

\[ \varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2}(u_{i,j} + u_{j,i}) \]

the strain tensor and $\sigma$ the stress tensor. We adopt the following notation for the normal and tangential components of the displacement vector and stress vector: $\nu_\nu = u \cdot \nu, \quad u_\tau = u - u_\nu \nu, \quad \sigma_\nu = (\sigma \nu) \cdot \nu, \quad \sigma_\tau = \sigma - \sigma_\nu \nu$, where $\nu$ is the outward unit normal vector to $\Gamma$.

To proceed with the variational formulation, we need some function spaces

\begin{align}
H &= (L^2(\Omega))^d, \\
Q &= \{ \tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega) \}, \\
H_1 &= (H^1(\Omega))^d.
\end{align}
$H$ and $Q$ are Hilbert spaces equipped with the respective inner products

$$(u,v)_H = \int_{\Omega} u_i v_i \, dx, \quad (\sigma, \tau)_Q = \int_{\Omega} \sigma_{ij} \tau_{ij} \, dx.$$ 

Let $V$ be the closed subspace of $H_1$ defined by

$$V = \{ v \in H_1 : v = 0 \text{ on } \Gamma_1 \}$$

and the set of admissible displacements fields given by

$$K = \{ v \in V : v \leq g \text{ on } \Gamma_3 \},$$

where $g \geq 0$. Since meas ($\Gamma_1$) > 0, the following Korn’s inequality holds [6],

$$\| \varepsilon(v) \|_Q \geq c_\Omega \| v \|_{H_1}, \quad \forall v \in V,$$

where $c_\Omega > 0$ is a constant which depends only on $\Omega$ and $\Gamma_1$. We equip $V$ with the inner product given by

$$(u,v)_V = (\varepsilon(u), \varepsilon(v))_Q$$

and let $\| \cdot \|_V$ be the associated norm. It follows from (2.6) that the norms $\| \cdot \|_{H_1}$ and $\| \cdot \|_V$ are equivalent and $(V, \| \cdot \|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant $d_\Omega > 0$ depending only on the domain $\Omega$, $\Gamma_1$ and $\Gamma_3$ such that

$$\| v \|_{L^2(\Gamma_3)^d} \leq d_\Omega \| v \|_V, \quad \forall v \in V.$$ 

In the study of the mechanical problem $P_1$, we assume that $\mathcal{E} = (\mathcal{E}_{ijkh}) : \Omega \times S_d \to S_d$ is a bounded symmetric positive definite fourth order tensor, i.e.,

$$\mathcal{E}_{ijkh} \in L^\infty(\Gamma_3), \quad 1 \leq i, j, k, h \leq d;$$

$$\mathcal{E} \sigma \cdot \tau = \sigma \cdot \mathcal{E} \tau, \quad \forall \sigma, \tau \in S_d, \text{ a.e. in } \Omega;$$

$$\text{There exists } \alpha > 0 \text{ such that}$$

$$\mathcal{E} \tau \cdot \tau \geq \alpha | \tau |^2 \quad \forall \tau \in S_d, \text{ a.e. in } \Omega.$$
We define the bilinear form \( a(\cdot, \cdot) \) on \( V \times V \) by
\[
a(u, v) = \int_{\Omega} \mathcal{E}(u) : \mathcal{E}(v) \, dx.
\]
It follows from \((2.8)\) that \( a \) is continuous and coercive, that is,
\[
\begin{align*}
(a) & \text{ there exists } M > 0 \text{ such that } \| a(u, v) \| \leq M \| u \|_V \| v \|_V \quad \forall u, v \in V, \\
(b) & \text{ there exists } m > 0 \text{ such that } a(v, v) \geq m \| v \|_V^2 \quad \forall v \in V.
\end{align*}
\]

The forces and the tractions are assumed to satisfy
\[
(2.10) \quad \phi_1 \in H, \quad \phi_2 \in (L^2(\Gamma_2))^d.
\]
Next, using Riesz’s representation theorem, we define \( f \in V \) by
\[
(f, v)_V = (\phi_1, v)_H + (\phi_2, v)_{(L^2(\Gamma_2))^d}.
\]
As in \([9]\) we assume that the normal compliance function \( p \) satisfies
\[
\begin{align*}
(a) & \quad p : [-\infty, g] \to \mathbb{R}, \\
(b) & \quad \text{there exists } L_p > 0 \text{ such that } \| p(r_1) - p(r_2) \| \leq L_p |r_1 - r_2|, \text{ for all } r_1, r_2 \leq g; \\
(c) & \quad (p(r_1) - p(r_2)) (r_1 - r_2) \geq 0, \text{ for all } r_1, r_2 \leq g; \\
(d) & \quad p(r) = 0 \text{ for all } r \leq 0.
\end{align*}
\]
Now we want to explain the physical sense of unilateral conditions \((2.5)\). Indeed, when \( u_\nu < 0 \), i.e., when there is separation between the body and the obstacle then the condition \((2.5)\) combined with hypotheses \((2.10)\) on the function \( p \) shows that the reaction of the foundation vanishes (since \( \sigma_\nu = 0 \)). When \( 0 \leq u_\nu < g \) then \( -\sigma_\nu = p(u_\nu) \) which means that the reaction of the foundation is uniquely determined by the normal displacement. When \( u_\nu = g \) then \( -\sigma_\nu \geq p(g) \) and \( \sigma_\nu \) is not uniquely determined. We note then that when \( g = 0 \), the condition \((2.5)\) becomes the classical Signorini contact condition without a gap
\[
\begin{align*}
u \leq 0, \quad \sigma_\nu \leq 0, \quad \sigma_\nu u_\nu = 0,
\end{align*}
\]
and when \( g > 0 \) and \( p = 0 \), condition (2.5) becomes the classical Signorini contact condition with a gap:

\[
\begin{align*}
    u_\nu &\leq g, \\
    \sigma_\nu &\leq 0, \\
    \sigma_\nu (u_\nu - g) &= 0.
\end{align*}
\]

The last two conditions are used to model the unilateral conditions with a rigid foundation. Finally the frictional conditions (2.5) represent the associated version of Coulomb’s law of dry friction.

We assume that the coefficient of friction satisfies

\[
(2.12) \quad \mu \in L^\infty (\Gamma_3) \text{ such that } \mu \geq 0 \text{ a.e. on } \Gamma_3.
\]

Next we define the functionals

\[
(2.13) \quad j_\nu : V \times V \to \mathbb{R}, \quad j_\tau : V \times V \to \mathbb{R}
\]

by

\[
\begin{align*}
    j_\nu (v, w) &= \int_{\Gamma_3} p(v_\nu) w_\nu \, da, \\
    j_\tau (v, w) &= \int_{\Gamma_3} \mu p(v_\nu) |w_\tau| \, da,
\end{align*}
\]

and let

\[
\begin{align*}
    j &= j_\nu + j_\tau.
\end{align*}
\]

Now, in order to establish the weak formulation of Problem \( P_1 \), we assume \( u \) is a smooth function satisfying (2.1)–(2.6). Indeed, let \( v \in V \) and multiply the equilibrium of forces (2.2) by \( v - u \), integrate the result over \( \Omega \) and use Green’s formula to obtain

\[
\int_{\Omega} \sigma (\varepsilon (v) - \varepsilon (u)) \, dx = \int_{\Omega} \phi_1 \cdot (v - u) \, dx + \int_{\Gamma} \sigma \nu \cdot (v - u) \, da.
\]

Taking into account the boundary conditions (2.3) and \( v = 0 \) on \( \Gamma_1 \), we see that

\[
\int_{\Gamma} \sigma \nu \cdot (v - u) \, da = \int_{\Gamma_2} \phi_2 \cdot (v - u) \, da + \int_{\Gamma_2} \sigma \nu \cdot (v - u) \, da.
\]

Moreover, we have

\[
\int_{\Gamma_3} \sigma \nu \cdot (v - u) \, da = \int_{\Gamma_3} \sigma_\nu (v_\nu - u_\nu) \, da + \int_{\Gamma_3} \sigma_\tau \cdot (v_\tau - u_\tau) \, da.
\]
But from frictional contact conditions (2.6) we have
\[ \sigma_t \cdot (v_t - u_t) + \mu p(u_\nu) \,(|v_t| - |u_t|) \geq 0, \; \forall v_t, \]
and we see that
\[
\int_{\Gamma_3} \sigma_\nu (v_\nu - u_\nu) \, da = \int_{\Gamma_3} (\sigma_\nu + p(u_\nu)) \,(v_\nu - u_\nu) \, da \\
- \int_{\Gamma_3} p(u_\nu) \,(v_\nu - u_\nu) \, da.
\]
Then we deduce that the function \( u \) satisfies the inequality
\[
a(u, v - u) + j(u, v) - j(u, u) \\
\geq (f, v - u)_V + \int_{\Gamma_3} (\sigma_\nu + p(u_\nu)) \,(v_\nu - u_\nu) \, da, \; \forall v \in V.
\]
On the other hand we observe that
\[
\int_{\Gamma_3} (\sigma_\nu + p(u_\nu))\,(v_\nu - u_\nu) \, da \\
= \int_{\Gamma_3} (\sigma_\nu + p(u_\nu))\,((v_\nu - g) - (u_\nu - g)) \, da \\
= \int_{\Gamma_3} (\sigma_\nu + p(u_\nu))\,(v_\nu - g) \, da \\
- \int_{\Gamma_3} (\sigma_\nu + p(u_\nu))\,(u_\nu - g) \, da.
\]
Then using the unilateral contact conditions (2.6) it follows that
\[
\int_{\Gamma_3} (\sigma_\nu + p(u_\nu))\,(v_\nu - g) \, da \geq 0, \; \forall v \in K,
\]
and
\[
\int_{\Gamma_3} (\sigma_\nu + p(u_\nu))\,(u_\nu - g) \, da = 0.
\]
Hence, we deduce that
\[
\int_{\Gamma_3} (\sigma_\nu + p(u_\nu))\,(v_\nu - u_\nu) \geq 0, \; \forall v \in K
\]
and finally, we obtain the following variational formulation of Problem $P_1$.

**Problem $P_2$.** Find a displacement field $u \in K$ such that

\begin{equation}
 a(u, v - u) + j(u, v) - j(u, u) \geq (f, v - u)_V,
\end{equation}

\forall v \in K.

### 3 Existence and uniqueness

The main results of this section is on the existence and uniqueness for the weak formulation $P_2$. One has the following theorem.

**Theorem 3.1.** Let (2.9), (2.10), (2.11) and (2.12) hold. Then there exists a constant $\mu_0 > 0$ such that Problem $P_2$ has a unique solution if

\[ \|\mu\|_{L^\infty(\Gamma_3)} < \mu_0. \]

The proof of Theorem 3.1 will be carried out in several steps. It is based on fixed points arguments. Indeed, let $\lambda \in K$ and consider the following variational problem.

**Problem $P_\lambda$.** Find $u_\lambda \in K$ such that

\begin{equation}
 (Au_\lambda, v - u_\lambda)_V + j_\tau(\lambda, v) - j_\tau(\lambda, u_\lambda) \geq (f, v - u_\lambda)_V,
\end{equation}

\forall v \in K,

where the operator $A : V \to V$ is defined as

\[ (Au, v)_V = a(u, v) + j_\nu(u, v). \]

We have the following result.

**Lemma 3.2.** For any $\lambda \in K$, Problem $P_\lambda$ has a unique solution $u_\lambda \in K$.

**Proof.** We use (2.9), (2.11)(b) and (2.11)(c) to show that the operator $A$ is strongly monotone and Lipschitz continuous. The functional $j_\tau(\lambda, \cdot) : V \to \mathbb{R}$ is a continuous seminorm; since $K$ is a nonempty closed convex subset of $V$ it follows from the theory of elliptic variational inequalities (see [4]) that the inequality (3.1) has a unique solution.

Next, Lemma 3.2 enables us to consider a mapping $S : K \to K$ defined by

\[ S(\lambda) = u_\lambda. \]
Lemma 3.3. There exists a constant $\mu_0 > 0$ such that if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, the mapping $S$ has a unique fixed point $\lambda^* \in K$ and $u_{\lambda^*}$ is a unique solution to Problem $P_2$.

Proof. For $\lambda_1$, $\lambda_2 \in K$ denote by $u_{\lambda_i}$ and $u_{\lambda_2}$ the corresponding solutions of inequality (3.1). Then $u_{\lambda_i} \in K$, $i = 1, 2$, satisfies the inequality

$$(A u_{\lambda_i}, v - u_{\lambda_i})_V + j_T (\lambda_i, v) - j_T (\lambda_i, u_{\lambda_i}) \geq (f, v - u_{\lambda_i})_V, \quad \forall v \in K.$$ 

Take $v = u_{\lambda_2}$ in the inequality satisfied by $u_{\lambda_1}$ and $v = u_{\lambda_1}$ in the inequality satisfied by $u_{\lambda_2}$, and adding the resulting inequalities to obtain

$$a (u_{\lambda_1} - u_{\lambda_2}, u_{\lambda_1} - u_{\lambda_2}) + \int_{\Gamma_3} (p(u_{\lambda_1}) - p(u_{\lambda_2})) (u_{\lambda_1} - u_{\lambda_2}) da \leq \int_{\Gamma_3} \mu \left( p(\lambda_1) - p(\lambda_2) \right) (|u_{\lambda_1}| - |u_{\lambda_2}|) da.$$ 

By (2.11) (c) we have

$$\int_{\Gamma_3} (p(u_{\lambda_1}) - p(u_{\lambda_2})) (u_{\lambda_1} - u_{\lambda_2}) da \geq 0.$$ 

Then it follows that

$$a (u_{\lambda_1} - u_{\lambda_2}, u_{\lambda_1} - u_{\lambda_2}) \leq \int_{\Gamma_3} \mu \left( p(\lambda_1) - p(\lambda_2) \right) (|u_{\lambda_1}| - |u_{\lambda_2}|) da.$$ 

Using (2.7) and (2.11) (b) we obtain

$$m \|u_{\lambda_1} - u_{\lambda_2}\|_V^2 \leq L_p d^2_\Omega \|\mu\|_{L^\infty(\Gamma_3)} \|\lambda_1 - \lambda_2\|_V \|u_{\lambda_1} - u_{\lambda_2}\|_V,$$

which implies

$$\|S(\lambda_1) - S(\lambda_2)\|_V \leq L_p \|\mu\|_{L^\infty(\Gamma_3)} \frac{d^2_\Omega}{m} \|\lambda_1 - \lambda_2\|_V.$$ 

Take

$$\mu_0 = \frac{m}{L_p d^2_\Omega}.$$ 

It follows that the mapping $S$ is a contraction if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$. Then it admits a unique fixed point $\lambda^*$ on $K$ and $u_{\lambda^*}$ is a unique solution of Problem $P_2$. \qed
4 Continuous dependence

Next, we investigate the behaviour of the weak solution to the problem $P_1$ with respect to perturbations of the normal compliance function $p$. For every $\alpha \geq 0$, let $p^\alpha$ be a perturbation of $p$ which satisfies respectively (2.11) with Lipschitz constant $L_p^\alpha$. Let us also introduce the functionals $j^\alpha$, which are obtained by replacing $p$ by $p^\alpha$ in $j$. We consider now the following problem.

**Problem $P_\alpha$.** For every $\alpha \geq 0$, find a displacement field $u^\alpha \in K$ such that

$$a (u^\alpha, v - u^\alpha) + j (u^\alpha, v) - j (u^\alpha, u^\alpha) \geq (f, v - u^\alpha)_V, \quad \forall v \in K.$$  

Using Theorem 3.1 we deduce that for each $\alpha \geq 0$ Problem $P_\alpha$ has a unique solution $u^\alpha$ for $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$. Suppose now that the contact function $p^\alpha$ satisfies the following assumptions.

There exists $\delta > 0$ and a function $\varphi_p : R_+ \to R_+$ such that

$$\begin{cases} 
(a) \ |p^\alpha (r) - p (r)| \leq \varphi_p (\alpha) \ |r|, & \forall r \leq g, \\
(b) \ \lim_{\alpha \to 0} \varphi_p (\alpha) = 0, \\
(c) \ L_p^\alpha \|\mu\|_{L^\infty(\Gamma_3)} + \delta < \frac{m}{d_1}.
\end{cases}$$  

We have the following result.

**Theorem 4.1.** Under the assumption (4.2) we have

$$u^\alpha \to u \text{ strongly in } V \text{ as } \alpha \to 0.$$  

**Proof.** Let $\alpha \geq 0$. Using (2.14) and (4.1) we obtain

$$a (u^\alpha - u, u^\alpha - u) \leq j (u, u^\alpha) - j (u, u) + j^\alpha (u^\alpha, u) - j^\alpha (u^\alpha, u^\alpha)$$

$$= \int_{\Gamma_3} (p (u_\nu) - p^\alpha (u_\nu^\alpha)) (u_\nu^\alpha - u_\nu) \ da$$

$$+ \int_{\Gamma_3} \mu (p (u_\nu) - p^\alpha (u_\nu^\alpha)) (|u_\nu^\alpha| - |u_\nu|) \ da.$$

Using (2.11)(c) we have

$$\int_{\Gamma_3} (p (u_\nu) - p^\alpha (u_\nu^\alpha)) (u_\nu^\alpha - u_\nu) \ da \leq 0,$$
then we get

\begin{align}
(4.4) \quad a(u^\alpha - u, u^\alpha - u) & \leq \int_{\Gamma_3} \mu \left( p(u_r) - p^\alpha (u^\alpha_r) \right) \left( |u^\alpha_r| - |u_r| \right) \, da \\
& \leq \|\mu\|_{L^\infty(\Gamma_3)} \|p(u_r) - p^\alpha (u^\alpha_r)\|_{L^2(\Gamma_3)} \\
& \quad \times \|u^\alpha - u\|_{(L^2(\Gamma_3))^2}.
\end{align}

Using (4.2)(a) we have

\[ \|p(u_r) - p^\alpha (u^\alpha_r)\|_{L^2(\Gamma_3)} \leq L_p^\alpha \|u^\alpha - u\|_{(L^2(\Gamma_3))^2} + \varphi_p (\alpha) \|u\|_{(L^2(\Gamma_3))^2}, \]

which implies that

\[ \int_{\Gamma_3} \mu \left( p(u_r) - p^\alpha (u^\alpha_r) \right) \left( |u^\alpha_r| - |u_r| \right) \, da \\
\leq \|\mu\|_{L^\infty(\Gamma_3)} \left( L_p^\alpha \|u^\alpha - u\|_{(L^2(\Gamma_3))^2}^2 \\
+ \varphi_p (\alpha) \|u\|_{(L^2(\Gamma_3))^2} \|u^\alpha - u\|_{(L^2(\Gamma_3))^2} \right). \]

Now, using the previous inequality and (4.4), we obtain by using (2.7), (2.8)(b) and (4.2)(c) that

\[ m \|u^\alpha - u\|^2_V \\
\leq d_1 \left( \frac{m}{d_2} - \delta \right) \|u^\alpha - u\|^2_V + d_2 \|\mu\|_{L^\infty(\Gamma_3)} \varphi_p (\alpha) \|u\|_V \|u^\alpha - u\|_V. \]

As a result, we obtain

\[ \delta \|u^\alpha - u\|_V \leq \varphi_p (\alpha) \|\mu\|_{L^\infty(\Gamma_3)} \|u\|_V. \]

So, we conclude (4.3). \(\square\)

5 Numerical approximation In this section we study the finite element approximation of the variational problem \(P_1\). Let a parameter \(h \to 0^+\) and \(V_h \subset V\) be a finite element subspace. Let \((K_h)_h\) be a family of non-empty closed convex subsets of \(V_h\) which approximates \(K\) in the following sense.

(i) \(\forall v \in K, \exists r_h v \in K_h\) such that \(r_h v \to v\) strongly in \(V\).
\( \forall v_h \in K_h \) with \( v_h \to v \) weakly in \( V \), then \( v \in K \).

Then, we formulate the following discrete problem as

\[
\begin{aligned}
\text{(5.1)} \\
\left\{ \begin{array}{l}
\text{Find } u_h \in K_h \text{ such that } \\
a(u_h, v_h - u_h) + j(u_h, v_h) - j(u_h, u_h) \\
\quad \geq (f, v_h - u_h), \quad \forall v_h \in K_h.
\end{array} \right.
\end{aligned}
\]

Under the assumptions of Theorem 3.1, the discrete problem (5.1) has a unique solution \( u_h \in K_h \) for \( k \prod_{3} ^{L_{3} (G_3)} < \mu_0 \).

Next, we focus on error analysis of the numerical solution. We first derive a Cea’s type inequality.

**Theorem 5.1.** Under the assumptions of Theorem 3.1, there exists a constant \( c > 0 \) such that

\[
(5.2) \quad \|u - u_h\|_V \leq c \inf_{v_h \in K_h} \left\{ \|u - v_h\|_V + \|u - v_h\|_{L^2(G_3)} + \|u - v_h\|_{L^2(G_3)} \right\}.
\]

**Proof.** We use (2.14) with \( v = u_h \) and (5.1) to get

\[
m \|u - u_h\|_V^2 \leq a(u - u_h, u - u_h) \\
\leq a(u - u_h, u - v_h) + a(u, v_h - u) + j(u, u_h) \\
\quad - j(u, u) + j(u_h, v_h) - j(u_h, u_h) - (f, v_h - u),
\]

i.e.,

\[
(5.3) \quad m \|u - u_h\|_V^2 \leq A_1 + A_2 + A_3 + A_4,
\]

where

\[
A_1 = a(u - u_h, u - v_h) \\
A_2 = a(u, v_h - u) + j(u, v_h) - j(u, u) - (f, v_h - u), \\
A_3 = j(u, u_h) - j(u_h, u_h) + j(u_h, u) - j(u, u), \\
A_4 = j(u_h, v_h) - j(u, v_h) + j(u, u) - j(u_h, u).
\]
Let us estimate each of the four terms. We estimate the terms $A_1$ and $A_2$ as

\begin{align}
|A_1| & \leq M \|u - u_h\|_V \|u - v_h\|_V, \\
|A_2| & \leq (M \|u\|_V + \|f\|_V) \|u - v_h\|_V \\
& \quad + \|p(u_v)\|_{L^2(\Omega_s)} \|u - v_h\|_{(L^2(\Omega_s))^d}.
\end{align}

To estimate the term $A_3$, we have

\begin{align}
A_3 &= \int_{\Gamma_3} (p(u_v) - p(u_{h\nu})) (u_{h\nu} - u_v) \, da \\
& \quad + \int_{\Gamma_3} \mu (p(u_v) - p(u_{h\nu})) \left( |u_h\tau| - |u\tau| \right) \, da,
\end{align}

and since

\[ \int_{\Gamma_3} (p(u_v) - p(u_{h\nu})) (u_{h\nu} - u_v) \, da \leq 0, \]

it follows that

\[ |A_3| \leq L_p \|\mu\|_{L^\infty(\Gamma_3)} \int_{\Gamma_3} |u_{h\nu} - u_v| \left| |u_h\tau| - |u\tau| \right| \, da. \]

Hence, using (2.7), we deduce

\[ |A_3| \leq d^2 L_p \|\mu\|_{L^\infty(\Gamma_3)} \|u - u_h\|_V^2. \]

Similarly, we have

\begin{align}
A_4 &= \int_{\Gamma_3} (p(u_{h\nu}) - p(u_v)) (v_{h\nu} - u_v) \, da \\
& \quad + \int_{\Gamma_3} \mu (p(u_{h\nu}) - p(u_v)) \left( |v_h\tau| - |u\tau| \right) \, da,
\end{align}

which implies

\begin{align}
|A_4| & \leq L_p \left( 1 + \|\mu\|_{L^\infty(\Gamma_3)} \right) \|u - u_h\|_{(L^2(\Omega_s))^d} \|u - v_h\|_{(L^2(\Omega_s))^d} \\
& \quad \leq d^2 L_p \left( 1 + \|\mu\|_{L^\infty(\Gamma_3)} \right) \|u - u_h\|_V \|u - v_h\|_{(L^2(\Omega_s))^d}. \]

\]
Now, we use the bounds (5.4)–(5.7) in (5.3) and apply Young’s inequality
\[ ab \leq \frac{\alpha a^2}{2} + \frac{b^2}{2\alpha}, \quad \forall \alpha > 0, \forall a, b \in \mathbb{R}, \]
to obtain
\[ \| u - u_h \|_V^2 \leq c \left\{ \| u - v_h \|_V^2 + \| u - v_h \|_{(\mathbb{L}^2(\Gamma_3))'}^2 \right. \]
\[ + \left. \| u - v_h \|_V + \| p(u) \|_{\mathbb{L}^2(\Gamma_3)} \| u - v_h \|_{(\mathbb{L}^2(\Gamma_3))'} \right\}. \]

So the inequality (5.2) follows from inequality (5.8). \( \square \)

In the following we derive an error estimate for a finite element approximation of (2.14). The error estimation and the convergence analysis are based on the inequality (5.2). Indeed, for definiteness let \( \Omega \) be a polygonal domain in \( \mathbb{R}^2 \). Then the boundary \( \Gamma \) consists of line segments. We also assume that the sets \( \Gamma_1 \cap \Gamma_2, \Gamma_1 \cap \Gamma_3 \) and \( \Gamma_2 \cap \Gamma_3 \) contain only a finite number of points. Write
\[ \Gamma_3 = \bigcup_{i=1}^{I} \Gamma_{3,i} \]
with each \( \Gamma_{3,i} \) being a line segment. We define the finite-element space \( V_h \) as
\[ V_h = \left\{ v_h \in V \cap [\mathbb{C}^0(\overline{\Omega})]^2 : v_h |_T \in [P_1(T)]^2, \forall T \in \mathcal{T}_h \right\} \]
where \( \mathcal{T}_h \) is a regular triangulation on \( \overline{\Omega} \) (see [5]) such that \( \overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} T \).

We suppose that each triangulation is compatible with the boundary decomposition \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \); that is, each point where the boundary condition changes is a node of a set \( T \). We recall that there exists an operator (see [5]) \( \pi_h : V \to V_h \) such that
\[ \| \pi_h v - v \|_V \leq c_1 h \| v \|_{(H^2(\Omega))^2}, \]
\[ \| \pi_h v - v \|_{\mathbb{L}^2(\Gamma_3)}^2 \leq c_1 h^2 \| v \|_{(H^2(\Omega))^2}, \]
for every \( v \in (H^2(\Omega))^2 \cap V \), where \( \pi_h v \) denotes, as usual, the \( V_h \)-interpolant of the function \( v \in V \). In the sequel we define the non-empty closed convex set \( K_h \) by
\[ K_h = \{ v_h \in V_h : v_{h|_T} \leq g \text{ on } \Gamma_3 \}. \]
We remark that $K_h \subset K$. Now, to obtain an error estimate, we need to make additional assumptions on solution regularity. Indeed, assume

\[(5.10) \quad u \in (H^2(\Omega))^2 \cap K,\]

then we have the following proposition.

**Proposition 5.2.** Suppose that (5.9) and (5.10) hold. Then

\[(5.11) \quad \|u - u_h\|_V \leq c h^{\frac{3}{4}} \left( \|u\|_{(H^2(\Omega))^2} + \|p(u_h)\|_{L^2(\Gamma_3)}^{\frac{1}{2}} \right),\]

where $c$ is a positive constant independent of $h$.

**Proof.** Using (5.8), we obtain

\[(5.12) \quad \|u - u_h\|_V \leq c \left\{ \|u - \pi_h u\|_V + \|u - \pi_h u\|_{(L^2(\Gamma_3))^2} \right\}
\]\
\[\quad + \|p(u_h)\|_{L^2(\Gamma_3)}^{\frac{1}{2}} \|u - \pi_h u\|_{(L^2(\Gamma_3))^2}^{\frac{1}{2}} \right\}.
\]

Therefore, by (5.9) and (5.12), the estimate (5.11) follows. \[\Box\]

The discrete problem (5.1) can be approximated by a fixed-point iteration method. Choosing an initial guess $u_h^0 \in K_h$ bounded uniformly, we define recursively a sequence $\{u_h^n\} \subset K_h$ by

\[(5.13) \quad (Au_h^{n+1}, v_h - u_h^{n+1})_V + j_T(u_h^n, v_h) - j_T(u_h^n, u_h^{n+1}) 
\geq (f, v_h - u_h^{n+1})_V, \quad \forall v_h \in K_h,
\]

where the operator $A$ was introduced in the proof of Theorem 3.1.

We have the following convergence result.

**Theorem 5.3.** Under the assumptions of Theorem 3.1, the iteration method (5.13) converges

\[\|u_h^n - u_h\|_V \to 0 \quad \text{as} \quad n \to \infty.\]

Moreover, there exists a constant $\delta \in (0,1)$ such that

\[(5.14) \quad \|u_h^n - u_h\|_V \leq c\delta^n.
\]
Proof. Take \( v_h = u_h^{n+1} \) in (5.1), we get the inequality

\[
(Au_h, u_h^{n+1} - u_h)_V + j_\tau (u_h, u_h^{n+1}) - j_\tau (u_h, u_h) \geq (f, u_h^{n+1} - u_h)_V,
\]

and take \( v_h = u_h \) in (5.13), we also get the inequality

\[
(Au_h^{n+1}, u_h - u_h^{n+1})_V + j_\tau (u_h^{n}, u_h) - j_\tau (u_h^{n}, u_h^{n+1}) \geq (f, u_h - u_h^{n+1})_V.
\]

Adding the resulting inequalities (5.15) and (5.16), we obtain

\[
(Au_h - Au_h^{n+1}, u_h - u_h^{n+1})_V \\
\leq j_\tau (u_h, u_h^{n+1}) - j_\tau (u_h, u_h) + j_\tau (u_h^{n}, u_h) - j_\tau (u_h^{n}, u_h^{n+1}).
\]

Then as in the proof of Lemma 3.3, we can derive the estimate

\[
\|u_h^{n+1} - u_h\|_V \leq \frac{d^2}{m} L_p \|\mu\| L_{\infty} (\Gamma_3) \|u_h^n - u_h\|_V.
\]

Now let

\[
\delta = \frac{d^2}{m} L_p \|\mu\| L_{\infty} (\Gamma_3).
\]

Under the condition

\[
\|\mu\| L_{\infty} (\Gamma_3) < \mu_0,
\]

\( \delta < 1 \), and the estimate (5.14) follows.

6 Conclusion In this work we have considered the problem of frictional contact between an elastic body and a foundation. The contact is modelled with normal compliance such that the penetration is restricted with unilateral constraint, and the associated version of Coulomb's law of dry friction. We present a weak formulation of the problem, and establish existence and uniqueness result using arguments of variational inequalities and fixed point theory. Moreover, we show the continuous dependence of the solution on the contact conditions. We also study the finite element approximation of the problem and derive an error estimate. Finally, we introduce an iterative method to solve the resulting finite element problem.
REFERENCES


Laboratoire de Systèmes Dynamiques, Faculté de Mathématiques, USTHB, BP 32 EL ALIA, Bab-Ezzouar, 16111, Algérie
E-mail address: ttouzaline@yahoo.fr