EXPANSIONS FOR FUNCTIONS OF DETERMINANTS OF POWER SERIES

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ABSTRACT. Let \( A(\epsilon) \) be an analytic \( n \times n \) matrix function of a scalar \( \epsilon \). We obtain expansions in powers of \( \epsilon \) for \( \det A(\epsilon) \) and smooth functions of it including its powers and its log. An application is given to multivariate statistics.

1 Introduction Suppose that \( A(\epsilon) \) is an \( n \times n \) complex matrix function of a complex scalar \( \epsilon \) expandable in the form

\[
A(\epsilon) = \sum_{k=0}^{\infty} a_k \frac{\epsilon^k}{k!},
\]

where \( \{a_k\} \) are \( n \times n \) complex matrices. Suppose also that \( \det a_0 \neq 0 \). We show how to obtain power series expansions for functions of \( \det A(\epsilon) \). As examples we give the coefficients in the expansions

\[
\log \det A(\epsilon) = \sum_{k=0}^{\infty} g_k \frac{\epsilon^k}{k!},
\]

\[
\det A(\epsilon) = \sum_{k=0}^{\infty} c_k \frac{\epsilon^k}{k!},
\]

\[
\left\{ \frac{\det A(\epsilon)}{\det A(0)} \right\}^\alpha = 1 + \sum_{k=1}^{\infty} e_k^* \frac{\epsilon^k}{k!}.
\]

Functions of the form (1.2)–(1.4) arise often in multivariate analysis (see Srivastava [10], Anderson [1], Kollo and von Rosen [8], Eaton [4], Härdle and Simar [6], Johnson and Wichern [7]).

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2 Main results

We first show how to obtain power series expansions for functions of
\[ g(\epsilon) = \log \det A(\epsilon), \]
say

\[ f(g(\epsilon)) = \sum_{k=0}^{\infty} b_k \frac{\epsilon^k}{k!}, \tag{2.1} \]

where \( f : C \to C \) is any function which is analytic about

\[ g_0 = g(0) = \log \det a_0 \tag{2.2} \]

and \( C \) is the set of complex numbers. By Taylor’s series, (2.1) holds with \( b_k = \frac{d}{d\epsilon} f(g(\epsilon)) |_{\epsilon=0} \), so that \( b_0 = f(g_0) \) for \( g_0 \) of (2.2). By Faà di Bruno’s formula \cite[2, page 137]{2},

\[ b_k = \sum_{j=1}^{k} f_j B_{kj}(g) \tag{2.3} \]

for \( k \geq 1 \), where \( f_j = f^{(j)}(g_0), g = (g_1, g_2, \ldots), g_j = g^{(j)}(0) \), and \( B_{kj}(g) \) is the partial exponential Bell polynomial tabled on pages 307–308 of \cite{2} to \( k = 12 \), and defined by

\[ \left( \sum_{k=1}^{\infty} \frac{g_k \epsilon^k}{k!} \right)^j = \sum_{k=j}^{\infty} B_{kj}(g) \frac{\epsilon^k}{k!} \]

for \( j \geq 1 \). So, in terms of \( f = (f_0, f_1, \ldots) \) and \( g \), the first seven coefficients in (2.3) are

\[ b_0 = f_0 = f(g_0), \tag{2.4} \]
\[ b_1 = f_1 g_1, \tag{2.5} \]
\[ b_2 = f_1 g_2 + f_2 g_1^2, \tag{2.6} \]
\[ b_3 = f_1 g_3 + f_2 (3g_1 g_2) + f_3 g_1^3, \tag{2.7} \]
\[ b_4 = f_1 g_4 + f_2 (4g_1 g_3 + 3g_2^2) + f_3 (6g_1^2 g_2) + f_4 g_1^4, \tag{2.8} \]
\[ b_5 = f_1 g_5 + f_2 (5g_1 g_4 + 10g_2 g_3) + f_3 (10g_1^2 g_4 + 15g_1 g_2^2) + f_4 (10g_1^3 g_3 + 15g_1 g_2 g_2^2) + f_5 g_1^5. \tag{2.9} \]
(2.10) \[ b_6 = f_1 g_6 + f_2 (6g_1 g_5 + 15g_2 g_4 + 10g_3^2) \]
\[ + f_3 (15g_1^2 g_4 + 60g_1 g_2 g_3 + 15g_2^3) \]
\[ + f_4 (20g_1^3 g_3 + 45g_1 g_2^2 g_2) + f_5 (15g_1^4 g_2) + f_6 g_6. \]

We now give expressions for \( g_j \) of (1.2). Set

\[ \mathbf{A}_k = \left( \frac{d}{d\epsilon} \right)^k \mathbf{A}(\epsilon), \quad \overline{\mathbf{A}}_k = \mathbf{A}_0^{-1} \mathbf{A}_k. \]

Since \( (d/d\epsilon)\mathbf{A}_0^{-1} = -\mathbf{A}_0^{-1} \mathbf{A}_1 \mathbf{A}_0 \), direct calculation gives

\[ g^{(1)}(\epsilon) = \text{tr} \overline{\mathbf{A}}_1 = \text{tr} \overline{\mathbf{A}}_1 \text{ say}, \]
\[ g^{(2)}(\epsilon) = \text{tr} (\overline{\mathbf{A}}_2 - \overline{\mathbf{A}}_1^2), \]
\[ g^{(3)}(\epsilon) = \text{tr} (\overline{\mathbf{A}}_3 - 3\overline{\mathbf{A}}_1 \overline{\mathbf{A}}_2 + 2\overline{\mathbf{A}}_1^3), \]

and so on. Set \( \overline{\mathbf{a}}_k = \mathbf{a}_0^{-1} \mathbf{a}_k. \) Since \( \overline{\mathbf{A}}_k|_{\epsilon=0} = \overline{\mathbf{a}}_k \), we have

(2.12) \[ g_1 = \text{tr} \overline{\mathbf{a}}_1, \]
(2.13) \[ g_2 = \text{tr} (\overline{\mathbf{a}}_2 - \overline{\mathbf{a}}_1^2), \]
(2.14) \[ g_3 = \text{tr} (\overline{\mathbf{a}}_3 - 3\overline{\mathbf{a}}_1 \overline{\mathbf{a}}_2 + 2\overline{\mathbf{a}}_1^3), \]
(2.15) \[ g_4 = \text{tr} (\overline{\mathbf{a}}_4 - 4\overline{\mathbf{a}}_1 \overline{\mathbf{a}}_3 - \overline{\mathbf{a}}_2^2 + 12\overline{\mathbf{a}}_1 \overline{\mathbf{a}}_2 - 6\overline{\mathbf{a}}_1^4), \]
(2.16) \[ g_5 = \text{tr} (\overline{\mathbf{a}}_5 - 5\overline{\mathbf{a}}_1 \overline{\mathbf{a}}_4 - 10\overline{\mathbf{a}}_2 \overline{\mathbf{a}}_3 + 20\overline{\mathbf{a}}_1^2 \overline{\mathbf{a}}_3 \]
\[ + 30\overline{\mathbf{a}}_1 \overline{\mathbf{a}}_2^2 - 60\overline{\mathbf{a}}_1 \overline{\mathbf{a}}_2 + 24\overline{\mathbf{a}}_1^3), \]
(2.17) \[ g_6 = \text{tr} \left( \overline{\mathbf{a}}_6 - 6\overline{\mathbf{a}}_1 \overline{\mathbf{a}}_5 - 15\overline{\mathbf{a}}_2 \overline{\mathbf{a}}_4 - 10\overline{\mathbf{a}}_3^2 + 30\overline{\mathbf{a}}_2^2 \overline{\mathbf{a}}_3 \]
\[ + 60 \sum_{i=1}^{2} \overline{\mathbf{a}}_i \overline{\mathbf{a}}_2 \overline{\mathbf{a}}_3 + 30\overline{\mathbf{a}}_2^3 - 120\overline{\mathbf{a}}_1 \overline{\mathbf{a}}_3 \]
\[ + 15\overline{\mathbf{a}}_1 \overline{\mathbf{a}}_2 \overline{\mathbf{a}}_3^2 - 120\overline{\mathbf{a}}_1 \overline{\mathbf{a}}_2^2 \right), \]

and so on, where

(2.18) \[ \sum_{i=1}^{2} \overline{\mathbf{a}}_i \overline{\mathbf{a}}_2 \overline{\mathbf{a}}_3 = \overline{\mathbf{a}}_1 (\overline{\mathbf{a}}_2 \overline{\mathbf{a}}_3 + \overline{\mathbf{a}}_3 \overline{\mathbf{a}}_2), \]
\[ \sum_{i=1}^{6} \mathbf{x}_i^2 \mathbf{x}_2 = 4 \mathbf{x}_1^2 \mathbf{x}_2^2 + 2 (\mathbf{x}_1 \mathbf{x}_2)^2. \]

To obtain the general term, consider first the univariate case \( n = 1 \): by (2.1) and (2.3) with \( f(g) = \log g \), \( f_j = (-1)^{j-1}(j-1)! g_j^0 \), and for \( k \geq 1 \),

\[ g_k = \sum_{j=1}^{k} f_j B_{kj}(\mathbf{a}) = \sum_{j=1}^{k} (-1)^{j-1}(j-1)! B_{kj}(\mathbf{a}), \]

where \( \mathbf{a} = (a_1, a_2, \ldots) \) and \( \mathbf{a} = (a_1, a_2, \ldots) \). Comparison with (2.12)–(2.17) shows that for general \( n \) for \( k \geq 1 \),

\[ g_k = \sum_{j=1}^{k} (-1)^{j-1}(j-1)! \mathbf{B}_{kj}(\mathbf{a}), \]

where \( \mathbf{B}_{kj}(\mathbf{a}) \) is \( \text{tr} \ B_{kj}(\mathbf{a}) \) with \( B_{1r} \) replaced by \( B_{1r} \), and \( B_{1r} \) is any sequence of matrices from \( \{\mathbf{B}_j\} \). One then simplifies using \( \text{tr} \ B_{1r}B_2 = \text{tr} B_2B_1 \). For example,

\[ \text{tr} (B_1^2 B_2) = \text{tr} B_1^2 B_2, \]
\[ \text{tr} (B_1 B_2 B_3) = 2^{-1} \text{tr} B_1 (B_2 B_3 + B_3 B_2), \]
\[ \text{tr} (B_1^2 B_2^2) = 6^{-1} \text{tr} \left( 4 B_1^2 B_2^2 + 2 (B_1 B_2)^2 \right), \]
\[ \text{tr} (B_1 B_2 \cdots B_r) = \text{tr} B_1 (B_2 \cdots B_r). \]

The second and third of these four equations give (2.18) and (2.19).

**Example 2.1.** Suppose \( \mathbf{A} = \mathbf{I}_n + \epsilon \mathbf{a} \) for \( \mathbf{I}_n \) the identity matrix and \( \mathbf{a} \) an \( n \times n \) matrix. Then for \( k \geq 1 \), \( g_k = (-1)^{k-1}(k-1)! \text{tr} \mathbf{a}^k \), so that

\[ \log \det (\mathbf{I}_n + \epsilon \mathbf{a}) = \text{tr} \log (\mathbf{I}_n + \epsilon \mathbf{a}), \]

where \( \log (\mathbf{I}_n + \epsilon \mathbf{a}) \) is defined as the matrix power series \( \sum_{k=1}^{\infty} (-1)^{k-1}(\mathbf{a})^k/k \).

This is a special case of \( \log \det \mathbf{A} = \text{tr} \log \mathbf{A} \),
a result that is given in Dunford and Schwarz [3, p. 1029] and Glazman and Ljubic [5, p. 140] or in an equivalent form by Perko [9, p. 16]. A proof for any nonsingular square matrix $A$ is easily obtained by defining $\log A$ in terms of the Jordan canonical form of $A$.

**Example 2.2.** Take $f(g) = \exp(g)$. So, $f_j \equiv \det a_0$ and

\[
\det A(e) = \sum_{k=0}^{\infty} c_k \frac{e^k}{k!},
\]

where $c_0 = \det a_0$, and by (2.3), for $k \geq 1$, $c_k = c_0 c_k^*$, where

\[
c_k^* = \sum_{j=1}^{k} B_{kj}(g) = B_k(g)
\]

say with $g$ given by (2.12)-(2.20). That is,

\[
\frac{\det A(e)}{\det a_0} = 1 + \sum_{k=1}^{\infty} c_k^* \frac{e^k}{k!}
\]

with $c_k^* = b_k$ of (2.3)-(2.10) with $f_j \equiv 1$. Comtet [2] calls $B_k(g)$ the complete exponential Bell polynomial.

**Example 2.3.** Take $f(g) = \exp(\alpha g)$ for some scalar $\alpha$. So, $f_j = \alpha^j$ and

\[
\left\{ \frac{\det A(e)}{\det a_0} \right\}^\alpha = 1 + \sum_{k=1}^{\infty} c_k^* \frac{e^k}{k!},
\]

where $c_k^* = b_k$ of (2.3)-(2.10) with $f_j = \alpha^j$.

By the above result, for $F(g) = \log g$ and $a_0 = I_n$,

\[
F(\det(A(e))) = F(1) + \sum_{k=1}^{\infty} G_k \frac{e^k}{k!},
\]

where

\[
G_k = \sum_{j=1}^{k} F^{(j)}(1)B_{kj}(a).
\]
By Faa di Bruno’s formula the same holds with $B_{kj}(a)$ replaced by $B_{kj}(c)$ for $c$ of (1.3). However, $B_{kj}(c) \neq B_{kj}(a)$ for $a_0 = I_n$, $k \geq 1$. So, (2) does not hold for general smooth $F$. For general smooth $F$ we do have in terms of $c_0 = \det a_0$ and $c$ of (1.3), even if $a_0 \neq I_n$,

$$F(\det(\epsilon)) = F(c_0) + \sum_{k=1}^{\infty} G_k \frac{c^k}{k!},$$

where

$$G_k = \sum_{j=1}^{k} F^{(j)}(c_0) B_{kj}(c).$$

However, unless $n$ is small and $A(\epsilon)$ is a finite series, that is a polynomial in $\epsilon$, these coefficients are better obtained using (1.2) and (2.2) since the coefficients $g$ of (1.2) each being the trace of a matrix, are more fundamental than the coefficients $c$ of (1.3) which are nonlinear functions of traces derived from $g$.

Our results have been presented in terms of power series. Alternatively, they can be written as derivatives of functions. So, (2.3) can be written as

$$(d/d\epsilon)^k f(\log \det A(\epsilon)) = \sum_{j=1}^{k} f^{(j)}(\log \det A(\epsilon)) B_{kj}(g)$$

for $k \geq 1$, where $g_k$ is given by (2.20) with $\overline{\epsilon}$ replaced by $\overline{A}$ of (2.11). For many purposes a first approximation will suffice:

$$F(\det A(\epsilon)) = F(\det A(0)) + \epsilon F^{(1)}(\det A(0)) \mathrm{tr} A(0)^{-1} A^{(1)}(0) + O(\epsilon^2).$$

3 Application Here, we present an example of an application to the theory of multivariate statistics. Suppose that $X_1$ and $X_2$ are independent $p$-variate normal random variables with means $\mu_1$ and $\mu_2$, respectively, and covariances $V_1$ and $V_2$, respectively. For example, $X_1$ has density

$$p_{X_1}(x) = (\det V)^{-1/2} (2\pi)^{-p/2} \exp \left\{ - (x - \mu_1)^T V^{-1} (x - \mu_1)/2 \right\};$$

Then $Y = X_1 + \epsilon X_2$ is $p$-variate normal with mean $\mu_1 + \epsilon \mu_2$ and covariance $V_1 + \epsilon^2 V_2$. From (1.2) it follows that the density of $Y$ is given in terms of that of $X_1$ by

$$\log p_Y(x) = \log p_{X_1}(x) - 2^{-1} \sum_{k=0}^{\infty} \epsilon^k d_k,$$
where
\[ d_{2j} = (-1)^j j^{-1} \text{tr} \left( V_1^{-1} V_2 \right)^j + (x - \mu_1)' b_j (x \mu_1) + \mu_2 b_{j-1} \mu_2, \]
\[ d_{2j+1} = -2 \mu_2 b_j (x - \mu_1), \]
\[ b_{-1} = 0, \quad b_j = (-1)^j \left( V_1^{-1} V_2 \right)^j V_1^{-1} \]
for \( j \geq 1. \)

REFERENCES


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