INCOMPLETE INFORMATION AND LEXICOGRAPHIC CHOICES: SOME CONTINUITY RESULTS

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ABSTRACT. We construct a model where choices are defined by preference relations induced by incomplete information on a set of commodities, whose elements are given by finite dimensional vectors of characteristics. Imposing natural conditions on the choice set allows for preferences to be continuously and additively representable on it. Lexicographic choices are shown to follow from continuous preferences and to be representable by continuous utility functions that are both additive and lexicographic. Since the lexicographic quality of choice derives from the incomplete information assumption, lexicographic preferences should not be excluded from economic equilibrium theory unless choices are made under perfect information.

1 Introduction There exists a non-studied incompatibility between the economic general equilibrium literature and the mathematical theory of choice caused by divergences in the assumed preference structures of decision makers. The reasons for these divergences are better understood if we consider first the similarities between both theories. Commodities, the objects of choice, are generally modeled in both cases following the consumer demand theory developed by Lancaster [22], where sets of commodities are defined as convex subsets of a finite dimensional compact metric space. In particular, a commodity is identified with a n-dimensional vector \((x_1, x_2, \ldots, x_n)\), where every \(x_i\) stands for the value of the \(i\)-th characteristic of the commodity. Anderson et al. [6] provide a comprehensive introduction to the, so called, address approach that

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defines commodities as points in a real space of characteristics. In order to define a complete order or preference relation, and hence a utility function, on the commodity set all its elements, and the entire characteristic vector of each one of them, must be known and comparable. That is, decision makers have all possible information describing the set of available commodities before making a choice.

The economic general equilibrium literature defines sufficient conditions under which, after including some additional assumptions on the endowments and preferences of decision makers, a competitive equilibrium exists, and studies its optimality properties. General (competitive) equilibrium models with differentiated commodities were introduced in the economic literature by Mas-Colell [24] and Jones [21]. On the other hand, the mathematical theory of choice, usually applied to psychological choice models with multi-attribute differentiated commodities, is mainly based on lexicographic choice rules (see [23, 28, 29, 30]). It is widely known that lexicographic choice rules, induced by lexicographic orders defined on a particular consumption set, cannot be represented by continuous utility functions (see, for example, [25]). Choices generated by lexicographic preferences are regarded as rational, but not further considered in economic analysis. The obvious reason for such omission is that exchange economies cannot be based on the lexicographic choice rules defined by decision theorists since a competitive equilibrium, with its corresponding welfare theorems, will not generally exist.

The purpose of this paper is to develop a theoretical framework where suitable conditions are studied to obtain lexicographic preferences which are representable by continuous utility functions on the given consumption set, allowing for the existence of an economic equilibrium. To this end, we relax the perfect information assumption common to the economic and choice theoretical literatures, whenever strategic settings are not considered. Both theories assume that decision makers have complete information regarding the characteristics of all commodities in the consumption set. However, perfect information should not be assumed unless information (as a commodity) is freely available in the economic system, i.e., costless, and decision makers are able to assimilate and consider all the characteristics of all the commodities in the consumption set before making a choice.

The first point was addressed by Grossman and Stiglitz [19], in a model of financial arbitrage where decision makers must pay to obtain precise information about the state of the system. Indeed, this idea has been further formalized by Allen [3, 4, 5], who treats information as an economic commodity, endowed with a topological structure, over
which a demand is defined by decision makers depending on their budget constraints. Therefore, the demand functions of decision makers on the set of commodities are conditioned by their information sets.

On the other hand, consider the case of an economic system where information is perfect and freely available. If the ability of decision makers to assimilate and use all available information to define their choices is limited, by imposing an exogenous bound on their information processing capacity—a common restriction in models of bounded rationality—, decision makers must develop heuristic optimal stopping rules defining incomplete information sets on which to base their decisions. In other words, decision makers will either exclude certain commodities of their consumption set from their information set, or consider only a subset on characteristics as informationally relevant. An ample theoretical and experimental analysis of bounded rationality and heuristic choice mechanisms is presented by Gigerenzer and Selten [18].

We study a model where, due to any of the previous reasons, commodities are chosen under incomplete information. Using partial information sets, lexicographic choices stem from continuous preference relations assigned a priori on the consumption set. The lexicographic choices obtained are shown to be representable by utility functions which turn out to be not only continuous but also both lexicographic and additive. As the lexicographic quality of choice is derived from the incompleteness of the information sets, we conclude that, from an economic point of view, the omission of lexicographic choice rules from the general equilibrium literature is fully justifiable only if these choices are made under perfect information.

The paper proceeds as follows. Sections 2 and 3 deal with the standard notation and basic assumptions needed to develop the model. Sections 4 and 5 define the partial information sets and induced maps and preferences, while Section 6 studies the conditions on the choice set that guarantee the continuity of the associated utility functions. The main results regarding the lexicographic nature of information and the properties of the corresponding preferences and utilities are presented in Section 7.

2 Preliminaries Let $X$ be a nonempty set. A preference relation on $X$ is a binary relation $R \subseteq X \times X$ satisfying

reflexivity: $\forall x \in X, \ (x, x) \in R$;

The assumption of reflexivity is actually redundant, since it is implied by that of completeness.
completeness: \( \forall x, y \in X, (x, y) \in R \lor (y, x) \in R \);
transitivity: \( \forall x, y, z \in X, (x, y) \in R \land (y, z) \in R \implies (x, z) \in R \).

Preference relations are usually denoted by the symbols \( \succeq \) or \( \geq \). We usually write \( x \succeq y \) in place of \( (x, y) \in \succeq \) and read: \( x \) is preferred or indifferent to \( y \).

The strict preference and the indifference relations associated to a preference relation \( \succeq \) are defined as follows:

\[
x \succ y \overset{def}{=} x \succeq y \land y \not\succeq x \quad \text{and} \quad x \sim y \overset{def}{=} x \succeq y \land y \succeq x.
\]

We read \( x \succ y \) as \( x \) is preferred to \( y \), while \( x \sim y \) is read \( x \) is indifferent to \( y \).

From the definition it is clear that preference relations are complete preorders. Such preference relations are usually referred to as rational in the economic literature. Hence, all the preference relations in this paper are rational.

A utility function representing \( \succeq \) is any function \( u : X \to \mathbb{R} \) such that

\[
\forall x, y \in X, \ x \succeq y \iff u(x) \geq u(y).
\]

We will denote by \((X, \tau)\) the set \( X \) when endowed with the topology \( \tau \).

If \((X, \tau)\) is a connected separable topological space, the existence of a continuous utility function representing \( \succeq \) is guaranteed if \( \succeq \) is also continuous with respect to \( \tau \) (see [11, 12]; Proposition 3.C.1 in [25] states the result when \( X \subseteq \mathbb{R}_+^L \), with \( L \in \mathbb{N} \)).

A preference relation \( \succeq \) on \( X \) is continuous with respect to \( \tau \) if for every \( x \in X \), the subsets

\[
[x, \to) = \{ y \in X : y \succeq x \} \quad \text{and} \quad (\leftarrow, x] = \{ y \in X : x \succeq y \}
\]

are closed subsets of \((X, \tau)\). In other words, \( \succeq \) is continuous with respect to \( \tau \) if \( \tau \) is finer than the order topology induced by \( \succeq \) (see [31]).

In general, the order topology induced by a complete preorder \( \geq \) on a set \( X \), denoted by \( \tau_{\geq} \), is the topology having as a subbase all subsets \((x, \to) = \{ y \in X : y \geq x \}\) and \((\leftarrow, x)] = \{ y \in X : x \geq y \}\), where \( x \in X \). All the intervals of the form \((a, b)\), where \( a, b \in X \) and \( a < b \), are open subsets of \((X, \tau_{\geq})\), while all \( \geq \)-rays of the form \((\leftarrow, x)] = \{ y \in X : x \geq y \}\) and \([x, \to) = \{ y \in X : y \geq x \}\), where \( x \in X \), are closed subsets of \((X, \tau_{\geq})\). In particular, for every \( x \in X \), the singleton \( \{ x \} \) is a closed...
subset of $X$. As a result, the order topology induced by $\geq$ is the smallest topology on $X$ with respect to which $\geq$ is continuous. Unless further assumptions are considered, this does not necessarily imply that the preference relation is representable by a continuous utility function. The lexicographic order $\succ_{\text{Lex}}$ on $\mathbb{R}^2$ is continuous with respect to the order topology $\tau_{\text{Lex}}$, but it is well-known that it cannot be represented by a continuous utility function (see Example 3.C.1 in [25]).

Given two natural numbers $i, n \in \mathbb{N}$, $i \leq n$ will be a short for $\{1, 2, \ldots, n\}$. The Cartesian product of $n$ nonempty sets $X_1, \ldots, X_n$ will be denoted by $\prod_{i \leq n} X_i$.

Henceforth, all Cartesian products are to be considered non-trivial (that is, $n \geq 2$).

The following definition recalls the notions of “induced preference relation,” “mutually preferentially independent factors,” and “essential factor” (see also Definition 4 in [12] and Section 4.1 in [17]).

**Definition 2.1.** Let $\prod_{i \leq n} X_i$ be the Cartesian product of $n$ nonempty sets endowed with a preference relation $\succ_i$. Fix $A \subseteq \{1, 2, \ldots, n\}$ and $(z_i)_{i \in A} \in \prod_{i \in A} X_i$. The preference relation given by $(z_i)_{i \in A}$, denoted by $\succ_{(z_i)_{i \in A}}$, is the preference relation induced by $\succ_i$ on $\prod_{i \in A} X_i$ when the element of $X_i$ is equal to $z_i$ for all $i \in A$. Then $n$ factors $X_1, X_2, \ldots, X_n$ are mutually preferentially independent if for every $A \subseteq \{1, 2, \ldots, n\}$ the preference relation $\succ_{(z_i)_{i \in A}}$ is independent of $((z_i)_{i \in A})$, that is,

$$\forall (z_i)_{i \in A}, (z'_i)_{i \in A} \in \prod_{i \in A} X_i, \quad \succ_{(z_i)_{i \in A}} \equiv \succ_{(z'_i)_{i \in A}}.$$ 

The factor $X_i$ is essential if, for some $(z_j)_{j \neq i}$, not all its elements are indifferent for the preference relation $\succ_{(z_j)_{j \neq i}}$ induced over $X_i$.

Let $\prod_{i \leq n} X_i$ be the Cartesian product of $n$ nonempty sets. We recall that a function $u : \prod_{i \leq n} X_i \to \mathbb{R}$ is called additive if there exist $n$ real-valued functions, $u_1, \ldots, u_n$, respectively, on $X_1, \ldots, X_n$, such that $u = \sum_{i \leq n} u_i$, that is,

$$\forall (x_1, \ldots, x_n) \in \prod_{i \leq n} X_i, \quad u(x_1, \ldots, x_n) = \sum_{i \leq n} u_i(x_i).$$

A preference relation $\succ$ on $\prod_{i \leq n} X_i$ is called additive if it can be represented by an additive utility function.

If $u : \prod_{i \leq n} X_i \to \mathbb{R}$ is an additive function, then for every nonempty set $Y$ and every function $f : Y \to \prod_{i \leq n} X_i$, we have $(u \circ f) = \sum_{i \leq n} (u_i \circ f)$. 

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where \( f_i : Y \rightarrow X_i \) defined by \( f_i(y) = \text{\( i \)-th coordinate of } f(y) \), is the \( i \)-th coordinate function of \( f \). Clearly, \((u \circ f)\) satisfies an additive-like property. Thus, abusing notation but in order to be formally consistent, we introduce the following extension of the notion of additivity.

**Definition 2.2.** Let \( \prod_{i \leq n} X_i \) be the Cartesian product of \( n \) nonempty sets, \( Y \) be a nonempty set and \( u : \prod_{i \leq n} X_i \rightarrow \mathbb{R} \). Given a function \( f : Y \rightarrow \prod_{i \leq n} X_i \), the composite function \( u \circ f \) will be called *additive* if \( u \) is additive.

In the case when \( \succeq \) is a preference relation of \( \prod_{i \leq n} X_i \) represented by \( u \), a preference relation \( \succeq_Y \) on \( Y \) will be called *additive* if there exists \( f : Y \rightarrow \prod_{i \leq n} X_i \) such that \( u \circ f \) represents \( \succeq_Y \) and it is additive.

Let \( \prod_{i \leq n} X_i \) be the Cartesian product of \( n \) nonempty sets endowed with a preference relation \( \succeq \). A utility function \( u : \prod_{i \leq n} X_i \rightarrow \mathbb{R} \) representing \( \succeq \) is called *lexicographic* (see Section 4.3 in [17]) if there exist \( n \) real-valued functions \( v_1, \ldots, v_n \), respectively, on \( X_1, \ldots, X_n \), such that

\[
\forall (x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \prod_{i \leq n} X_i, \\
u(x_1, \ldots, x_n) \succ u(y_1, \ldots, y_n) \\
\iff (v_1(x_1), \ldots, v_n(x_n)) \succ_{\text{Lex}} (v_1(y_1), \ldots, v_n(y_n)).
\]

For the sake of completeness, recall that for every

\[
(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{R}^n, \\
(a_1, \ldots, a_n) \succ_{\text{Lex}} (b_1, \ldots, b_n) \iff a_1 > b_1 \lor [a_1 = b_1, a_2 > b_2] \lor \cdots \lor [(\forall i \leq n-1, a_i = b_i), a_n > b_n].
\]

In the lexicographic case, the order of the factors \( X_1, \ldots, X_n \), that is the enumeration of them, is tacitly very important. In a preference sense, the existence of a lexicographic utility roughly means that \( X_1 \) dominates \( X_2 \), \( X_2 \) dominates \( X_3 \), and so on. Such an order also implies that, in a costly information gathering environment with a binding budget constraint, decision makers would start acquiring information about the dominant factors, proceeding with the remaining ones only if indifferent among the former factors.

In the case when \( \prod_{i \leq n} X_i \) is the product of \( n \) finite sets, appropriate but strong independence conditions must hold for the existence of lexicographic utilities to imply that of additive utilities (see Section 4.3 in
The converse is neither generally true, nor has been adequately studied. We shall investigate some natural conditions under which additive utilities happen to be lexicographic, and vice versa.

As in the additive case, we need to generalize the notion of lexicographity to all utility functions, not only those defined on Cartesian products.

**Definition 2.3.** Let $\prod_{i \leq n} X_i$ be the Cartesian product of $n$ nonempty sets endowed with a preference relation $\succeq$ represented by the utility function $u : \prod_{i \leq n} X_i \to \mathbb{R}$. Let $Y$ be a nonempty set and $f : Y \to \prod_{i \leq n} X_i$. The composite function $u \circ f$ will be called lexicographic if $u$ is lexicographic.

A preference relation $\succeq_Y$ on $Y$ will be called lexicographic if there exists $f : Y \to \prod_{i \leq n} X_i$ such that $u \circ f$ represents $\succeq_Y$ and it is lexicographic.

For additional topological concepts and standard results the reader may refer to [16] or [26].

### 3 Main assumptions

Let $\mathcal{G}$ denote the set of all goods, or commodities. Fix $n \geq 2$. For every $i \leq n$, let $X_i$ represent the set of all possible variants for the $i$-th characteristic or attribute of any commodity in $\mathcal{G}$. An element $x_i^G \in X_i$ specifies the $i$-th characteristic of a given commodity $G \in \mathcal{G}$. Let $X$ denote the Cartesian product $\prod_{i \leq n} X_i$.

We will refer to each $X_i$ as the $i$-th characteristic factor space and to $X$ as the characteristic space. Assuming perfect information on the set of all goods, the bijective map $\varphi : \mathcal{G} \to X$ defined by $\varphi(G) = (x_1^G, x_2^G, \ldots, x_n^G)$, for every $G \in \mathcal{G}$, identifies every good with a $n$-tuple. Note that it is possible for $X$ to contain vectors of characteristics giving place to goods that do not necessarily exist. That is, decision makers may have their preferences defined also over non-existing goods. While the given incomplete information framework provides an intuitive justification for this assumption, no theoretical constraint prevents, a priori, a decision maker from defining her preferences over “any” set of goods.

**Assumption 1.** For every $i \leq n$, $X_i$ is a connected and separable topological space.

**Assumption 2.** $X$ is endowed with the product topology, $\tau_p$, and a preference relation $\succeq$ continuous with respect to $\tau_p$.

**Assumption 3.** $\{X_i : i \leq n\}$ is a system of mutually preferentially independent factors and each factor $X_i$ is essential.
For every $A \subseteq \{1, 2, \ldots, n\}$, let $A^c = \{1, 2, \ldots, n\} \setminus A$. Denote by $\succeq_{A^c}$, $\succ_{A^c}$ and $\sim_{A^c}$ the preference, strict preference and indifference relations induced on $\prod_{i \in A^c} X_i$ by any point $(z_i)_{i \in A} \in \prod_{i \in A} X_i$. In particular, for every $i \leq n$, $\succeq_i$, $\succ_i$ and $\sim_i$ stand for the preference, strict preference and indifference relations induced on $X_i$ by any point $(z_j)_{j \neq i} \in \prod_{j \neq i} X_j$.

By Assumptions 1, 2 and 3 (see Theorem 3 in [12]), there exists a continuous utility function $u : X \rightarrow \mathbb{R}$ representing $\succeq$ and defined by

$$
\forall x = (x_1, \ldots, x_n) \in X,
\quad u(x) = \sum_{i \leq n} u_i(x_i),
$$

where for every $i \leq n$, $u_i : X_i \rightarrow \mathbb{R}$ is a continuous utility function representing $\succeq_i$.

Through the remaining of the paper, $u, u_1, \ldots, u_n$ are fixed such that the above property is satisfied. \footnote{An alternative approach to the problem of representation of preference relations on sets of goods could involve the use of a totally ordered semigroup in place of our Cartesian product $X$. Regarding the existence conditions of additive continuous utility functions in this alternative setting, the reader may refer, among others, to [9] and [10].}

**Definition 3.1.** Given any function $f : \mathcal{G} \rightarrow X$, the $f$-induced preference relation on $\mathcal{G}$, denoted by $\succeq_f$, is defined as follows:

$$
\forall G_1, G_2 \in \mathcal{G},
\quad G_1 \succeq_f G_2 \Leftrightarrow f(G_1) \succeq f(G_2).
$$

Hence, given a function $f : \mathcal{G} \rightarrow X$,

$$
\forall G_1, G_2 \in \mathcal{G},
\quad G_1 \succeq_f G_2 \iff u(f(G_1)) \geq u(f(G_2)).
$$

It is easy to check that $\succeq_f$ is a preference relation, which justifies the definition above.

In particular, the (bijective) identification $\varphi$ induces the preference relation $\succeq_{\varphi}$ on $\mathcal{G}$, defined as follows. For every $G, G' \in \mathcal{G}$,

$$
G \succeq_{\varphi} G' \iff \varphi(G) \succeq \varphi(G'),
$$

that is,

$$
G \succeq_{\varphi} G' \iff u(x^G_1, \ldots, x^G_n) \geq u(x'^G_1, \ldots, x'^G_n).
$$
Assumption 4. \( G \) is endowed with the weak topology\(^3\) induced by the function \( \varphi \), \( \tau_{\varphi} \), and the preference relation \( \succeq_{\varphi} \).

The topology \( \tau_{\varphi} \) on \( G \) is finer than the order topology induced by \( \succeq_{\varphi} \), as well as the topology \( \tau_p \) on \( X \) is finer than the order topology induced by \( \succeq \). In particular, the preference relation \( \succeq_{\varphi} \) is continuous with respect to \( \tau_{\varphi} \). The map \( \varphi \) is also open (images of open subsets of \((G, \tau_{\varphi})\) are open subsets of \((X, \tau_p)\)). The following proposition is now easy to check.

**Proposition 3.2.** The map \( \varphi \) is an order homeomorphism of \((G, \tau_{\varphi})\) into \((X, \tau_p)\).

For every \( i \leq n \), let \( \varphi_i \) denote the \( i \)-th coordinate function of \( \varphi \), that is,
\[
\varphi_i : G \to X_i : x \mapsto x_i^G.
\]

**Corollary 3.3.** For every \( i \leq n \), \( \varphi_i \) is continuous.

**Proof.** By the continuity of \( \varphi = \prod_{i \leq n} \varphi_i \) (Theorem 18.4 in [26]). \( \square \)

**Proposition 3.4.** \( u \circ \varphi \) is an additive continuous utility function representing \( \succeq_{\varphi} \).

**Proof.** The function \( u \circ \varphi \) is additive by Definition 2.2 and continuous by construction (both \( u \) and \( \varphi \) are continuous). \( \square \)

The following assumption guarantees the existence of maximal and minimal elements for the relation \( \succeq \).

**Assumption 5.** For every \( i \leq n \), \( X_i \) is a compact space.

By Assumption 5, the product space \((X, \tau_p)\) is compact (apply the well-known Tychonof Theorem). Thus, by the Extreme Value Theorem, there exist \( x_M, x_m \in X \) such that \( x_M \succeq x \succeq x_m \) for all \( x \in X \). \( x_M \) and \( x_m \) are \( \succeq \)-maximal and a \( \succeq \)-minimal element of \( X \), respectively.  \(^4\)

\(^3\)That is, the coarsest topology on \( G \) with respect to which \( \varphi \) is continuous.

\(^4\)Many authors have focused their attention on the problem of the existence of maximal (and hence minimal) elements of binary relations on compact sets and several interesting approaches can be found in the literature. A short history on the subject appears in [1], where a characterization of the existence of maximal elements of acyclic binary relations generalizing sufficient conditions introduced by Bergstrom [7] and Walker [32] is presented. A survey of the sufficient conditions proposed to obtain maximal elements of binary relations on compact sets is given in [8], while variations of results by Bergstrom [7], Walker [32], and Sonnenschein [27] for generalized lexicographic relations are the main results of Hougaard and Tvede [20].
The existence of maximal and minimal elements for the product \( X \) implies the existence of maximal and minimal elements for each factor \( X_i \). It easily follows from the mutually preferentially independence of the factors (see Assumption 3) that the \( i \)-th coordinates of \( x_M \) and \( x_m \), that is \( (x_M)_i \) and \( (x_m)_i \), are a \( \succeq_i \)-maximal and a \( \preceq_i \)-minimal element, respectively. 5

According to the identification of \( G \) with \( X \) (via the map \( \varphi \)), under perfect information the decision maker must be able to find at least one maximal (minimal) good delivering her the highest (lowest) possible utility. These goods are not necessarily represented by \( x_M = ((x_M)_1, \ldots, (x_M)_n) \) or \( x_m = ((x_m)_1, \ldots, (x_m)_n) \), respectively, since the existence of \( \varphi^{-1}(x_M) \) or \( \varphi^{-1}(x_m) \) is not guaranteed. As previously noted, the decision maker is able to define her preferences and utility over goods described by particular vectors of characteristics that do not need to correspond to any existing one.

Finally, we assume the decision maker to be endowed with a subjective probability (density) function over each characteristic factor \( X_i \). Abusing notation, each \( X_i \) can be considered a random variable.

**Assumption 6.** For every \( i \leq n \), \( \mu_i : X_i \to [0, 1] \) is a nonatomic probability density function if \( X_i \) is absolutely continuous, and a non-degenerate probability function if \( X_i \) is discrete. 6

Clearly, we do not consider atomic probability density functions or degenerate probability functions, since they do not necessarily induce risk on the choices made by the decision maker.

The functions \( \mu_1, \ldots, \mu_n \) must be interpreted as the subjective “beliefs” of the decision maker. For \( i \leq n \), \( \mu_i(Y_i) \) is the subjective probability that a randomly observed good from \( G \) displays an element \( x_i \in Y_i \subseteq X_i \) as its \( i \)-th characteristic. 7

To complete our initial setting, we need to define a set of values assigned by the decision maker to the unknown components of the vector of characteristics. This is particularly useful if she had to compare

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5Following [1], we could have assumed that for every \( i \leq n \) and every \((z_j)_{j \neq i} \in \prod_{j \neq i} X_j \), \( X_i \) was endowed with a \( \succeq_{(z_j)_{j \neq i}} \)-upper compact topology such that \( \succeq_{(z_j)_{j \neq i}} \) was upper semi-continuous. This assumption which is weaker than Assumption 5 guarantees the existence of a maximal element for each \( X_i \), and consequently for the product space \( X \), without necessarily involving the utility function \( u \). However, this approach would have caused the initial mathematical context and the corresponding notations to be uselessly more complicated considering the main purposes of the present paper.

6For these and other concepts commonly used in statistical decision theory; see [13].

7Note that the functions \( \mu_1, \ldots, \mu_n \) can be assumed either independent or correlated, without this fact affecting our results.
between two or more goods whose sets of known characteristics differ. Consider the case where the decision maker must compare the $i$-th characteristic between two goods, using the preorder defined by $\succeq_i$, but only has information on one of the goods. Following the standard economic theory of choice under uncertainty (see [25]), the decision maker must assign to the unknown $i$-th characteristic the $i$-th certainty equivalent value induced by her subjective probability (density) function $\mu_i$.

**Definition 3.5.** Let $i \leq n$. The *certainty equivalent* of $\mu_i$ and $u_i$, denoted by $c_i$, is a characteristic in $X_i$ that the decision maker is indifferent to accept in place of the expected one to be obtained through $(\mu_i, u_i)$.

In other words, for every $i \leq n$, $c_i$ is an element of $X_i$ whose utility $u_i(c_i)$ equals the expected value of $u_i$. Hence,

$$c_i \in u_i^{-1}\left(\int_{X_i} u_i(x_i) \mu_i(x_i) \, dx_i\right)$$

if $X_i$ is absolutely continuous, and

$$c_i \in u_i^{-1}\left(\sum_{x_i \in X_i} u_i(x_i) \mu_i(x_i)\right),$$

if $X_i$ is discrete.

In the current setting, the existence of the $i$-th certainty equivalent characteristic defined by the decision maker in $X_i$ is guaranteed by the continuity of the utility function $u_i$ and the connectedness of the factor space $X_i$. 8

This fact does not necessarily imply the existence of a good whose $i$-th characteristic corresponds to the $i$-th certainty equivalent. Nevertheless, as pointed out at the beginning of this section, the decision maker has a preference defined on the good represented by $(c_1, \ldots, c_n)$.

**Definition 3.6.** The *certainty equivalent good*, denoted by $C$, is the good in $G$ represented by $(c_1, \ldots, c_n)$ under the identification map $\varphi$, that is,

$$C = \varphi^{-1}(c_1, \ldots, c_n).$$

8Removing the assumption of connectedness for $X_i$ or that of continuity for $u_i$, it is not difficult to provide examples of pairs $(\mu_i, u_i)$ on the set $X_i$ such that $c_i$ does not exist. In these cases, the decision maker can fix an element of $X_i$ whose utility provides the subjectively closest approximation to the expected value (that is, $\int_{X_i} u_i(x_i) \mu_i(x_i) \, dx_i$, or $\sum_{x_i \in X_i} u_i(x_i) \mu_i(x_i)$); see [14] and [15].

Clearly, any approximation process generates a bias on the choice of the decision maker. However, all our strictly set-theoretical results (see Sections 4 and 5) would remain unaffected by this fact.
The use of certainty equivalent values implies that if the known characteristic delivers a higher (lower) utility than the corresponding subjective certainty equivalent value, the decision maker prefers the good defined by the former (latter) one.  

4 Information sets and info-evaluation maps Let \( P(\mathcal{G}) \) denote the power set of \( \mathcal{G} \). For every \( \mathcal{H} \in P(\mathcal{G}) \) and every \( i \leq n \), let

\[
I_{\mathcal{H},i} = \{ x^G_j : G \in \mathcal{H}, j \leq i \}
\]

Clearly, for every \( i \leq n \), \( I_{\emptyset,i} = I_{\emptyset} = \emptyset \).  

**Definition 4.1.** Let \( m \in \mathbb{N} \), \( m \leq n \). An information set is a set of the form

\[
\bigcup \{ I_{\mathcal{G},i_h} : h \leq m \},
\]

where for \( h \leq m \), \( \mathcal{G}_h \in P(\mathcal{G}) \) and \( i_h \leq n \). We will write that \( \bigcup \{ I_{\mathcal{G},i_h} : h \leq m \} \) is given if for every \( h \leq m \), the decision maker knows the first \( i_h \) characteristics of all goods belonging to \( \mathcal{G}_h \).

In particular, “\( I_{\mathcal{G},m} \) is given” if perfect information is available; and “\( I_{\emptyset} \) is given” if no information is available.

Finally, “\( I \) is a given information set” if there exist a finite family \( \{ \mathcal{G}_h : h \leq m \} \subseteq P(\mathcal{G}) \) and a finite sequence \( (i_h : h \leq m) \) of natural numbers \( \leq n \) such that \( I = \bigcup \{ I_{\mathcal{G},i_h} : h \leq m \} \).

An information set \( I \) is uniquely determined by a finite family \( \{ \mathcal{G}_h : h \leq m \} \subseteq P(\mathcal{G}) \) and a finite sequence \( (i_h : h \leq m) \) of natural numbers \( \leq n \).

Without loss of generality, we can work under the following assumption.

**Assumption 7.** For every information set \( I \), the family \( \{ \mathcal{G}_h : h \leq m \} \) and the finite sequence \( (i_h : h \leq m) \) determining \( I \) satisfy the following requirement:

\[
\forall h, k \leq m \ (h \neq k \rightarrow (\mathcal{G}_h \cap \mathcal{G}_k = \emptyset \land i_h \neq i_k)).
\]

\(^9\)The use of certainty equivalents is frequent in the literature on economic uncertainty. In this regard, several applications of certainty equivalents to different areas of economic theory can be found in [2].
Remark 4.2. The previous definition is quite natural from both an economic and a set-theoretical point of view. Economically speaking, due to the order of dominance of the characteristic factors $X_1, \ldots, X_n$, it is reasonable to assume that a resource constrained decision maker would collect the information about every single good following the same order, that is, looking first for the first characteristic, then for the second one, and so on. Set-theoretically, we would like to observe that it would not make any difference to define an information set by using an infinite subfamily $\{G_\alpha : \alpha \leq \kappa\}$ of $P(G)$ and hence, an infinite sequence $(i_\alpha : \alpha \leq \kappa)$ of natural numbers $\leq n$. In fact, in the case when $\kappa$ is an infinite cardinal, by the Pigeons’ Principle, there exists a finite sequence $(i_h : h \leq m)$ of natural numbers $\leq n$ whose support is the same as $(i_\alpha : \alpha \leq \kappa)$, i.e., for every $\alpha \leq \kappa$, there exists $h \leq m$ such that $i_\alpha = i_h$. Thus, the information set determined by $\{G_\alpha : \alpha \leq \kappa\}$ and $(i_\alpha : \alpha \leq \kappa)$, would be exactly the same as the one determined by $\{\bigcup \{G_\alpha : i_\alpha = i_h\} : h \leq m\}$ and $(i_h : h \leq m)$.

Even if the underlying family of a given information set is always a finite subfamily $\{G_h : h \leq m\}$ of $P(G)$, there is no limitation on the cardinality of every $G_h$. Thus, the information set can contain quite a quantity of information. For example, the information set $I = \bigcup \{I_{G_h,i_h} : h \leq m\}$ has cardinality $|I| = \sum_{h \leq m} i_h|G_h|$.

It is easy to check that:

**Proposition 4.3.** Let $I$ be a given information set. $I$ is finite if and only if the subfamily of $P(G)$ determining $I$ consists of finite subsets of $G$.

**Definition 4.4.** Let $I = \bigcup \{I_{G_h,i_h} : h \leq m\}$ be given. For every $j \leq n$, the function $\psi^I_j : G \rightarrow X$ defined by

$$\psi^I_j(G) = \begin{cases} x^G_j, & \text{if } G \in G_h \text{ and } j \leq i_h, \\ c_j, & \text{otherwise,} \end{cases}$$

where $c_j$ is the $j$-th info-evaluation function determined by $I$. The product function $\prod_{j \leq n} \psi^I_j : G \rightarrow X$ defined by

$$\left( \prod_{j \leq n} \psi^I_j \right)(G) = (\psi^I_1(G), \ldots, \psi^I_n(G)),$$

where $G \in G$, and denoted by $\psi^I$ is the info-evaluation map determined by $I$. 
Remark 4.5. (a) If $I_{\sigma}$ is given (that is, under the no information assumption), $\psi_{I_{\sigma}}$ is the constant function defined by $\psi_{I_{\sigma}}(G) = (c_1, \ldots, c_n)$, whenever $G \in \mathcal{G}$.

(b) For every $i \leq n$, $\psi_{I_{\sigma}^{(i)}} = \psi_{I_{\sigma}}$, where $C = \varphi^{-1}(c_1, \ldots, c_n)$ is the certainty equivalent good.

5 Preferences induced by info-evaluation maps

Given any information set $I$ and any $\mathcal{H} \in P(\mathcal{G})$, $\psi^I | \mathcal{H}$ denotes the restriction of the info-evaluation map $\psi^I$ to $\mathcal{H}$. Similarly, $\psi^I_i | \mathcal{H}$ denotes the restriction of the $i$-th info-evaluation function $\psi^I_i$ to $\mathcal{H}$.

Let $I = \bigcup \{I_{h_0,h} : h \leq m\}$ be given. Let $A(I) = \{j \in \mathbb{N} : i_0 < j \leq n\}$, where $i_0 = \max \{i_h : h \leq m\}$. For every $G \in \mathcal{G}$, we have

$$\psi^I(G) = (\psi^I_1(G), \ldots, \psi^I_{i_0}(G), c_{i_0+1}, \ldots, c_n) \in \prod_{j \leq i_0} X_j \times \prod_{i_0 < j \leq n} \{c_j\}.$$ 

The Cartesian product $\prod_{j \leq i_0} X_j$, in particular, remains ordered with the induced relation $\preceq_{A(I)}$, see Definition 2.1. Such an ordering allows to define a preference relation on $\mathcal{G}$ as follows.

$$\forall H, H' \in \mathcal{G}, \ H \succeq_{\psi^I} H' \iff \left( \prod_{j \leq i_0} \psi^I_j(H) \succeq_{A(I)} \prod_{j \leq i_0} \psi^I_j(H') \right).$$

For compatibility reasons, it is important to note that the preference relation $\succeq_{\psi^I}$ induced by $\succeq_{A(I)}$ coincides with the $\psi^I$-induced preference relation, $\succeq_{\psi^I}$, on the set $\mathcal{G}$; see Definition 3.1. In fact, given $H, H' \in \mathcal{G}$, we have

$$H \succeq_{\psi^I} H' \iff \psi^I(H) \succeq_{A(I)} \psi^I(H') \iff \sum_{i \leq n} u_i(\psi^I_i(H)) \geq \sum_{i \leq n} u_i(\psi^I_i(H')) \iff \sum_{i \leq i_0} u_i(\psi^I_i(H)) \geq \sum_{i \leq i_0} u_i(\psi^I_i(H')) \iff \left( \prod_{i \leq i_0} \psi^I_i(H) \succeq_{A(I)} \prod_{i \leq i_0} \psi^I_i(H') \right) \iff H \succeq_{\psi^I} H'.$$

By Definition 2.2 and by the additivity of the utility function $u$ when restricted to each image set of the form $\psi^I(G)$, the following statement is trivial.

**Proposition 5.1.** Let $I$ be a given information set and $\mathcal{H} \in P(\mathcal{G})$. Then, $u \circ (\psi^I | \mathcal{H})$ is an additive utility function representing $\succeq_{\psi^I}$ over $\mathcal{H}$.
Corollary 5.2. Let $I$ be a given information set. Then, $u \circ \psi^I$ is an additive utility function representing $\succeq_{\psi^I}$ over $\mathcal{G}$.

Note that Proposition 5.1 and Corollary 5.2 are actually two equivalent statements. According to these two results, it is plausible to assume that decision makers will choose among the goods in $\bigcup_{h \leq m} \mathcal{G}_h$, once the information available is provided by $\bigcup \{I_{\mathcal{G}_h, \delta_h} : h \leq m \}$. However, it should be evident that a decision maker can always choose the certainty equivalent good if the goods defined by the information set, namely the elements of $\bigcup_{h \leq m} \mathcal{G}_h$, provide her with a lower utility. In this case, the decision maker expects (based on her subjective probability densities) to obtain a higher utility from choosing any of the unknown goods, whose characteristics are defined by the certainty equivalent values. This justifies the following definition.

Definition 5.3. Let $I$ be a given information set and $\{\mathcal{G}_h : h \leq m \}$ be the subfamily of $P(\mathcal{G})$ underlying $I$. The choice set relative to $I$, denoted by $\mathcal{C}(I)$, is the union $\bigcup_{h \leq m} \mathcal{G}_h \cup \{C\}$.

Suppose a generic information set $I$ is given. The preference relation $\succeq_{\psi^I}$ is in general different from the preference relation $\succeq_{\varphi}$ induced by $\succeq$ on $\mathcal{G}$ under perfect information (see Section 3). Decision maker’s preference over all goods changes according to the available information, but remains always rational and numerically well-represented by a utility function. The choice sets to which the decision maker restricts her attention varies in a natural way, too. Moreover, Proposition 5.1 implies:

Corollary 5.4. Let $I$ be a given information set. Then, $u \circ (\psi^I|_{\mathcal{C}(I)})$ is an additive utility function representing $\succeq_{\psi^I}$ over $\mathcal{C}(I)$.

The continuity of utility functions determined by a given information set, such as $u \circ (\psi^I|_{\mathcal{C}(I)})$, is studied in the following section. We show that continuity holds if the associated choice set has further topological properties, namely that of being a finite union of closed subsets of $\mathcal{G}$.

6 Continuity of info-evaluation maps  Let $I$ be a given information set such that $\mathcal{C}(I)$ is finite. It is straightforward to show that $\psi^I|_{\mathcal{C}(I)}$ is continuous. Hence, by Corollary 5.4, $u \circ (\psi^I|_{\mathcal{C}(I)})$ is an additive continuous utility function representing $\succeq_{\psi^I}$ over $\mathcal{C}(I)$.

We aim to show how the continuity of $\psi^I|_{\mathcal{C}(I)}$ can also be obtained after relaxing the finiteness assumption on the choice set, $\mathcal{C}(I)$. 
Let $I = \bigcup\{I_{\mathcal{G}_h,i_h} : h \leq m\}$ be given. It can be easily checked that the info-evaluation map determined by $I$, $\psi^I$, is the “union” of the family of maps $\{\psi^I_{\mathcal{G}_h,i_h} : h \leq m\} \cup \{\psi^I_{\mathcal{G}_h} : h \leq m\}$, whose domains are pairwise disjoint by Assumption 7. More precisely,

$$
\psi^I(G) = \begin{cases} 
\psi^I_{\mathcal{G}_h,i_h}|_{\mathcal{G}_h}(G), & \text{if } \exists h \leq m : G \in \mathcal{G}_h, \\
(c_1, \ldots, c_n), & \text{otherwise.}
\end{cases}
$$

whenever $G \in \mathcal{G}$.

**Lemma 6.1.** Let $I = \bigcup\{I_{\mathcal{G}_h,i_h} : h \leq m\}$ be given. If, for every $h \leq m$:

(a) $\mathcal{G}_h$ is a closed subset of $\mathcal{G}$;
(b) $\psi^I_{\mathcal{G}_h,i_h}|_{\mathcal{G}_h}$ is continuous;
then, the map $\psi^I|_{\mathcal{C}(I)}$ is continuous.

**Proof.** For every $h \leq m$, $\psi^I|_{\mathcal{G}_h} = \psi^I_{\mathcal{G}_h,i_h}|_{\mathcal{G}_h}$. By Assumption 7 and the pasting lemma (Theorem 18.3 or Exercise 18.9 in [23]), it follows that $\psi^I|_{\mathcal{C}(I)}$ is continuous.

**Lemma 6.2.** Let $\mathcal{H} \in P(\mathcal{G})$ and let $I_{\mathcal{H},i}$ be given. Then $\psi^{I_{\mathcal{H},i}}|_{\mathcal{H}}$ is continuous.

**Proof.** If either $\mathcal{H} = \emptyset$ or $\mathcal{H} = \{C\}$, it is trivial (see Remark 4.5).

If $\mathcal{H} \neq \emptyset$ and $\mathcal{H} \neq \{C\}$, then for every $j \leq i$, $\psi_j^{I_{\mathcal{H},i}}|_{\mathcal{H}} = \varphi_j|_{\mathcal{H}}$; while for $i < j \leq n$, $\psi_j^{I_{\mathcal{H},i}}|_{\mathcal{H}}$ is the constant function $H \in \mathcal{H} \to c_j \in X_j$. Hence $\psi^{I_{\mathcal{H},i}}|_{\mathcal{H}} = \prod_{j \leq n} \psi_j^{I_{\mathcal{H},i}}|_{\mathcal{H}}$ is the product of continuous functions.

Lemmas 6.1 and 6.2 yield:

**Proposition 6.3.** Let $I$ be a given information set. If either

(i) $\mathcal{C}(I) \neq \mathcal{G}$ and $\mathcal{C}(I)$ is the disjoint union of closed subsets of $\mathcal{G}$;

or

(ii) $\mathcal{C}(I) = \mathcal{G}$ and $I = I^{\mathcal{G},i}$ for some $i \leq n$;

then $\psi^I|_{\mathcal{C}(I)}$ is continuous.

**Proof.** If (i) holds, then apply Lemma 6.1 and Lemma 6.2. If (ii) holds, then apply Lemma 6.2.
The assumption that \( I = I^{G,i} \) for some \( i \leq n \), in the case when \( C(I) = G \), cannot be removed or weakened. Suppose, for instance, that \( I = \bigcup \{I_{G,ih} : h \leq m \} \) is given such that \( \bigcup \{G_h : h \leq m \} = G \) and \( m > 1 \). By Assumption 7, \( G_h \cap G_k = \emptyset \) and \( i_h \neq i_k \) provided that \( h \neq k \). The \( G_h \)’s cannot be all closed subsets of \( G \), otherwise \( G \) would be a disconnected space, which contradicts the connectedness deriving from Proposition 3.1. Consequently, (a) of Lemma 6.1 is not satisfied and \( \psi^I = \psi^I|_G \) is not necessarily continuous.

It is worth to underline the fact that the continuity of the whole map \( \psi^I \), guaranteed by assumption (ii) of Proposition 6.3, does not in general hold. Suppose, for instance that an information set \( I \) is given such that (i) of Proposition 6.3 is satisfied. The set \( G \cap C(I) \) cannot be possibly closed in \( G \), this contradicting again the connectedness of \( G \). Hence, although the restriction map \( \psi^I|_{G\cap C(I)} \) is continuous (it is the map constantly equal to \( (c_1,c_2,\ldots,c_n) \)), the pasting lemma does not apply to the whole map \( \psi^I \).

In order to further clarify the previous comments we provide the following example.

**Example.** Let \( X_1 = [0,2] \) and \( X_2 = [0,1] \), endowed with the standard Euclidean topology, be the 1st and 2nd characteristic spaces. Let \( X = X_1 \times X_2 \) and \( u : X \to \mathbb{R} \) be defined by \( u(x_1,x_2) = x_1 + x_2 \). Consider on \( X \) the preference order \( \succeq \) defined by

\[
(x_1,x_2) \succeq (x'_1,x'_2) \overset{def}{\iff} x_1 + x_2 \geq x'_1 + x'_2.
\]

It can be easily checked that \( \succeq \) is continuous with respect to the product topology on \( X \) and \( u \) is a continuous utility function representing \( \succeq \). Endow \( X_1 \) and \( X_2 \) with the continuous uniform probability density functions \( \mu_1 \) and \( \mu_2 \), respectively. Clearly, given that \( u_2(x_2) = x_2 \), the certainty equivalent value \( c_2 \) coincides with the expected utility value, which is equal to \( \frac{3}{4} \). Finally, let \( G = X_1 \times X_2 \) and \( \varphi \) be the identity map. (Thus, \( G \) coincides with the characteristic space \( X \).) Suppose the following information set is given:

\[
I = I^{G_1,1} \cup I^{G_2,2}
\]

where

\[
G_1 = [0,1] \times \left[ 0, \frac{3}{4} \right] \quad \text{and} \quad G_2 = G \setminus G_1.
\]

Then, \( C(I) = G \) but \( I \neq I^{G,1} \) and \( I \neq I^{G,2} \) (compare with Proposition 6.3(ii)). The info-map \( \psi^I \) sends each good \((x_1,x_2) \in G_1 \) to the
pair \( (x_1, \frac{2}{3}) \in [0, 1] \times \left\{ \frac{2}{3} \right\} \), and each good \( (x_1, x_2) \in \mathcal{G}_2 \) to itself. We claim that \( \psi^f \) is not continuous. Indeed, consider the set \( \left( \frac{1}{2}, \frac{2}{3} \right) \times \left( \frac{1}{3}, \frac{2}{3} \right) \).

This set is an open subset of \( X \), but its preimage via \( \psi^f \) is the set \( \left( \left( \frac{1}{2}, 1 \right] \times \left[ 0, \frac{2}{3} \right) \cup \left( 1, \frac{2}{3} \right) \times \left( \frac{1}{3}, \frac{2}{3} \right) \right) \), which is not open in \( \mathcal{G} \).

Proposition 6.3 and Corollary 5.4 yield the main continuity result.

**Theorem 6.4.** Let \( I \) be a given information set. If either (i) or (ii) of Proposition 6.3 holds, then \( \psi^f|_{\mathcal{C}(I)} \) is continuous and \( u \circ \left( \psi^f|_{\mathcal{C}(I)} \right) \) is an additive continuous utility function representing \( \succeq_{\psi^f} \) over \( \mathcal{C}(I) \).

**Corollary 6.5.** Let \( I \) be a given information set. If \( \mathcal{C}(I) \) is a finite subset of \( \mathcal{G} \), then \( \psi^f|_{\mathcal{C}(I)} \) is continuous and \( u \circ \left( \psi^f|_{\mathcal{C}(I)} \right) \) is an additive continuous utility function representing \( \succeq_{\psi^f} \) over \( \mathcal{C}(I) \).

**Proof.** The singletons of \( \mathcal{G} \) are closed subsets. Furthermore, \( \mathcal{C}(I) \neq \mathcal{G} \). If not, \( \mathcal{G} \) would be a totally disconnected space; a contradiction (see Proposition 3.1). Hence, (i) of Proposition 6.3 is satisfied.

7 Lexicographic information sets: when additive utilities are lexicographic

**Definition 7.1.** Let \( m \in \mathbb{N}, m \leq n \). An information set \( \bigcup \{ I_{\mathcal{G}_h, i_h} : h \leq m \} \) is called lexicographic if \( \forall h \leq m, \forall H \in \mathcal{G}_h : i_h = \min \{ i \leq n : x_{i,h}^H \neq (x_M)_i \} = \min \{ i \leq n : u_i(x_{i,h}^H) < u_i((x_M)_i) \} \).

In other words, if a lexicographic information set \( I = \bigcup \{ I_{\mathcal{G}_h, i_h} : h \leq m \} \) is given, the decision maker knows that for every \( h \leq m \), the characteristics \( x_1, x_2, \ldots, x_{i_h-1} \) of each of the goods in \( \mathcal{G}_h \) are indifferent to, and hence have the same utility as, the corresponding maximal ones \( (x_M)_1, (x_M)_2, \ldots, (x_M)_{i_h-1} \). At the same time \( x_{i_h} \) is less preferred to \( (x_M)_{i_h} \) and hence delivers less utility than \( (x_M)_{i_h} \). In symbols:

\[ \forall i < i_h, \ u_i(x_i) = u_i((x_M)_i) \quad \text{and} \quad u_{i_h}(x_{i_h}) < u_{i_h}((x_M)_{i_h}). \]

In particular, the information set \( I^{h,j} \), where \( j \leq n \) and \( \mathcal{H} \in \mathcal{P}(\mathcal{G}) \), is lexicographic if

\[ \forall H \in \mathcal{H}, \ j = \min \{ i \leq n : u_i(x_H^H) < u_i((x_M)_i) \}. \]

which means that the decision maker does not only know the value of the first \( j \) characteristics of all the goods in \( \mathcal{H} \), but is also indifferent
between each of their first $j − 1$ characteristics and the corresponding maximal ones. Hence, the utility of the goods in $H$ depends only on the value of their $j$-th characteristic.

Let $I = \bigcup \{ I_{g_{h},i_{h}} : h \leq m \}$ be a lexicographic information set, with $m > 1$. For every $h, k \leq m$, let

$$
\alpha(h, k) = \min \{ i_{h}, i_{k} \} \quad \text{and} \quad \beta(h, k) = \max \{ i_{h}, i_{k} \}.
$$

Consider the following condition on the information set $I$.

(*) For every $h, k \leq m$, $h \neq k$,

$$
H \in G_{\alpha(h, k)} \land K \in G_{\beta(h, k)} \iff \sum_{i=\alpha(h, k)}^{\beta(h, k)} u_{i}(x_{i}^{H}) \leq \sum_{i=\alpha(h, k)}^{\beta(h, k)} u_{i}(x_{i}^{K})
$$

Note that condition (*) cannot be considered on $I$ provided that $m = 1$, namely if $I = I_{g_{h},i_{h}}$, with $j \leq n$ and $H \in P(G)$.

We claim that the specific form of a lexicographic information set $I$ together with condition (*), where necessary, force the additive utility function $u \circ (\psi^{I}_{\cdot} |_{C(I)})$ to be also lexicographic on $C(I)$.

**Proposition 7.2.** Let $I = \bigcup \{ I_{g_{h},i_{h}} : h \leq m \}$ be a given lexicographic information set. Suppose that $m > 1$ implies that condition (*) holds on $I$. Then, for every $H, K \in C(I)$, the following are equivalent:

(a) $\sum_{i \leq n} u_{i}(\psi_{i}^{I}(H)) \geq \sum_{i \leq n} u_{i}(\psi_{i}^{I}(K))$.

(b) $(u_{1}(\psi_{1}^{I}(H)), \ldots, u_{n}(\psi_{n}^{I}(H)) \geq_{\text{Lex}} (u_{1}(\psi_{1}^{I}(K)), \ldots, u_{n}(\psi_{n}^{I}(K)))$.

Proof. If $m = 1$, then $I = I_{g_{h},i_{h}}$, for some $j \leq n$ and $H \in P(G)$. Clearly, $C(I) = H$ and, for every $H \in C(I)$,

$$
u(\psi_{1}^{I}(H), \ldots, \psi_{n}^{I}(H)) = u((x_{M})_{1}^{H}, \ldots, (x_{M})_{j-1}^{H}, x_{j}^{H}, c_{j+1}, \ldots, c_{n}).
$$

Hence, for every $H, K \in C(I)$,

$$
\sum_{i \leq n} u_{i}(\psi_{i}^{I}(H)) \geq \sum_{i \leq n} u_{i}(\psi_{i}^{I}(K)) \iff u_{j}(x_{j}^{H}) \geq u_{j}(x_{j}^{K})
$$

$$
\iff (u_{1}(\psi_{1}^{I}(H)), \ldots, u_{n}(\psi_{n}^{I}(H))) \geq_{\text{Lex}} (u_{1}(\psi_{1}^{I}(K)), \ldots, u_{n}(\psi_{n}^{I}(K))).
$$

Consider now the case when $m > 1$ and fix $H, K \in C(I)$. If $H, K \in G_{h}$, for $h \leq m$, then proceed as for the case $m = 1$. If $H \in G_{h}$ and $K \in G_{k},$
where \( h \neq k \), then by condition (\( \ast \)),

\[
\sum_{i \leq n} u_i(\psi_i^I(H)) \geq \sum_{i \leq n} u_i(\psi_i^I(K)) \iff \sum_{i=\alpha(h,k)}^{\beta(h,k)} u_i(x_i^H) \geq \sum_{i=\alpha(h,k)}^{\beta(h,k)} u_i(x_i^K)
\]

\[
\iff (u_1(\psi_1^I(H)), \ldots, u_n(\psi_n^I(H))) \succeq_{\text{lex}} (u_1(\psi_1^I(K)), \ldots, u_n(\psi_n^I(K)))).
\]

\[\square\]

Theorem 6.4 and Proposition 7.2 lead to the main result of the paper.

**Theorem 7.3.** Let \( I = \bigcup \{ I_{\mathcal{G}_h,i_h} : h \leq m \} \) be a given lexicographic information set. If either

(i) \( \mathcal{C}(I) \neq \mathcal{G}, \mathcal{C}(I) \) is the disjoint union of closed subsets of \( \mathcal{G} \) and \( m > 1 \) implies that condition (\( \ast \)) holds on \( I \);

or

(ii) \( \mathcal{C}(I) = \mathcal{G} \) and \( I = I_{\mathcal{G},i} \) for some \( i \leq n \);

then, \( u_0(\psi^I_{\mathcal{C}(I)}) \) is both an additive and lexicographic, continuous utility function representing \( \succ_\psi^I \) over \( \mathcal{C}(I) \).

**Corollary 7.4.** Let \( I = \bigcup \{ I_{\mathcal{G}_h,i_h} : h \leq m \} \) be a given lexicographic information set. If:

(a) \( \mathcal{C}(I) \neq \mathcal{G} \) and \( \mathcal{C}(I) \) is the disjoint union of closed subsets of \( \mathcal{G} \);

(b) for every \( h \leq m \) and every \( H \in \mathcal{G}_h \), \( x_i^H = c_h \);

then, \( u_0(\psi^I_{\mathcal{C}(I)}) \) is both an additive and lexicographic, continuous utility function representing \( \succ_\psi^I \) over \( \mathcal{C}(I) \).

**Proof.** When \( m > 1 \), (b) implies condition (\( \ast \)) on \( I \). Apply Theorem 7.3 with assumption (i). \[\square\]

**Corollary 7.5.** Let \( I = \bigcup \{ I_{\mathcal{G}_h,i_h} : h \leq m \} \) be a given lexicographic information set. If

(a) \( \mathcal{C}(I) \) is finite;

(b) \( m > 1 \) implies that condition (\( \ast \)) holds on \( I \);

then, \( u_0(\psi^I_{\mathcal{C}(I)}) \) is both an additive and lexicographic, continuous utility function representing \( \succ_\psi^I \) over \( \mathcal{C}(I) \).

**Proof.** By (a), \( \mathcal{C}(I) \neq \mathcal{G} \). Apply Theorem 7.3 with assumption (i). \[\square\]
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