A PREY-PREDATOR MODEL WITH
ALTERNATIVE PREY: MATHEMATICAL
MODEL AND ANALYSIS

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ABSTRACT. This paper describes a prey predator model with type II functional response where harvesting of each species is taken into consideration. Discrete type gestational delay of predators is incorporated and its effect on the dynamical behaviour of the model system is analyzed. It is shown that harvesting efforts may be used to control the prey predator system considered. We have also studied the Hopf bifurcation of the model system in the neighbourhood of the co-existing equilibrium point considering delay as a variable bifurcation parameter. Numerical simulations are given to verify the analytical results.

1 Introduction Mathematical modelling of exploitation of biological resources is still a very interesting field of research. Mathematical modelling is frequently an evolving process. Systematic mathematical analysis can often lead to better understanding of bioeconomic models. The exposed discrepancies in turn lead to the necessary modifications. The final model may or may not be free of any significant discrepancies but the analysis of the final model can thus be expected to reveal important and nontrivial features of the system. In the last few decades, interest has been growing steadily in the designing and studying of mathematical models of population interactions. There is a tendency among researchers to rush into the analysis of an existing model without knowing the background or motivation of such model, or rush into the final models and indulge in the analysis of certain mathematical properties of such models. There are several such literatures which mainly in-
clude prey predator interactions with the effects of time delay. Toaha et al. [15] studied analytically the necessary conditions of harvesting to ensure the existence of the equilibrium points and the stabilities of a model which consists of time delay and two populations subjected to constant effort of harvesting. They proved that time delay can induce instability and a Hopf bifurcation may appear. They also related the stable equilibrium point for the model with harvesting to profit function problem and found that there exists a critical value of the effort that maximizes the profit and the equilibrium point also remains stable. The effect of constant rate harvesting on the dynamics of predator-prey systems has been investigated by Dai and Tang [4], Myerscough et al. [13] and Xiao and Ruan [16], and they obtained very rich and interesting dynamical behaviours. Feng [6] considered a differential equation system with diffusion and time delays which models the dynamics of predator-prey interactions within three biological species. The results obtained by him explicitly present the effects of all the environmental data (growth rates and interaction rates) on the ultimate bounds of the three biological species and numerical simulations of the model were given to demonstrate the pattern of dynamics (extinction, persistence, and permanence) in the ecological model. Nakaoka et al. [14] examined a Lotka-Volterra predator-prey system with delays. They concluded that a positive equilibrium of the system is globally asymptotically stable for small delays and the system exhibits some chaotic behavior when time delay becomes large. They analytically determined critical values of time delay through which the system undergoes a Hopf bifurcation. The dynamics of a prey-predator system, where predator population has two stages, juvenile and adult with harvesting were modelled using a system of delay differential equation by Kar [9]. He concluded that both the delay and harvesting effort may play a significant role on the stability of the system. Yafia [17] established an explicit algorithm for determining the direction of the Hopf bifurcation and the stability or instability of the bifurcating branch of periodic solutions through considering a model with one delay. A model in which the revenue is generated from fishing and the growth of fish depends upon the plankton which in turn grows logistically is developed by Dhar et al. [5]. They further formulated the model with delay in digestion of plankton by fish and found the threshold value of conversional parameter for Hopf-bifurcation. Kar and Matsuda [10], and Kar and Pahari [11] discussed the predator prey model with time delay and analyzed the effects of time delay on model dynamics such as the time delay may change the stability of equilibrium points and even cause a switching of stabilities.
From the above literature survey it is very relevant to point out that no attempt has been made to study the dynamics of a prey predator system in the presence of an alternative resource by taking into account functional response in the model. In the present paper we, therefore, model the dynamics of a prey-predator system where predator population partially dependent on a logistically growing resource with functional response. Also we have considered a gestation delay of predator population and harvesting of both the species. The main objective of this paper is to examine the effects of both fishing effort and delay on the model dynamics. Conditions for the existence of nonnegative equilibria, criterion for their local and global stability behaviour are obtained. It is also proved that the time delay can cause a stable equilibrium to become unstable and even a switching of stabilities may occur.

2 The model formulation

We consider a prey-predator model and it is assumed that the dynamics of both prey and predator population follow logistic law of growth. Let us assume $x$ and $y$ are, respectively, the size of prey and predator population at time $t$. The predator population consumes the prey population with the functional response, known as Holling type II functional response and contributes to its growth rate. It is considered that prey and predator population are continuously harvested and hence the growth of prey and predator population are directly affected by harvesting. Keeping these aspects in view, the dynamics of the system may be governed by the following system of differential equations

$$\frac{dx}{dt} = r_1 x \left(1 - \frac{x}{K_1}\right) - \frac{mxy}{1 + ax} - h_1(t),$$

$$\frac{dy}{dt} = r_2 y \left(1 - \frac{y}{K_2}\right) + \frac{amxy}{1 + ax} - h_2(t),$$

where $r_1$ and $r_2$ are, respectively, the intrinsic growth rate of prey and predator population. $K_1$ and $K_2$ are their respective environmental carrying capacities, $m$ is the conversion of biomass constant, $a$ is Michaelis-Menten constant, $h_1(t)$ and $h_2(t)$ are, respectively, the harvesting from prey and predator population at time $t$, $\alpha$ is the maximum value of per capita reduction rate of $x$ due to $y$.

Let us now consider this harvested prey predator system with time delay due to gestation. It is inherently assumed that all the metabolic energy a predator obtains through its food used for logistic growth which...
ultimately enhance the predator population. Here the predator population
consumes the prey population at a constant rate $\alpha$, but the reproduc-
tion of predators after predating the prey population is not instantaneous
thus it will be incorporated by some time lag required for gestation of
predators. Suppose the time interval between the moments when an
individual prey is killed and the corresponding biomass is added to the
predator population is considered as the time delay $\tau$. Under this
assumption the model becomes

\[
\begin{align*}
\frac{dx}{dt} &= r_1x \left(1 - \frac{x}{K_1}\right) - \frac{mxy}{1 + ax} - h_1(t), \\
\frac{dy}{dt} &= r_2y \left(1 - \frac{y}{K_2}\right) + \frac{\alpha mxy(t - \tau)y(t - \tau)}{1 + ax(t - \tau)} - h_2(t).
\end{align*}
\]

The functional form of harvest is generally considered using the phrase
catch-per-unit-effort (CPUE) hypothesis (Clark [3]) to describe an
assumption that catch per unit effort is proportional to the stock level.
Thus we consider $h_1 = q_1E_1x$ and $h_2 = q_2E_2y$, where $E_1$ and $E_2$
are, respectively, the harvesting effort used to harvest from prey and
predator population, $q_1$ and $q_2$ are, respectively, the catchability co-
efficient of prey and predator population. Here in this model we assume
$q_1 = q_2 = 1$.

Thus the final model becomes

\[
\begin{align*}
\frac{dx}{dt} &= r_1x \left(1 - \frac{x}{K_1}\right) - \frac{mxy}{1 + ax} - E_1x, \\
\frac{dy}{dt} &= r_2y \left(1 - \frac{y}{K_2}\right) + \frac{\alpha mxy(t - \tau)y(t - \tau)}{1 + ax(t - \tau)} - E_2y,
\end{align*}
\]

with initial conditions $x(\theta), y(\theta) \geq 0$, $\theta \in [-\tau, 0)$, and $x(0) > 0$, $y(0) > 0$.

For a nonlinear delay system, there are two types of stabilities: ab-
solute stability (independent of the delay) and conditional stability (de-
pending on the delay). Here we consider two cases separately with and
without time delay.

**Case 1. $\tau = 0$**

For $\tau = 0$, the system (1) becomes

\[
\begin{align*}
\frac{dx}{dt} &= r_1x \left(1 - \frac{x}{K_1}\right) - \frac{mxy}{1 + ax} - E_1x, \\
\frac{dy}{dt} &= r_2y \left(1 - \frac{y}{K_2}\right) + \frac{\alpha mxy}{1 + ax} - E_2y,
\end{align*}
\]
Clearly, \( P_0(0,0) \) is the trivial equilibrium point of the system (2) and all other possible equilibrium points of the system (2), considered in the first quadrant of \( x-y \) plane, are as follows:

\[
\begin{align*}
P_1(\bar{x},0) & \quad \text{where } \bar{x} = \frac{K_1}{r_1}(r_1 - E_1), \\
P_2(0,\bar{y}) & \quad \text{where } \bar{y} = \frac{K_2}{r_2}(r_2 - E_2), \\
P_3(x^*,y^*) & \quad \text{where } y^* = \frac{K_2(r_2 - E_2)}{r_2} + \frac{K_2\alpha mx^*}{r_2(1 + ax^*)},
\end{align*}
\]

and \( x^* \) satisfying the following cubic equation

\[
(3) \quad Ax^3 + Bx^2 + Cx^1 + D = 0
\]

where

\[
\begin{align*}
A &= a^2 r_1 \frac{K_1}{r_1} > 0, \\
B &= a \left( \frac{2r_1}{K_1} - a(r_1 - E_1) \right), \\
C &= \frac{r_1}{K_1} + \frac{amK_2(r_2 - E_2) + K_2\alpha m^2}{r_2} - 2a(r_1 - E_1), \\
D &= \frac{mK_2(r_2 - E_2)}{r_2} - (r_1 - E_1).
\end{align*}
\]

For the existence of the equilibrium points \( P_1(\bar{x},0) \) and \( P_2(0,\bar{y}) \) it is assumed that

\[
(4) \quad E_1 < r_1 \quad \text{and} \quad E_2 < r_2.
\]

Equation (3) has a unique positive solution if any one of the following inequalities holds:

(i) \( B < 0, \quad C < 0, \quad D < 0, \)
(ii) \( B > 0, \quad C < 0, \quad D < 0, \)
(iii) \( B > 0, \quad C > 0, \quad D < 0. \)

Condition (i) holds if \( E_1 < r_1 - M_1 \) where

\[
M_1 = \max \left[ \frac{2r_1}{aK_1} \frac{1}{2a} \left( \frac{r_1}{K_1} \right) \frac{amK_2(r_2 - E_2) + K_2\alpha m^2}{r_2} + \frac{mK_2(r_2 - E_2)}{r_2} \right].
\]
Condition (ii) holds if
\[
\left( r_1 - \frac{2r_1}{aK_1} \right) < E_1 < (r_1 - M_2)
\]
where
\[
M_2 = \max \left[ \frac{1}{2a} \left( \frac{r_1 K_1}{K_1} + \frac{amK_2(r_2 - E_2) + K_2am^2}{r_2} \right), \frac{mK_2(r_2 - E_2)}{r_2} \right].
\]
Condition (iii) holds if
\[
(r_1 - M_3) < E_1 < \left( r_1 - \frac{mK_2(r_2 - E_2)}{r_2} \right)
\]
where
\[
M_3 = \max \left[ \frac{2r_1}{aK_1}, \frac{1}{2a} \left( \frac{r_1 K_1}{K_1} + \frac{amK_2(r_2 - E_2) + K_2am^2}{r_2} \right) \right].
\]

Once we get the unique positive solution of \(x^*\) from equation (3) it is easy to get the interior positive solution of \(y^*\).

Thus, for a set of parameters of model system (2) different equilibria exist for different levels of the harvesting efforts.

Before studying the stability of the model system, we show that the solutions of the model system are bounded in a finite region initiating at \((x(0), y(0))\).

**Theorem 1.** All the solutions of the system (2) which start in \(\mathbb{R}_+^2\) are uniformly bounded.

**Proof.** Let \((x(t), y(t))\) be any solution of the system with positive initial conditions. Now we define the function \(W = x + \alpha y\). Therefore, time derivative gives
\[
\frac{dW}{dt} = \frac{dx}{dt} + \frac{1}{\alpha} \frac{dy}{dt}
\]
\[
= r_1 x \left( 1 - \frac{x}{K_1} \right) - \frac{mxy}{1 + ax} - E_1 x + \frac{r_2 y}{1} \left( 1 - \frac{y}{K_2} \right) + \frac{mxy}{1 + ax} - \frac{E_2 y}{\alpha}
\]
\[
= r_1 x - \frac{r_1}{K_1} x^2 - E_1 x + \frac{r_2 y}{\alpha K_2} y^2 - \frac{E_2}{\alpha} y.
\]
Now for each \( v > 0 \), we have
\[
\frac{dW}{dt} + v W = r_1 x - \frac{r_1 K_1}{r_1} x^2 - E_1 x + \frac{r_2}{\alpha} y - \frac{r_2}{\alpha K_2} y^2 - \frac{E_2}{\alpha} y + v x + v y.
\]
Thus,
\[
\frac{dW}{dt} + v W \leq u,
\]
where
\[
u = \frac{r_1}{4 K_1} \left( \frac{E_1 K_1}{r_1} - K_1 - \frac{v K_1}{r_1} \right)^2 + \frac{r_2}{4 \alpha K_2} \left( \frac{E_2 K_2}{r_2} - K_2 - \frac{v K_2}{r_2} \right)^2.
\]
Applying the theory of differential inequality (Birkoff and Rota [1]), we obtain
\[
0 \leq W(x, y) \leq \frac{u}{v} + \frac{W(x(0), y(0))}{e^{vt}} \text{ and for } t \to \infty, \ 0 \leq W \leq \frac{u}{v}.
\]
Thus, all the solutions of the system (4) enter into the region
\[
B = \left\{ (x, y) : 0 \leq W \leq \frac{u}{v} + \epsilon, \text{ for any } \epsilon > 0 \right\}.
\]
This completes the proof.

3 Stability analysis  In this section we discuss the stability of different equilibrium points of the system (2), by computing the variational matrix at various equilibrium points and using the Routh-Hurwitz criterion [12].

At \( P_0(0, 0) \), we find the roots of the characteristic equation are \( (r_1 - E_1) \) and \( (r_2 - E_2) \). Thus, the trivial equilibrium \( P_0(0, 0) \) is stable if \( E_1 > r_1 \) and \( E_2 > r_2 \).

For the boundary equilibrium \( P_1(\bar{x}, 0) \), the eigenvalues of the characteristic equation are \( \lambda_1^1 \) and \( \lambda_1^2 \) where
\[
\lambda_1^1 = -(r_1 - E_1), \quad \lambda_1^2 = (r_2 - E_2) + \frac{\alpha m K_1 (r_1 - E_1)}{r_1 + a K_1 (r_1 - E_1)}.
\]
It is noted that \( P_1(\bar{x}, 0) \) is stable if
\[
E_1 < r_1 \quad \text{and} \quad E_2 > r_2 + \frac{\alpha m K_1 (r_1 - E_1)}{r_1 + a K_1 (r_1 - E_1)}.
\]
Again, for the boundary equilibrium \( P_2(0, \hat{y}) \), the eigenvalues of the characteristic equation are \( \lambda_1^2 \) and \( \lambda_2^2 \) where

\[
\lambda_1^2 = -(r_2 - E_2), \quad \lambda_2^2 = (r_1 - E_1) - \frac{mK_2}{r_2}(r_2 - E_2).
\]

Therefore, \( P_2(0, \hat{y}) \), is stable if

\[
E_2 < r_2 \quad \text{and} \quad E_1 > r_1 - \frac{mK_2}{r_2}(r_2 - E_2).
\]

It is observed that even in the absence of prey the predator may exists in its equilibrium level and this is happened due to the alternative source of prey.

The characteristic equation for the interior equilibrium \( P_3(x^*, y^*) \) is given by

\[
(5) \quad \mu^2 + f(x^*(E_1, E_2), y^*(E_1, E_2))\mu + g(x^*(E_1, E_2), y^*(E_1, E_2)) = 0,
\]

where

\[
f(x^*(E_1, E_2), y^*(E_1, E_2)) = \frac{r_1x^*}{K_1} + \frac{r_2y^*}{K_2} - \frac{may^*x^*}{(1 + ax^*)^2}
\]

and

\[
g(x^*(E_1, E_2), y^*(E_1, E_2)) = \frac{r_1r_2x^*y^*}{K_1K_2} + \frac{mx^*y^*}{(1 + ax^*)^2} \left( \frac{am}{1 + ax^*} - \frac{ar_2y^*}{K_2} \right).
\]

Therefore, stability of the interior equilibrium \( P_3(x^*, y^*) \) depends on the sign of \( f(x^*(E_1, E_2), y^*(E_1, E_2)) \) and \( g(x^*(E_1, E_2), y^*(E_1, E_2)) \). If

1. \((H_1)\) \( f(x^*(E_1, E_2), y^*(E_1, E_2)) > 0 \) and \( g(x^*(E_1, E_2), y^*(E_1, E_2)) > 0 \), then \( P_3(x^*, y^*) \) is a stable node or a stable spiral.
2. \((H_2)\) \( f(x^*(E_1, E_2), y^*(E_1, E_2)) < 0 \) and \( g(x^*(E_1, E_2), y^*(E_1, E_2)) > 0 \), then \( P_3(x^*, y^*) \) is an unstable node or an unstable spiral.
3. \((H_3)\) \( g(x^*(E_1, E_2), y^*(E_1, E_2)) < 0 \), then \( P_3(x^*, y^*) \) is a saddle point.
4. \((H_4)\) \( f(x^*(E_1, E_2), y^*(E_1, E_2)) = 0 \), then there are some limit cycles.

Now we want to study the global behaviour of the system (2). We state the following theorem.
Theorem 2. Let us define $\rho(E_1, E_2) = \frac{r_1}{E_1} - ay^*$ and assume that the positive equilibrium is locally stable. If $\rho(E_1, E_2) > 0$, then it is globally asymptotically stable.

Proof. To show the global stability of system (2), we define a Lyapunov function as follows:

$$V(x, y) = L_1 \left( x - x^* - x^* \ln \frac{x}{x^*} \right) + L_2 \left( y - y^* - y^* \ln \frac{y}{y^*} \right)$$

where $L_1$ and $L_2$ are positive constants to be determined in the subsequent steps. It can be easily verified that the function $V$ is zero at the equilibrium $P_3(x^*, y^*)$ and is positive for all other positive values of $x$ and $y$.

The time derivative of along the trajectories of (2) is

$$\frac{dV}{dt} = L_1 \frac{x - x^*}{x} \frac{dx}{dt} + L_2 \frac{y - y^*}{y} \frac{dy}{dt}$$

$$= L_1 \frac{x - x^*}{x} \left[ r_1 x \left( 1 - \frac{x}{K_1} \right) - \frac{mxy}{1 + ax} - E_1 x \right]$$

$$+ L_2 \frac{y - y^*}{y} \left[ r_2 y \left( 1 - \frac{y}{K_2} \right) + \frac{mxy}{1 + ax} - E_2 y \right]$$

$$= - \left[ \frac{r_1 L_1}{K_1} (x - x^*)^2 + \frac{r_2 L_2}{K_2} (y - y^*)^2 \right]$$

$$- m \left[ (L_1 - L_2 \alpha) (x - x^*) (y - y^*) + L_1 a (x - x^*) (y^* - y^*) \right]$$

$$= - \left[ \left( \frac{r_1 L_1}{K_1} - L_1 a y^* \right) (x - x^*)^2 + \frac{r_2 L_2}{K_2} (y - y^*)^2 \right.$$  

$$+ m (L_1 - L_2 \alpha + L_1 a x^*) (x - x^*) (y - y^*) \right].$$

Choosing $L_1 = 1$ and $L_2 = \frac{1 + ax^*}{\alpha}$, a little algebraic manipulation yields

$$\frac{dv}{dt} = - \left[ \left( \frac{r_1}{K_1} - a y^* \right) (x - x^*)^2 + \frac{r_2 (1 + ax^*)}{K_2 \alpha} (y - y^*)^2 \right].$$

Thus, if $\rho(E_1, E_2) > 0$, then $\frac{dv}{dt} < 0$ and $\frac{dv}{dt} = 0$ if and only if $x = x^*$ and $y = y^*$. This completes the proof. \qed
The condition $\rho(E_1, E_2) > 0$ is plotted in the parametric space $(E_1, E_2)$ corresponding to the numerical values: $r_1 = 1.8$, $r_2 = 2.5$, $K_1 = 30$, $K_2 = 20$, $m = 0.5$, $a = 3$, $\alpha = 1.5$, $E_2 = 2$.

The plot of $\rho(E_1, E_2)$ indicates that the assumption $\rho(E_1, E_2) > 0$ makes some sense.

In the next section we will study the uniform persistence of the system (2) using the average Lyapunov function. By the permanence or persistence of a system, we mean that all the species are present and none of them will go to extinction. The persistence of a system have been studied by Freedman [7] and Butler et al. [2] and some other authors.

**Theorem 3.** The system (2) is uniformly persistent if $E_1 < r_1$, $E_2 < r_2$ and $(r_1 - E_1) > \frac{mK_1}{r_2}(r_2 - E_2)$.

**Proof.** To show the uniform persistence of system (2). We define an average Lyapunov function as follows:

$$\phi(X) = x^c y^d$$

where $c$ and $d$ are positive constants, $\phi(x)$ is non-negative function defined in $\mathbb{R}_+^2$ and $X$ is a function of $x$ and $y$. After differentiating, we have

$$\theta(X) = \frac{\dot{\phi}(x)}{\phi(x)} = \frac{\dot{x}}{x} + \frac{\dot{y}}{y} = c \left[ r_1 \left( 1 - \frac{x}{K_1} \right) - \frac{my}{1 + ax} - E_1 \right]$$
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\[ + d \left[ r_2 \left(1 - \frac{y}{K_2} \right) + \frac{amx}{1+ax} - E_2 \right]. \]

Hence, the uniform persistence exists, if there exists \( c \) and \( d \), such that \( \theta(X) \) is positive at \( P_0(0,0) \), \( P_1(x,0) \) and \( P_2(0,y) \). Now

(i) \[ \theta(P_0(0,0)) = c(r_1 - E_1) + d(r_2 - E_2) > 0 \]
if \( E_1 < r_1 \) and \( E_2 < r_2 \),

(ii) \[ \theta(P_1(x,0)) = d \left[ (r_2 - E_2) + \frac{amK_1(r_1 - E_1)}{r_1 + aK_1(r_1 - E_1)} \right] > 0 \]
if \( E_1 < r_1 \) and \( E_2 < r_2 \),

(iii) \[ \theta(P_2(0,y)) = c \left[ (r_1 - E_1) - \frac{mK_2}{r_2}(r_2 - E_2) \right] > 0 \]
if \( (r_1 - E_1) > \frac{mK_2}{r_2}(r_2 - E_2) \).

Thus, if \( E_1 < r_1, E_2 < r_2 \) and \( (r_1 - E_1) > \frac{mK_2}{r_2}(r_2 - E_2) \), then \( \theta(X) > 0 \) for all equilibrium points. This completes the proof. \( \square \)

4 Bifurcation in the parametric space \((E_1, E_2)\) In this section, we investigate the existence and stability of interior equilibrium in the parametric space \((E_1, E_2)\). For this purpose the conditions for existence of interior equilibrium, local stability and global stability are plotted in the parametric space \((E_1, E_2)\). After plotting the aforesaid conditions in the parametric space \((E_1, E_2)\), it is observed that the parametric space \((E_1, E_2)\) can be divided into three regions which consist of eleven subregions. Let us now check in how many regions the interior equilibrium is locally asymptotically stable. To analyze the system behaviour we assign numerical value of the parameters as follows: \( r_1 = 1.8, \ r_2 = 2.5, \ K_1 = 30, \ K_2 = 20, \ m = 0.5, \ a = 3, \ \alpha = 1.5 \). For the purpose of computations we mainly use the software MATLAB 7.0.

**Region-A:** \[ \{(E_1, E_2) \in \mathbb{R}_+^2 : I \cup IX \cup X \cup XI \} \]

\[ I : f(x^*(E_1, E_2), y^*(E_1, E_2)) > 0, \quad g(x^*(E_1, E_2), y^*(E_1, E_2)) > 0, \]
\[ B < 0, \quad C > 0, \quad D > 0. \]
\[ IX : f(x^*(E_1, E_2), y^*(E_1, E_2)) > 0, \quad g(x^*(E_1, E_2), y^*(E_1, E_2)) > 0, \]
FIGURE 2: Existence and stability regions of the equilibrium points in the parametric space \((E_1, E_2)\) corresponding to the numerical values as mentioned earlier.

\[ B < 0, \quad C < 0, \quad D < 0. \]

\[ X : f(x^*(E_1, E_2), y^*(E_1, E_2)) > 0, \quad g(x^*(E_1, E_2), y^*(E_1, E_2)) > 0, \quad B < 0, \quad C < 0, \quad D > 0. \]

\[ XI : f(x^*(E_1, E_2), y^*(E_1, E_2)) > 0, \quad g(x^*(E_1, E_2), y^*(E_1, E_2)) > 0, \quad B < 0, \quad C > 0, \quad D > 0. \]

In each subregion of the region \(A\) we find that \(\text{Trace } J < 0\) and \(\det J > 0\), therefore, the interior equilibrium is a stable node or a stable spiral. The following figures ensure the stability of interior equilibrium for an arbitrarily taken point \((E_1, E_2)\) in each subregion of the region \(A\).

Phase plane trajectories of Figures 3, 4, 5 and 6 are drawn with different initial levels. Trajectories clearly indicate that the interior equilibrium is asymptotically stable.

**Region-B:** \(\{(E_1, E_2) \in \mathbb{R}_+^2 = II \cup III\}\)

\[ II : f(x^*(E_1, E_2), y^*(E_1, E_2)) > 0, \quad g(x^*(E_1, E_2), y^*(E_1, E_2)) < 0, \quad B < 0, \quad C > 0, \quad D > 0. \]

\[ III : f(x^*(E_1, E_2), y^*(E_1, E_2)) < 0, \quad g(x^*(E_1, E_2), y^*(E_1, E_2)) > 0, \quad B > 0, \quad C > 0, \quad D > 0. \]
FIGURE 3: For \((E_1, E_2) = (0.72289, 1.5261)\) taken from subregion-I.

FIGURE 4: For \((E_1, E_2) = (0.56225, 2.49)\) taken from subregion-II.
FIGURE 5: For \((E_1, E_2) = (0.16064, 2.008)\) taken from subregion-X.

FIGURE 6: For \((E_1, E_2) = (1.4458, 2.49)\) taken from subregion-XI.
Interior equilibrium does not exist in the region-B.

**Region-C:** \( \{(E_1, E_2) \in \mathbb{R}^2_+ = IV \cup V \cup VI \cup VII \cup VIII\} \) where

- **IV:** \( f(x^*(E_1, E_2), y^*(E_1, E_2)) < 0, \quad g(x^*(E_1, E_2), y^*(E_1, E_2)) < 0, \quad B > 0, \quad C > 0, \quad D < 0. \)
- **V:** \( f(x^*(E_1, E_2), y^*(E_1, E_2)) > 0, \quad g(x^*(E_1, E_2), y^*(E_1, E_2)) < 0, \quad B > 0, \quad C > 0, \quad D < 0. \)
- **VI:** \( f(x^*(E_1, E_2), y^*(E_1, E_2)) < 0, \quad g(x^*(E_1, E_2), y^*(E_1, E_2)) < 0, \quad B > 0, \quad C > 0, \quad D < 0. \)
- **VII:** \( f(x^*(E_1, E_2), y^*(E_1, E_2)) < 0, \quad g(x^*(E_1, E_2), y^*(E_1, E_2)) < 0, \quad B < 0, \quad C < 0, \quad D < 0. \)
- **VIII:** \( f(x^*(E_1, E_2), y^*(E_1, E_2)) > 0, \quad g(x^*(E_1, E_2), y^*(E_1, E_2)) < 0, \quad B < 0, \quad C < 0, \quad D < 0. \)

It is found in each subregion of the region C that for an arbitrary value of \((E_1, E_2)\) the predator population is reached to its extinction though the prey population exists. Following figures ensure the analysis.

**FIGURE 7:** For \((E_1, E_2) = (2.8514, 3.1727)\) taken from subregion-V.
Variation of prey and predator biomass in Figures 7, 8, 9 and 10 with the increasing time clearly indicate that the predator population goes to extinction.
5 Numerical study of the system behaviour

In order to study the numerical behaviour of the model system, the parameters of the system (2) are assigned by the following numerical values: \( r_1 = 1.8, r_2 = 2.5, K_1 = 30, K_2 = 20, m = 0.5, \alpha = 3, \alpha = 1.5, E_1 = 1.2, E_2 = 2 \).

Figure 11 depicts the isoclines of prey and predator population when the fishing effort used to harvest predator population is considered to be constant \( E_2 = 2 \). The isoclines are obtained for different fishing efforts used to harvest prey population, i.e., \( E_1 = 0.6, E_1 = 0.8, E_1 = 1, E_1 = 1.2, E_1 = 1.4 \). When fishing effort associated to the predator population is assumed to be constant, the exploitation of the prey population is simultaneously done by fishing agency and a constant rate by the predator population. As a result the equilibrium point for the prey population gradually decreases when the respective fishing effort used to harvest prey population is simultaneously increased. Consequently the prey population collapses for a small time interval compare to the situation where both the population simultaneously harvested.

Figure 12 depicts the isoclines of prey and predator population when the fishing effort used to harvest prey population is considered to be constant \( E_1 = 1.2 \). The isoclines are obtained for different fishing efforts used to harvest predator population, i.e., \( E_2 = 1.5, E_2 = 1.75, E_2 = 2, E_2 = 2.25, E_2 = 2.5 \). It is observed that the predator isoclines are gradually decreases with the increasing fishing effort used to harvest predator population though effort on prey population is constant. This
is due to the fact that the reproduction of predators after predating the prey population is not instantaneous. Thus the predator population is ultimately reached to its extinction limit for a particular fishing effort.

FIGURE 11: Isoclines of prey and predator population when $E_2 = 2$ is constant.

FIGURE 12: Isoclines of prey and predator population when $E_1 = 1.2$ is constant.
6 Analysis of delay model We shall now investigate the dynamics of delay system (1).

Let \((x^*, y^*)\) be the only interior equilibrium of system (1) and let \(X = x - x^*\) and \(Y = y - y^*\) be the perturbed variables. After removing
the nonlinear terms and by using equilibrium conditions, we obtain the linear variational system as

\[
\frac{dX}{dt} = r_1 \left( 1 - \frac{2x^*}{K_1} \right) - \frac{m y^*}{(1 + ax^*)^2} - E_1 X - \frac{m x^*}{1 + ax^*} Y,
\]

\[
\frac{dY}{dt} = \frac{\alpha my^*}{(1 + ax^*)^2} X(t - \tau) + \left[ r_2 \left( 1 - \frac{2y^*}{K_2} \right) - E_2 \right] Y
+ \frac{\alpha mx^*}{1 + ax^*} Y(t - \tau).
\]

The characteristic equation of the linearized system (6) is a transcendental equation of the following form

\[
\Delta(\lambda, \tau) = \lambda^2 + P \lambda + Q + (R \lambda + S)e^{-\lambda \tau} = 0
\]

where

\[
P = \frac{m y^*}{(1 + ax^*)^2} + E_1 + E_2 - r_1 \left( 1 - \frac{2x^*}{K_1} \right) - r_2 \left( 1 - \frac{2y^*}{K_2} \right),
\]

\[
Q = \left( r_2 \left( 1 - \frac{2y^*}{K_2} \right) - E_2 \right) \left( r_1 \left( 1 - \frac{2x^*}{K_1} \right) - \frac{m y^*}{(1 + ax^*)^2} - E_1 \right),
\]

\[
R = -\frac{\alpha m x^*}{1 + ax^*},
\]

\[
S = \frac{\alpha m x^*}{1 + ax^*} \left( r_1 \left( 1 - \frac{2x^*}{K_1} \right) - E_1 \right).
\]

It is known that the steady state is asymptotically stable if all roots of the characteristic equation (7) have negative real parts. Now we can find the condition for nonexistence of delay induced instability by using the following theorem given by Gopalsamy [8].

**Theorem 4.** Necessary and sufficient conditions for \(P_3(x^*, y^*)\) to be locally asymptotically stable in the presence of a time delay \(\tau\) are

(i) the real parts of all the roots of \(\Delta(\lambda, \tau) = 0\) are negative;

(ii) for all real \(\omega\) and for \(\tau > 0\), \(\Delta(i \omega, \tau) \neq 0\), where \(i = \sqrt{-1}\).

If \(i \omega\) is a root of equation (7), then

\[-\omega^2 + Q + S \cos \omega \tau + R \omega \sin \omega \tau + i(P \omega + R \cos \omega \tau - S \sin \omega \tau) = 0.\]
Separating real and imaginary parts we get

\[ S \cos \omega \tau + R \omega \sin \omega \tau = \omega^2 - Q, \]
\[ R \omega \cos \omega \tau - S \sin \omega \tau = -P \omega. \]

(8)

Squaring and adding both of equation (8) we finally have

\[ \omega^4 - (R^2 - P^2 + 2Q)\omega^2 + (Q^2 - S^2) = 0. \]

(9)

Now by Descartes’ rule of sign, for an equation with real coefficient, it is evident from (9) that if

\[ (P^2 - R^2 - 2Q) > 0 \text{ and } (Q^2 - S^2) > 0, \]

then equation (9) does not have positive roots. Therefore, characteristic equation (7) does not have purely imaginary roots. Since conditions in ensure that all roots of equation (5) have negative real parts, by Theorem 4 stated above, it follows that the roots of equation (7) have negative real parts as well.

Again, it is observed that if

\[ (Q^2 - S^2) < 0, \]

then equation (9) has a unique positive root \( \omega_0^2 \). Substituting \( \omega_0^2 \) in (8) and solving for \( \tau \), we get

\[ \tau_n = \frac{1}{\omega_0} \cos^{-1} \left\{ \frac{S(\omega_0^2 - Q) - P R \omega_0^2}{R^2 \omega_0^4 + S^2} \right\} + \frac{2n\pi}{\omega_0}, \quad n = 0, 1, 2, \ldots. \]

(10)

Again, if

\[ (R^2 - P^2 + 2Q) > 0, \quad (Q^2 - S^2) > 0 \text{ and } (R^2 - P^2 + 2Q)^2 > 4(Q^2 - S^2), \]

then equation (9) has two positive roots

\[ \omega_{\pm}^2 = \frac{(R^2 - P^2 + 2Q) \pm \sqrt{(R^2 - P^2 + 2Q)^2 - 4(Q^2 - S^2)}}{2}. \]

Substituting \( \omega_{\pm}^2 \) in (9) we get

\[ \tau_k = \frac{1}{\omega_{\pm}} \cos^{-1} \left\{ \frac{S(\omega_{\pm}^2 - Q) - P R \omega_{\pm}^2}{R^2 \omega_{\pm}^4 + S^2} \right\} + \frac{2k\pi}{\omega_{\pm}}, \quad k = 0, 1, 2, \ldots. \]
From (7) we obtain

\[
\frac{d\lambda}{dt} = \frac{(R\lambda^2 + S\lambda)e^{-\lambda\tau}}{2\lambda + P + (R - R\lambda\tau - S\tau)e^{-\lambda\tau}}.
\]

Thus, we have

\[
\text{sign} \left\{ \frac{d}{d\tau} \text{Re} (\lambda) \right\}_{\lambda=i\omega} = \text{sign} \{ \omega^2[2\omega^2 - (R^2 - P^2 + 2Q)] \} = (R^2 - P^2 + 2Q)^2\omega^2 - 2(Q^2 - S^2).
\]

Let us now compute the following transversality conditions:

\[
\left[ \frac{d}{d\tau} \text{Re} (\lambda) \right]_{\tau = \tau_0, \omega = \omega_0} = \left[ \omega^4 - (Q^2 - S^2) \right] > 0,
\]

\[
\left[ \frac{d}{d\tau} \text{Re} (\lambda) \right]_{\tau = \tau^*_{+}, \omega = \omega_0} = \frac{1}{2} \left\{ ((R^2 - P^2 + 2Q)^2 - 4(Q^2 - S^2)) + \sqrt{(R^2 - P^2 + 2Q)^2 - 4(Q^2 - S^2)} \right\} > 0,
\]

\[
\left[ \frac{d}{d\tau} \text{Re} (\lambda) \right]_{\tau = \tau^*_{-}, \omega = \omega_0} = \frac{1}{2} \sqrt{(R^2 - P^2 + 2Q)^2 - 4(Q^2 - S^2)} \times \left[ \sqrt{(R^2 - P^2 + 2Q)^2 - 4(Q^2 - S^2)} - (R^2 - P^2 + 2Q) \right] < 0.
\]

Hence, the transversality conditions are verified. We summarize the above analysis by the following theorem.

**Theorem 5.** (i) If \((H_1)\) and \((H_5)\) hold, then all roots of equation (7) have negative real parts for all \(\tau \geq 0\).

(ii) If \((H_1)\) and \((H_6)\) hold, then equilibrium \((x^*, y^*)\) is asymptotically stable for \(\tau < \tau_0\) and unstable for \(\tau > \tau_0\). Further, as \(\tau\) increases through \(\tau_0\), \((x^*, y^*)\) bifurcates into small amplitude periodic solutions, where

\[
\tau_0 = \frac{1}{\omega_0} \cos^{-1} \left\{ \frac{S(\omega_0^2 - Q) - PR\omega_0^2}{R^2\omega_0^4 + S^2} \right\}.
\]

(iii) If \((H_1)\) and \((H_7)\) hold, then there is a positive integer \(m\) such that there are \(m\) switches from stability to instability and to stability.
other words, when \( \tau \in [0, \tau_0^+), (\tau_0^-, \tau_1^+), \ldots, (\tau_{m-1}^-, \tau_m^+), \) the equilibrium is stable and when \( \tau \in [\tau_0^+, \tau_0^-), (\tau_1^+, \tau_1^-), \ldots, (\tau_{m-1}^+, \tau_m^-), \) the equilibrium is unstable. Therefore, there are bifurcations at \( (x^*, y^*) \) when
\[
\tau = \frac{\pi}{k}, \quad k = 0, 1, 2, 3, \ldots. 
\]
The positive equilibrium is unstable for \( \tau > \tau_m^+ \) where \( m \) is the minimum nonnegative integer at which, for the first time
\[
\tau_{m+1}^+ = \frac{\theta^+ + (m + 1)2\pi}{\omega_+} \leq \tau_m^- = \frac{\theta^- + m2\pi}{\omega_-},
\]
i.e., \( m \) is the minimum nonnegative integer satisfying
\[
m \geq \frac{(\theta^+ + 2\pi)\omega_- - \theta^- \omega_+}{2\pi(\omega_+ - \omega_-)}.
\]
It is clear from the above discussion that the delay has a great impact on the dynamics of the system.

As an example, we consider the system
\[
\frac{dx}{dt} = 1.5x \left( 1 - \frac{x}{80} \right) - \frac{0.5xy}{1 + 2x} - 1.2x,
\]
\[
\frac{dy}{dt} = 1.2y \left( 1 - \frac{y}{60} \right) + \frac{3.9 \times 0.5x(t - \tau)y(t - \tau)}{1 + 2x(t - \tau)} - 2y.
\]
When \( \tau = 0 \), we see from Figure 15, that the equilibrium \((7.792, 5.92)\) is a stable node. Now we wish to study the effect of the delay \( \tau \) on the dynamics of the model. By using Theorem 5, it is found that there is a critical value \( \tau_0 = 0.27 \). At \( \tau_0 = 0.27 \) the interior equilibrium \((x^*, y^*) = (5.594, 4.746)\). The equilibrium \((x^*, y^*) = (5.594, 4.746)\) is asymptotically stable for \( \tau < 0.27 \) becomes unstable for \( \tau > 0.27 \), and there is a small amplitude periodic solution.

Therefore, if we consider \( \tau = 0.17 \), then it is observed from Figures 16 and 17 that \( P_3(x^*, y^*) \) is locally asymptotically stable and the populations \( x, y \) converge to their steady states in finite time. Now, if we gradually increase the value of \( \tau \) keeping other parameters fixed, then we have got a critical value \( \tau = 0.27 \) such that \( P_3(x^*, y^*) \) loses its stability as \( \tau \) passes through \( \tau_0 \). Figures 18 and 19 clearly show the result. It is also noted that if we consider \( \tau = 0.37 \), then it is evident from Figures 20 and 21 that the positive equilibrium \( P_3(x^*, y^*) \) is unstable and there is a periodic orbit near \( P_3(x^*, y^*) \).

The effects of stability of equilibrium points of the model system with the increasing time and particular harvesting efforts are shown in
FIGURE 15: Phase plane trajectories of prey and predator biomass beginning with different initial levels. Trajectories clearly indicate that the interior equilibrium $(x^*, y^*) = (7.792, 5.92)$ is asymptotically stable.

FIGURE 16: Variation of prey and predator biomass with the increasing time when $\tau = 0.17$. 
FIGURE 17: Phase plane trajectories of prey and predator biomass beginning with different initial levels when $\tau = 0.17$.

FIGURE 18: Variation of prey and predator biomass with the increasing time when $\tau = 0.27$. 
FIGURE 19: Phase plane trajectories of prey and predator biomass beginning with different initial levels when $\tau = 0.27$.

FIGURE 20: Variation of prey and predator biomass with the increasing time when $\tau = 0.37$. 
Figures 22–27 for a fixed value of \( \tau \). It is clear from Figures 22–27 that for a given value of harvesting effort has great influence on stability of equilibrium points of the model system. Figure 22 depicts that prey and predator population remains stable if we consider the harvesting efforts \( E_1 = 1.4 \) and \( E_2 = 2.2 \) for \( \tau = 0.17 \) but in this figure it is noted that the predator population tends to its extinction. It is evident from Figure 23 that for the harvesting efforts \( E_1 = 2 \) and \( E_2 = 3 \) both the populations reach to their extinction though \( \tau = 0.17 \) remains same. Again, it is observed that for the harvesting efforts \( E_1 = 0.5, E_2 = 1.5 \) and \( \tau = 0.17 \), the populations are not only stable and but also exist with a greater steady state value.

We have already found that the interior equilibrium becomes unstable for \( \tau = 0.37 \) but from Figure 25 it is observed that interior equilibrium becomes stable for the changed harvesting efforts \( E_1 = 1.5 \) and \( E_2 = 2.5 \) though \( \tau = 0.37 \) remains same. Here in this figure it is also observed that the predator population has already reached to its extinction and with the increasing time prey population also reaches to its extinction where as in Figure 26 it is clearly shown that though both the populations are collapsed for \( E_1 = 2, E_2 = 3 \) and \( \tau = 0.37 \), in finite time. It is noted from Figure 27 that for \( E_1 = 0.5, E_2 = 1.5 \) and \( \tau = 0.37 \), the populations exist for finite time and clearly stable, though it is unstable for \( E_1 = 1.2 \) and \( E_2 = 2 \) (see Figures 20 and 21).
FIGURE 22: Variation of prey and predator biomass with the increasing time when $\tau = 0.17$ and $E_1 = 1.4, E_2 = 2.2$.

FIGURE 23: Variation of prey and predator biomass with the increasing time when $\tau = 0.17$ and $E_1 = 2, E_2 = 3$. 
FIGURE 24: Variation of prey and predator biomass with the increasing time when \( \tau = 0.17 \) and \( E_1 = 0.5, E_2 = 1.5 \).

FIGURE 25: Variation of prey and predator biomass with the increasing time when \( \tau = 0.37 \) and \( E_1 = 1.5, E_2 = 2.5 \).
FIGURE 26: Variation of prey and predator biomass with the increasing time when $\tau = 0.37$ and $E_1 = 2, E_2 = 3$.

FIGURE 27: Variation of prey and predator biomass with the increasing time when $\tau = 0.37$ and $E_1 = 0.5, E_2 = 1.5$. 
7 Conclusion  In the present paper we have considered a deterministic model of a prey-predator system having alternative prey and harvesting of each species. We have stated and proved several results giving criterion for the existence of several equilibrium points, stability and bifurcations for the model system. We have obtained important results for stability of the differential equations and the delay differential equation model system in order to study the effect of gestation delay on various dynamic behaviours with different levels of harvesting efforts. It is observed that harvesting efforts have the ability to derive a stable equilibrium point to an unstable one and an unstable equilibrium to a stable one. Here, it is noted that time delay also plays an important role to the dynamics of the system.

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