ABSTRACT. This paper deals with stability analysis of a delayed SIS epidemiological model with disease-induced mortality and nonlinear incidence rate. Conditions are derived under which there can be no change in stability. Using the discrete time delay as a bifurcation parameter it is found that Hopf bifurcation occurs when the delay passes through a critical value. In case of the single delay a formula for determining the stability of the periodic solutions is given by using the centre manifold theory and the normal form method. Results are verified by computer simulations.

1 Introduction In modelling the spread of infections the population is usually considered to be subdivided into disjoint epidemiological classes (or compartments) of individuals in relation to the infectious disease: susceptible individuals, $S$, exposed individuals, $E$, infectious individuals, $I$ and removed individuals, $R$. The development of the infection is represented by transitions between these classes. The number of compartments included depends on the disease being modelled. If we take into account that the acquired immunity to reinfection is virtually non existent and hence recovered individuals pass directly back to the corresponding susceptible class then we deal with so called SIS models. These situations can be described, at least up to a crude first approximation, by a simple system of first order ordinary differential equations for the rates of transfer from one compartment to another. Studies of the dynamical properties of such models usually consist of finding constant equilibrium solutions, and then conducting a linearized analysis to determine their stability with respect to small disturbances.
In order to have more realism, it is often necessary taking into consideration that the present dynamics, the present rate of change of the state variables depends not only on the present state of the processes but also on the history of the phenomenon, on past values of the state variables. Thus, for some disease transmission models the incorporation of time delay effect is necessary: in case of, e.g., tuberculosis, influenza, measles, on adequate contact with an infective, a susceptible individual becomes infected but is not yet infective (cf. [12]). However, there are other aspects which motivate the incorporation of time delay effect into the system describing disease transmission, namely, it can also be used to model a period of temporal immunity (in a model including $R$), a latent period (in a model including $E$) or a maturation period (cf. [15] and the references therein). If delay effects must be considered then these models are formulated as a system of functional differential and/or integral equations. Delays in epidemic models can destabilize the originally stable equilibrium, so that periodic solutions arise by Hopf bifurcation. For a survey of epidemiological models with delays we refer to [14].

The time evolution model, we are dealing with was proposed in [1]. This is a $SIS$ epidemiological model with disease-induced mortality and bilinear incidence rate, in which the population is assumed to be divided in two interacting classes of individuals, namely susceptibles $S$ and infectives $I$. This model has, contrary to the most classical models (cf. [17]), a variable demographic structure: the total population, the sum of the numbers in all compartments was here assumed to be not constant.

Our paper is organized as follows. In Section 2, we introduce the model. In Section 3 we summarize the results about the model without delay regarded the existence and global stability of possible equilibria. The main result of this paper is given in Section 4. We examine the stability of the constant equilibria of the delayed system and the occurrence of periodic orbits. The stability of these orbits in a special case is determined by constructing a centre manifold and by applying the normal form method. In Section 5 we give a short discussion.

2 The model Microparasites can have pronounced effects on the growth characteristics of their host population. The work [1] focuses on microparasitic infections that are directly transmitted among invertebrate hosts and deals among others with the following first order ODE
system
\[
\begin{align*}
\dot{S} &= f_S(S, I) := a(S + I) - bS - \beta SI + \gamma I, \\
\dot{I} &= f_I(S, I) := \beta SI - (\alpha + b + \gamma)I.
\end{align*}
\]
Here, the dot means differentiation with respect to time \(t\); \(S(t)\) and \(I(t)\) are the densities of the populations of mature susceptible and infective hosts at time \(t\). \(a > 0\) represents the birth rate of the population (it is assumed that all newborns are susceptible and the birth rate is independent of whether or not the host is infected); \(b > 0\) and \(\alpha > 0\) are parameters associated with the death rate of susceptible and infective classes; \(\gamma > 0\) is the rate at which infected individuals recover and again become susceptible to re-infection; \(\beta > 0\) denotes the per capita horizontal transfer rate between a single susceptible and a single infected individual. A schematic representation of the model is shown in Figure 1.

The rate of change of the total population of hosts, \(N := S + I\), is obtained by adding equations in (1) to give
\[
\dot{N} = rN - \alpha I
\]
where \(r := a - b\) is the intrinsic growth rate of the disease-free host population. Thus, the total population is not a constant, but rather a dynamic variable (cf. [7, 10, 13]). Models with variable population size are interesting from several reasons. Instead of one threshold given by the basic reproduction number, these models can involve several thresholds that determine the asymptotic behaviour (cf. [23]).

In order to have a realistic model, we need to take into account that in SIS epidemiologic models, susceptibles become infectious after a sufficient contact with an infective. Therefore, it is reasonable to assume that the migration of the individuals from the class of susceptibles into the one of infected is subject to delay. We assume furthermore that the offspring of susceptible and infective parents are immune to the disease for a period, after which they become susceptible, hence the inflow of newborns into the susceptible class is also subject to time delay. Thus, the following system will be considered
\[
\begin{align*}
\dot{S} &= a(S(-\tau_2) + I(-\tau_2)) - bS - \beta SI + \gamma I, \\
\dot{I} &= \beta S(-\tau_1)I(-\tau_1) - (\alpha + b + \gamma)I,
\end{align*}
\]
with initial conditions
\(S_0(\vartheta) = \varphi_1(\vartheta) \geq 0, \ I_0(\vartheta) = \varphi_2(\vartheta) \geq 0 \ (\vartheta \in [-\tau, 0]),\ S_0(0) > 0, \ I_0(0) > 0\)
where $\varphi = (\varphi_1, \varphi_2)$ belongs to the Banach space $\mathcal{C} := C([-\tau, 0], (\mathbb{R}_0^+)^2)$ equipped with the norm $||\varphi|| := \max \{|\varphi(\psi)| : \psi \in [-\tau, 0]\}$ where $| \cdot |$ is any norm in $\mathbb{R}^2$ and $\tau := \max\{\tau_1, \tau_2\}$.

To the authors’ knowledge, (1) has not yet been studied with the delay as it is in (3), still it was subject of delay by many papers. In [6] the delayed epidemic model

$$\begin{align*}
\frac{dS}{dt} &= -rS + bS(t-\tau) + pb'I(t-\tau) - kS(t)I(t) \\
\frac{dI}{dt} &= -r'I(t) + qb'I(t-\tau) + kS(t)I(t) \\
\end{align*}$$

with $b = \delta e^{-\tau r}$, $b' = \delta' e^{-\tau' r}$ and $S(t) = S_0(t)$, $I(t) = I_0(t)$ ($-\tau \leq t \leq 0$), $S_0, I_0 \in \mathcal{C}$ has been investigated. A global stability analysis is given for the model when the maturation time is zero and the epidemiological effects of vertical transmission are discussed. Special cases of models with maturation delays, incubation delays and spatial diffusion...
are analyzed. In [27] the delayed model of the form

\[
\begin{align*}
\frac{dS}{dt} &= (b - r)S(t) + pb'I(t - \tau_1) - KS(t)I(t), \\
\frac{dI}{dt} &= -r'I(t) + qb'I(t - \tau_2) + KS(t)I(t),
\end{align*}
\]

with \( S(t) = S_0(t), I(t) = I_0(t) \) \((-\tau \leq t \leq 0), S_0, I_0 \in C \) has been considered. Using the Nyquist criterion on the characteristic equation, an estimate on the length of delays is given for which a system that is stable in the absence of delays remains stable. Further, conditions are derived under which there can be no change of stability.

3 The no-delay case In this section we summarize the results concerning existence and stability of equilibria. For a general system of autonomous ODEs this can be determined by using the saddle-node and Andronov-Hopf curves. A general method for finding these curves is the parametric representation method (cf. [32]). Here the system without delay is relatively simple hence the equilibria can be determined explicitly.

It is easy to see that system (1) has the trivial equilibrium \((0, 0)\) for all parameter values (in case of \( r < 0 \) this is the unique steady state); disease free equilibria (boundary equilibria) \((K, 0)\) (with optional \( K > 0 \), i.e., the whole \( S \)-axis consists equilibria) as long as the birth and death rates of susceptibles are identical: \( a = b \), i.e., the intrinsic growth rate of the host population is zero: \( r = 0 \); and one equilibrium with positive coordinates (endemic equilibrium)

\[
(\bar{S}, \bar{I}) := \left( \frac{T}{b'}, (a + b - a, a - b) \right) \quad \text{with} \quad T := \frac{a + b + \gamma}{a + b - a}
\]

provided that

\[
R_0 > 1
\]

holds, where we have defined the threshold \( R_0 := \alpha/r; R_0 \) is the basic reproductive rate of the parasite. Clearly, the inequality (7) implies \( r > 0 \). As we will see in the next paragraph, the proportion to 0 and 1 of the numbers \( r \) and \( R_0 \), respectively, determines the local asymptotic stability of these equilibria. If the parasite is sufficiently pathogenetic, i.e., \( R_0 > 1 \), it regulates the host population at a stable equilibrium level.
 Conversely, if $R_0 < 1$ then the disease is not able to regulate the host population to a stable level.

If the birth and death rates of susceptibles are identical, i.e., $a = b$ holds, the slope of the trajectories is given by

$$\frac{dI}{dS} = \frac{\dot{I}}{\dot{S}} = \frac{\beta S - (\alpha + a + \gamma)}{a + \gamma - \beta S}.$$ 

Thus, if $S > (\alpha + a + \gamma)/\beta$, then $\dot{S} < 0$ and $\dot{I} > 0$, if $(\alpha + \gamma)/\beta < S < (\alpha + a + \gamma)/\beta$, then $\dot{S} < 0$ and $\dot{I} < 0$, and if $S < (\alpha + a + \gamma)/\beta$, then $\dot{S} > 0$ and $\dot{I} < 0$. Furthermore, the maximum number of infectives occurs when $S = (\alpha + a + \gamma)/\beta$. A standard stability analysis based on the Jacobian

$$J(S, I) := \begin{bmatrix} -\beta I + a - b & -\beta S + (a + \gamma) \\ \beta I & \beta S - (a + b + \gamma) \end{bmatrix}$$

for the equilibria shows that in case of $r \neq 0$

- the origin is asymptotically stable if and only if the intrinsic growth rate of the disease-free host population is negative, i.e., $r < 0$ holds: the eigenvalues of the matrix $J(0, 0) = \begin{bmatrix} a - b & a + \gamma \\ 0 & -(a + b + \gamma) \end{bmatrix}$ are $a - b$ and $-(a + b + \gamma)$;

- the endemic equilibrium is asymptotically stable if it exists, i.e., if $R_0 > 1$, the characteristic polynomial of the matrix

$$J(S, I) = \begin{bmatrix} (b - a) & \frac{a + \gamma}{\alpha + b - a} \\ a + b - a & -(a + b - a) \end{bmatrix} \begin{bmatrix} (a - b)T \\ 0 \end{bmatrix}$$

is

$$z^2 + (a - b)\frac{a + \gamma}{\alpha + b - a} \cdot z + (a - b)(a + b + \gamma)$$

which is clearly stable if inequality (7) holds.

The phase space portraits for different values of the parameters are presented in Figure 2.

It can be shown that in case of $R_0 > 1$ the endemic equilibrium is also globally asymptotically stable. To prove this, one has to find a suitable Lyapunov function (e.g., using the so called Beretta-Capasso approach (cf. [10])) or to show that any positive orbit in the positive
FIGURE 2: Phase portraits of the system (1) for $r < 0$ (top), $r = 0$ (middle) and $r > 0$ with $R_0 > 1$ (bottom).
quadrant of the phase space $\Omega$ is bounded (cf. [30]), and to rule out
the existence of periodic orbits in $\Omega$. The latter result is easy to prove,
because $h(S, I) := 1/SI$ ($S, I > 0$) is a Dulac-function for system (1)
(cf. [16]): for $f := (f_S, f_I)$
\[
\text{div}(hf) = \frac{\partial(hf_S)}{\partial S} + \frac{\partial(hf_I)}{\partial I} = -\frac{\gamma + a}{S^2} < 0 \quad (S > 0, I > 0)
\]
holds.

4 The model with delay In this section we focus on investigat-
ing the stability of equilibria and Hopf bifurcation from the endemic
equilibrium of the system (3).

Using, e.g., step method (cf. [20]) it is easy to see that for every
initial function $\phi$ there exists a unique solution to equations (3). Ap-
plying Theorem 1.2 from [4], we conclude that the solutions to (3) are
nonnegative for any nonnegative initial condition.

4.1 Stability of equilibria Clearly, the equilibria $(0, 0)$, $(K, 0)$ (with
$K > 0$) and $(\bar{S}, \bar{I})$ of (1) are steady states of (3), too. Now, we are going
to determine the stability of equilibria $(0, 0)$ and $(\bar{S}, \bar{I})$ for (3). The
variational system of (3) with respect to the solution $(\bar{S}, \bar{I})$ (cf. [26])
assumes the form
\[
\begin{bmatrix}
\dot{u} \\
\dot{v}
\end{bmatrix} =
\begin{bmatrix}
-b - \beta \bar{I} & \gamma - \beta \bar{S} \\
0 & -(\alpha + b + \gamma)
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
\beta \bar{I} & \beta \bar{S}
\end{bmatrix}
\begin{bmatrix}
u(-\tau_1) \\
v(-\tau_1)
\end{bmatrix}
+ \begin{bmatrix}
a & a \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
u(-\tau_2) \\
v(-\tau_2)
\end{bmatrix}.
\]
Thus, the linearized system

- at $(0, 0)$ is
\[
\begin{bmatrix}
\dot{u} \\
\dot{v}
\end{bmatrix} =
\begin{bmatrix}
-b & \gamma \\
0 & -(\alpha + b + \gamma)
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix}
+ \begin{bmatrix}
a & a \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
u(-\tau_2) \\
v(-\tau_2)
\end{bmatrix},
\]

- at $(\bar{S}, \bar{I})$ it has the form
\[
\begin{bmatrix}
\dot{u} \\
\dot{v}
\end{bmatrix} =
\begin{bmatrix}
-b - \beta \bar{I} & \gamma - \beta \bar{S} \\
0 & -(\alpha + b + \gamma)
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
\beta \bar{I} & \beta \bar{S}
\end{bmatrix}
\begin{bmatrix}
u(-\tau_1) \\
v(-\tau_1)
\end{bmatrix}
+ \begin{bmatrix}
a & a \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
u(-\tau_2) \\
v(-\tau_2)
\end{bmatrix}.
\]
The characteristic function of equation (3) can be obtained by substituting the trial solution \((u(\cdot), v(\cdot)) := (c_u, c_v) \exp(z)\) into the linearized system (cf. \([21]\)):

- at \((0, 0)\):

\[
\Delta_{(0,0)}(z; \tau_1, \tau_2) := \det \begin{bmatrix}
-b + ae^{-z\tau_2} - z & \gamma + ae^{-z\tau_2} \\
0 & -(\alpha + \beta + \gamma) - z
\end{bmatrix}
\]

\[
\equiv \Delta^{(1)}(0,0)(z) \cdot \Delta^{(2)}_{(0,0)}(z; \tau_2)
\]

\[
\equiv (z + \alpha + \beta + \gamma) \cdot (z - ae^{-z\tau_2} + b),
\]

and

- at \((S, T)\):

\[
\Delta_{(S,T)}(z; \tau_1, \tau_2)
\]

\[
:= \det \begin{bmatrix}
-b - (a - b)T + ae^{-z\tau_2} - z & -(\alpha + b) + ae^{-z\tau_2} \\
(a - b)Te^{-z\tau_1} & (\alpha + b + \gamma) (e^{-z\tau_1} - 1) - z
\end{bmatrix}
\]

\[
\equiv z^2 + Az + B + Ce^{-z\tau_1} + De^{-z\tau_2} + Eze^{-z\tau_1} + Fze^{-z\tau_2} + G e^{-z(\tau_1+\tau_2)}
\]

where

\[
A := \frac{(\alpha + b)^2 - ab + \alpha \gamma}{\alpha + b - a}, \quad D := -a(\alpha + b + \gamma),
\]

\[
B := [\alpha a + \gamma(a - b)]T, \quad E := -(\alpha + b + \gamma),
\]

\[
C := -[\gamma(a - b) + b(\alpha + b - a)]T, \quad F := -a,
\]

\[
G := a(\alpha + 2b - 2a)T.
\]

4.1.1 The stability of the trivial equilibrium Clearly, the characteristic equation \(\Delta_{(0,0)}(z; \tau_1, \tau_2) = 0\) is stable if and only if \(\Delta^{(2)}_{(0,0)}(z; \tau_2)\) is stable, because the \(\Delta^{(1)}_{(0,0)}(\cdot)\) is stable for all parameter values. For \(\tau_1 \in [0, +\infty)\), \(\tau_2 = 0\) the function \(\Delta^{(2)}_{(0,0)}(\cdot; \tau_1, \tau_2)\) is identical with the characteristic function of \(J(0,0)\). The behaviour of characteristic function of type \(\Delta^{(2)}_{(0,0)}(\cdot; \tau_2)\) has been described in detail e.g., in \([22]\) (cf. \([3],[20, p. 339]\)), [18]. It is easy to examine its stability due to its simple structure. Thus, we can prove the following.
Theorem 4.1. If \( a \neq b \), then delay does not change the stability of the trivial equilibrium, i.e.,

1. if \( a > b \), then \((0, 0)\) is an unstable equilibrium of system (1) and is unstable for (3), too;
2. if \( a < b \), then \((0, 0)\) is a stable equilibrium of system (1) and remains stable for (3).

Proof. Step 1 If \( a > b \) then \((0, 0)\) is an unstable equilibrium of (3) for \( \tau_1 = \tau_2 = 0 \). It remains unstable for \( \tau_1 \geq 0, \tau_2 > 0 \) because, as a consequence of the Bolzano Theorem, \( \Delta^{(2)}_{(0,0)}(z; \tau_2) \) has a positive root: \( \Delta^{(2)}_{(0,0)}(0; \tau_2) = b - a < 0, \Delta^{(2)}_{(0,0)}(z; \tau_2) : R \to R \) is a continuous function with \( \lim_{z \to +\infty} \Delta^{(2)}_{(0,0)}(z; \tau_2) = +\infty \).

Step 2 If \( b > a \) then \((0, 0)\) is a stable equilibrium of (3) for \( \tau_1 = \tau_2 = 0 \).

Clearly, if \( \tau_2 > 0 \) and \( \Delta^{(2)}_{(0,0)}(z; \tau_2) \) has purely imaginary roots \( \pm \omega (\omega > 0) \), then

\[
    b - a \cos(\omega \tau_2) + i[\omega + a \sin(\omega \tau_2)] = 0,
\]

i.e., \( \omega^2 = a^2 - b^2 \leq 0 \), which is a contradiction.

4.1.2 The stability of the equilibrium \((S, I)\) We turn now to the characteristic equation \( \Delta_{(S,I)}(z; \tau_1, \tau_2) = 0 \). Equation of this type has already been studied by many authors. Using the Nyquist-plot technique (cf. \[29, 35\]) in \[19\] was proved that \((S, I)\) is an asymptotically stable equilibrium of (3) if

\[
    (9) \quad B + C + D + G > 0 \quad \text{and} \quad \tau_1 + \tau_2 < \frac{A + E + F}{|C| + |D| + |G|}
\]

holds, furthermore conditions where derived that imply no change in stability. In \[333\] an estimate of the delays was given for which stability will persist:

\[
    (10) \quad B + C + D + G > 0 \quad \text{and} \quad A + E + F > (G + 0.22|C|)\tau_1 + (G + 0.22|D|)\tau_2.
\]

In order to examine the stability of \( \Delta_{(S,I)}(z; \tau_1, \tau_2) \) we deal with the following two cases:
1. to fix one of the two delays at zero, we consider the other one as a parameter and find intervals for the delay where the characteristic equation is stable;
2. to make the two delays equal: \( \tau := \tau_1 = \tau_2 \) and examine the reduced characteristic equation from point of view of stability.

Clearly, in this section, the condition (7) will be assumed everywhere which implies that

\[
A + E + F = \frac{(a-b)(a+\gamma)}{\alpha+b-a} > 0,
\]

\[
B + C + D + G = (a-b)(\alpha+b+\gamma) > 0
\]

hold. Hence, for \( \tau_1 = \tau_2 = 0 \), the equilibrium \((0,0)\) is unstable, the polynomial \( \Delta_{(S,I)}(z;0,0) \), i.e., (8) is stable and, as a consequence, \((S,T)\) is asymptotically stable as an equilibrium of system (3). Furthermore, for all \( \tau_1, \tau_2 > 0 \), \( z = 0 \) is not a root of \( \Delta_{(S,I)}(z;\tau_1,\tau_2) = 0 \).

In the first case the characteristic equation \( \Delta_{(S,I)}(z;\tau_1,\tau_2) = 0 \) reduces to

\[
\Delta_{(S,I)}(z;\tau,0) \equiv \begin{align*}
\Delta_{(S,I)}(z;\tau,0) & \equiv z^2 + (A + F + Ee^{-\tau})z \\
& + B + D + (C + G)e^{-\tau} = 0 \quad (\tau_2 = 0),
\end{align*}
\]

respectively,

\[
\Delta_{(S,I)}(z;0,\tau) \equiv \begin{align*}
\Delta_{(S,I)}(z;0,\tau) & \equiv z^2 + (A + E + Fe^{-\tau})z + B + C \\
& + (D + G)e^{-\tau} = 0 \quad (\tau_1 = 0),
\end{align*}
\]

and in the second case,

\[
\Delta_{(S,I)}(z,\tau) \equiv \begin{align*}
\Delta_{(S,I)}(z,\tau) & \equiv z^2 + Az + B + (C + D)e^{-\tau} + (E + F)ze^{-\tau} + Ge^{-2\tau}.
\end{align*}
\]

The case of the single delay Now, in the first case using the Mikhailov criterion we are going to estimate the domains of the delay \( \tau \) for which the endemic equilibrium is stable. In order to find stability regions for the quasipolynomial \( \Delta(\omega,\tau) \) it is enough to investigate the change of the argument of \( \Delta(\omega,\tau) \) as \( \omega \) increases from 0 to +\( \infty \) which is formulated in detail in the following
Lemma 4.1 (Mikhailov (cf. [25, 26])). Let $P$ and $Q$ be polynomials of degree $m$ and $n$, respectively, with $m > n$, $\tau > 0$, and assume that the quasi-polynomial

$$\Delta(z, \tau) \equiv P(z) + Q(z) \exp(-z\tau)$$

has no roots on the imaginary axis. Then $\Delta(\cdot, \tau)$ is stable, i.e., all of its roots have negative real part if and only if

$$[\arg \Delta(i\omega, \tau)]_{\omega=0}^{\omega=+\infty} = \frac{\pi}{2} \cdot \deg P(i\omega),$$

i.e., the argument of $\Delta(i\omega, \tau)$ increases $m\pi/2$ as $\omega$ increases from 0 to $+\infty$.

Thus, we are able to prove the following

Theorem 4.2. If

- $\tau_2 = 0$, 

$$\gamma \neq \alpha + b - 2a \quad \text{and} \quad \tau_1 < \tau_{s1} := \frac{a + \gamma}{(\alpha + b + \gamma)[2a - \alpha - b + \gamma]}$$

respectively

- $\tau_1 = 0$, 

$$\tau_2 < \tau_{s2} := \frac{a + \gamma}{a(\alpha + b + \gamma)},$$

then $(S, \bar{T})$ is a stable equilibrium of (3).

**Proof.** Equations $\Delta_{(S, \bar{T})}(z; \tau, 0) = 0$ and $\Delta_{(S, \bar{T})}(z; 0, \tau) = 0$ imply in case of $\tau_1 = \tau$, $\tau_2 = 0$

$$P_1(z) \equiv z^2 + (A + F)z + B + D, \quad Q_1(z) \equiv C + G + Ez,$$

and in case of $\tau_1 = 0$, $\tau_2 = \tau$

$$P_2(z) \equiv z^2 + (A + E)z + B + C, \quad Q_2(z) \equiv D + G + Fz,$$

respectively. Thus, $\Delta_{(S, \bar{T})}(\cdot; \tau, 0)$ or $\Delta_{(S, \bar{T})}(\cdot; 0, \tau)$ are stable if and only if

$$[\arg \Delta_{(S, \bar{T})}(i\omega; \tau, 0)]_{\omega=0}^{\omega=+\infty} = \pi \quad \text{or} \quad [\arg \Delta_{(S, \bar{T})}(i\omega; 0, \tau)]_{\omega=0}^{\omega=+\infty} = \pi.$$
respectively. We have

\[
\Re \left( \Delta_{SIT}(\omega; \tau, 0) \right) = -\omega^2 + B + D \\
+ (C + G) \cos(\omega \tau) + E\omega \sin(\omega \tau)
\]

\( \Re \left( \Delta_{SIT}(\omega; \tau, 0) \right) = (A + F)\omega \\
- (C + G) \sin(\omega \tau) + E\omega \cos(\omega \tau),
\]

respectively.

\[
\Re \left( \Delta_{SIT}(\omega; 0, \tau) \right) = -\omega^2 + B + C \\
+ (D + G) \cos(\omega \tau) + F\omega \sin(\omega \tau)
\]

\( \Re \left( \Delta_{SIT}(\omega; 0, \tau) \right) = (A + E)\omega \\
- (D + G) \sin(\omega \tau) + F\omega \cos(\omega \tau).
\]

Moreover,

\[
\Re \left( \Delta_{SIT}(0; \tau, 0) \right) = \Re \left( \Delta_{SIT}(0; 0, \tau) \right) \\
= B + D + C + G = (a - b)(\alpha + b + \gamma) > 0,
\]

\( \Re \left( \Delta_{SIT}(0; \tau, 0) \right) = \Re \left( \Delta_{SIT}(0; 0, \tau) \right) = 0.
\]

Hence, if \( \Im \left( \Delta_{SIT}(\omega; \tau, 0) \right) > 0 \) \( (\omega > 0) \) respectively \( \Im \left( \Delta_{SIT}(\omega; 0, \tau) \right) > 0 \) \( (\omega > 0) \), then because of \( \lim_{\omega \to +\infty} \Re \left( \Delta_{SIT}(\omega; \tau, 0) \right) = -\infty \) respectively \( \lim_{\omega \to +\infty} \Re \left( \Delta_{SIT}(\omega; 0, \tau) \right) = -\infty \), the change of the argument of \( \Delta_{SIT}(\omega; \tau, 0) \), respectively of \( \Delta_{SIT}(\omega; 0, \tau) \) is equal to \( \pi \).

Namely, in this case

\[
\sin \left( \arg \Delta(\omega, \tau) \right) = \frac{\Im \left( \Delta(\omega, \tau) \right)}{\sqrt{\Re^2 \left( \Delta(\omega, \tau) \right) + \Im^2 \left( \Delta(\omega, \tau) \right)}} \to 0 \quad (\omega \to +\infty),
\]

\[
\cos \left( \arg \Delta(\omega, \tau) \right) = \frac{\Re \left( \Delta(\omega, \tau) \right)}{\sqrt{\Re^2 \left( \Delta(\omega, \tau) \right) + \Im^2 \left( \Delta(\omega, \tau) \right)}} \to -1 \quad (\omega \to +\infty)
\]

with \( \Delta(\omega, \tau) \in \{ \Delta_{SIT}(\omega; \tau, 0), \Delta_{SIT}(\omega; 0, \tau) \} \).
Substituting \( w := \omega \tau \) in (18), respectively (19) and multiplying the result by \( \tau \), we obtain
\[
\tau \mathfrak{N} \left( \Delta_{\left( \frac{\omega}{\tau}; 0 \right)} \right) = (A + F)w - (C + G)\tau \sin(w) + Ew \cos(w),
\]
respectively,
\[
\tau \mathfrak{N} \left( \Delta_{\left( \frac{\omega}{\tau}; 0 \right)} \right) = (A + E)w - (D + G)\tau \sin(w) + Fw \cos(w).
\]
For every \( w \geq 0 \), we have
\[
Ew \cos(w) \geq -|E|w,
\]
\[
(C + G)\tau \sin(w) \leq \tau|C + G|w,
\]
respectively,
\[
Fw \cos(w) \geq -|F|w,
\]
\[
(D + G)\tau \sin(w) \leq \tau|D + G|w.
\]
Therefore,
\[
\tau \mathfrak{N} \left( \Delta_{\left( \frac{\omega}{\tau}; 0 \right)} \right) \geq w (A + F - |E| - \tau|C + G|),
\]
respectively,
\[
\tau \mathfrak{N} \left( \Delta_{\left( \frac{\omega}{\tau}; 0 \right)} \right) \geq w (A + E - |F| - \tau|D + G|).
\]
It is easy to calculate that
\[
A + F - |E| = \frac{(a - b)(a + \gamma)}{\alpha + b - a} > 0
\]
and
\[
|C + G| = (a - b)|2a - \alpha - b + \gamma| T,
\]
respectively,
\[
A + E - |F| = \frac{(a - b)(a + \gamma)}{\alpha + b - a} > 0 \quad \text{and} \quad |D + G| = a(a - b)T.
\]
Thus, if
\[
\tau_1 < \frac{A + F - |E|}{|C + G|} = \frac{a + \gamma}{(\alpha + b + \gamma)(2a - \alpha - b + \gamma)} \quad (\gamma \neq \alpha + b - 2\alpha),
\]
respectively,
\[
\tau_2 < \frac{A + E - |F|}{|D + G|} = \frac{a + \gamma}{\alpha(\alpha + b + \gamma)},
\]
then all roots of \( \Delta_{(\mathcal{S}, \mathcal{T})}(\cdot; \tau_1, 0) \) respectively of \( \Delta_{(\mathcal{S}, \mathcal{T})}(\cdot; 0, \tau_2) \) have negative real part which means stability.

Remark 4.1. In certain cases our estimates for the stable intervals of \( \tau_i \ (i \in \{1, 2\}) \) are better than the ones in [19], respectively in [33]. For example, for the parameter values \( a = 0.0600, b := 0.0080, \beta := 0.0056, \gamma := 0.0200 \) and \( \alpha = 0.3200 \) (9), respectively (10) imply
\[
\tau_1 + \tau_2 < 0.3710 \quad \text{respectively} \quad \tau_1 < 0.8751, \quad \tau_2 < 0.7246
\]
while (16), respectively (17) gives the following values
\[
\tau_1 < \tau_{s_1} = 1.2228, \quad \tau_2 < \tau_{s_2} = 3.8314.
\]

The case of the multiple delay Now we are going to examine the stability of the characteristic equation in the second case where \( \Delta_{(\mathcal{S}, \mathcal{T})}(\cdot, \tau) \) in (14) is not yet a quasipolynomial. For easier reference we quote here the simplified version of Corollary 3.3. from [36] about the stability of the characteristic function of the form
\[
(20) \quad \Delta(z, \tau) \equiv p(z) + q(z)e^{-\tau z} + r(z)e^{-2\tau z},
\]
where \( p, q \) and \( r \) are polynomials for which \( \deg(r) < \deg(q) \) holds, is to be used.

Lemma 4.2. For \( y \in \mathbb{R} \) define
\[
a(y) := R(r(iy)) + R(p(iy)), \quad b(y) := z(r(iy)) - z(p(iy)),
\]
\[
c(y) := z(r(iy)) + z(p(iy)), \quad d(y) := R(p(iy)) - R(r(iy)),
\]
\[
e(y) := R(q(iy)), \quad f(y) := z(q(iy)).
\]
and suppose that in (20) the polynomial \( \Delta(\cdot, 0) \) is stable. If
\[
D(y) := a(y)d(y) - b(y)e(y) \neq 0 \quad (0 \neq y \in \mathbb{R}),
\]
then \( \Delta(\cdot, \tau) \) is stable—i.e., \( \Delta(\cdot, \tau) \) has no root \( \omega \) (\( \omega > 0 \))—for any given delay \( \tau \) if and only if there is no \( y > 0 \) for which
\[
[e(y)d(y) - f(y)b(y)]^2 + [e(y)e(y) - f(y)a(y)]^2 - \|D(y)\|^2 = 0
\]
holds.

In what follows, the role of the polynomials \( p, q \) and \( r \) will be played by
\[
\begin{align*}
p(z) & \equiv z^2 + Az + B, \\
q(z) & \equiv (E + F)z + C + D, \\
r(z) & \equiv G.
\end{align*}
\]
Hence, using these notations (14) takes the form of (20).

Now we are able to prove the following.

Theorem 4.3. If conditions \( A^2 > 2B \) and \( B^2 > G^2 \) are fulfilled, moreover, \( F_3 \) has a positive root \( \omega \) where
\[
F_3(y) := y^8 + c_6 y^6 + c_4 y^4 + c_2 y^2 + c_0 \quad (y \in \mathbb{R})
\]
with coefficients defined as
\[
\begin{align*}
c_6 & := 2(A^2 - 2B) - (E + F)^2, \\
c_4 & := A^4 - (C + D)^2 + 2B(3B + (E + F)^2) - A^2(4B + (E + F)^2) \\
& \quad + 2G((E + F)^2 - G), \\
c_2 & := 2(B^2 - G^2)(A^2 - 2B) + 2(C + D)(C + D - A(E + F))(B - G) \\
& \quad - (A(C + D) - (E + F)(B + G))^2, \\
c_0 & := (B - G)^2(B - C - D + G)(B + C + D + G),
\end{align*}
\]
then the endemic equilibrium \( (S, I) \) may lose its stability and eventually undergo a Hopf bifurcation as \( \tau \) increases and passes through a critical value, i.e., there may occur a small amplitude periodic solution with period approximately equal to \( 2\pi/\omega \).
Proof. Clearly,
\begin{align*}
a(y) &= G + B - y^2, \quad b(y) = -Ay, \\
c(y) &= A\omega, \quad d(y) = B - G - y^2, \quad (y \in \mathbb{R}) \\
e(y) &= C + D, \quad f(y) = (E + F)y.
\end{align*}
Therefore,
\[
\mathcal{D}(y) = a(y)d(y) - b(y)c(y) \\
= y^4 + (A^2 - 2B)y^2 + B^2 - G^2 \quad (y \in \mathbb{R}).
\]
Thus, \(A^2 > 2B\) and \(B^2 > G^2\) imply that \(\mathcal{D}(y) > 0\) for all \(y \in \mathbb{R}\).
Denoting the left-hand side in (22) by \(\mathcal{F}_3(y)\) for \(y \in \mathbb{R}\) a straightforward computation shows that \((\mathcal{S}, T)\) may lose its stability only if \(\mathcal{F}_3\) has a positive root.

4.2 Hopf bifurcation from the equilibrium \((\mathcal{S}, T)\)

In this section we apply the Hopf bifurcation theorem to show the existence of nontrivial periodic solutions to (3). We use the delay as a parameter of bifurcation. Similarly to stability investigations we deal with the two cases: first, it is assumed that one of the two delays is equal to zero, second, equating the two delays we show the existence of a limit cycle for (3). In the case of the single delay the stability of the bifurcating periodic solution will be also examined.

4.2.1 The case of single delay

To obtain stability switch one needs to have an imaginary root of \(\Delta_{(\mathcal{S}, T)}(\cdot; \tau, 0)\), respectively \(\Delta_{(\mathcal{S}, T)}(\cdot; 0, \tau)\). Let \(z = \omega \omega > 0\). Then
\[
\Delta_{(\mathcal{S}, T)}(i\omega; \tau, 0) = 0, \quad \text{respectively} \quad \Delta_{(\mathcal{S}, T)}(i\omega; 0, \tau) = 0
\]
implies
\[
(25) \quad |P_i(i\omega)| = |Q_i(i\omega)|, \quad \text{respectively} \quad |P_2(i\omega)| = |Q_2(i\omega)|.
\]
Thus, (25) determines a set of possible values of \(\omega\). We define the auxiliary function
\[
\mathcal{F}_i(y) := |P_i(iy)|^2 - |Q_i(iy)|^2 \quad (y \in \mathbb{R}) \quad \text{for} \quad i \in \{1; 2\}
\]
and look for stability switches that may occur when \(\mathcal{F}_i(\omega) = 0\) for some \(\omega_1 > 0\) (\(i = 1\) respectively \(i = 2\)). But before we proceed we quote here a lemma for easier reference about the stability of the characteristic function of the form in (15).
Lemma 4.3. (cf. [5, 11]) Let $p$ and $q$ be analytic functions in a right half-plane $\Re(z) > -c$ ($c > 0$) which satisfy the following conditions:

- $p$ and $q$ have no common imaginary root;
- $p(-iy) = p(iy)$, $q(-iy) = q(iy)$;
- $p(0) + q(0) \neq 0$;
- $\limsup \{|q(z)/p(z)| : |z| \to \infty, \Re(z) \geq 0\} < 1$;
- for all $y \in \mathbb{R}$, $\mathcal{F}(y) := |p(iy)|^2 - |q(iy)|^2$ has at most a finite number of real zeros, furthermore, let $\Delta(z, \tau) := p(z) + q(z)\exp(-z\tau)$.

Then the following statements are true.

1. If $\mathcal{F}$ has no positive roots, then no stability switch may occur, i.e., if $\Delta(\cdot, \tau)$ is stable at $\tau = 0$, it remains stable for all $\tau \geq 0$, whereas if it is unstable at $\tau = 0$, it remains unstable for all $\tau \geq 0$.

2. If $\mathcal{F}$ has at least one positive root and each of them is simple, then as $\tau$ increases, a finite number of stability switches may occur, and eventually $\Delta(\cdot, \tau)$ becomes unstable, i.e., there is a $\tau^* > 0$ such that for all $\tau > \tau^*$, $\Delta(\cdot, \tau)$ is unstable.

Clearly, $P_1$ and $Q_1$ ($\in \{1; 2\}$) are trivially analytic functions in a right half-plane $\Re(z) > -c$ ($c > 0$) (they are polynomials) satisfying the conditions of Lemma 4.3 and for $i \in \{1, 2\}$

$$\mathcal{F}_i(y) = y^4 + \kappa_i y^2 + \zeta_i \quad (y \in \mathbb{R})$$

where the coefficients are

$$\kappa_1 := (A + F)^2 - 2(B + D) - E^2 = \frac{(a - b)^2(a + \gamma)^2}{(a + b - a)^2} > 0,$$

$$\zeta_1 := (B + D + C + G)(B + D - C - G)$$

$$= (a - b)^2(a + b - a)T^2(3a - \alpha - b + 2\gamma),$$

respectively

$$\kappa_2 := (A + E)^2 - 2(B + C) - F^2$$

$$= \frac{(a - b)}{(a + b - a)^2} \times [a^2(2a + b) + a (2(\alpha + b)(\alpha + b + \gamma) + \gamma^2)$$

$$- a^3 - 2(\alpha + b)^2(\alpha + \gamma) - b\gamma^2],$$
\[ \zeta_2 := (B + C + D + G)(B + C - D - G) \]
\[ = (a - b)T^2[(\alpha + b)^2 - a^2] > 0. \]

Thus, we are able to prove the following.

**Theorem 4.4.** If \( \tau_2 = 0 \) and \( \alpha > 3a - b + 2\gamma \), then there exists a pair \( \tau_{\text{crit}}; \omega_1 > 0 \) such that \( (S, T) \) undergoes a Poincaré-Andronov-Hopf bifurcation as \( \tau_1 \) increases and passes through \( \tau_{\text{crit}} \), i.e., \( (S, T) \) loses its stability and there occurs a small amplitude periodic solution with period approximately \( 2\pi/\omega_1 \).

**Proof.** Clearly, in this case \( \zeta_1 < 0 \). Therefore \( \mathcal{F}_1 \) has unique positive root

\[ \omega_1 = \sqrt{\frac{\kappa_1^2 - 4 \zeta_1}{2}} \]

and \( |Q_1(\omega_1)|^2 = (C + G)^2 + E^2\omega_1^2 > 0 \). Thus, \( z_1(\tau_{\text{crit}}) := \omega_1 \) is a root of \( \Delta_{\mathcal{F}_1}(\cdot; \tau_{\text{crit}}, 0) \). Separating the real and imaginary parts, it follows that

\[
\cos(\omega_1 \tau_{\text{crit}}) = -\frac{\Re(P_1(\omega_1))\Re(Q_1(\omega_1)) + \Im(P_1(\omega_1))\Im(Q_1(\omega_1))}{|Q_1(\omega_1)|^2} \\
= \frac{(C + G)(\omega_1^2 - B - D) - E(A + F)\omega_1^2}{(C + G)^2 + E^2\omega_1^2},
\]

respectively

\[
\sin(\omega_1 \tau_{\text{crit}}) = -\frac{\Im(P_1(\omega_1))\Re(Q_1(\omega_1)) - \Re(P_1(\omega_1))\Im(Q_1(\omega_1))}{|Q_1(\omega_1)|^2} \\
= \frac{E\omega_1^3 + [(A + F)(C + G) - E(B + D)]\omega_1}{(C + G)^2 + E^2\omega_1^2}.
\]

Thus, at the critical value

\[ \tau_{\text{crit}} = \frac{1}{\omega_1} \cos^{-1} \left( \frac{(C + G)(\omega_1^2 - B - D) - E(A + F)\omega_1^2}{(C + G)^2 + E^2\omega_1^2} \right) \]

of the delay \( \tau = \tau_1 \) the endemic equilibrium may lose its stability.

To check if the equilibrium \( (S, T) \) loses its stability, one has to determine the sign of the derivative with respect to \( \tau \) of the real part of the
smooth extension of the root $z_1(\tau_{\text{crit}})$. Let us denote by $z_1(\tau)$ the root of $\Delta_{\tau_{\text{crit}}}f(\cdot, \tau, 0)$ that assumes the value $\omega_1$ at $\tau_{\text{crit}}$ and by

\begin{equation}
D(z, \tau) \equiv P_1(z) + Q_1(z) \exp(-z\tau),
\end{equation}

the characteristic quasipolynomial in (12) as a function of the delay parameter $\tau$. Since $D(z_1(\tau_{\text{crit}}), \tau_{\text{crit}}) = D(\omega_1, \tau_{\text{crit}}) = 0$ and $\omega_1$ is a simple root of the quasipolynomial $D(\cdot, \tau_{\text{crit}})$, the smooth function $z_1$ is uniquely determined by $D(z_1(\tau), \tau_{\text{crit}}) \equiv 0$, $z_1(\tau_{\text{crit}}) = \omega_1$. From the implicit function theorem it follows that

\begin{equation*}
\frac{\partial}{\partial z} \frac{\partial}{\partial \omega_1} D(\omega_1, \tau_{\text{crit}}) = \frac{\omega_1 Q_1'(\omega_1)}{P_1'(\omega_1) \exp(\omega_1 \tau_{\text{crit}}) + Q_1'(\omega_1) - \tau_{\text{crit}} Q_1(\omega_1)}.
\end{equation*}

Due to a result in [11], we have

\begin{equation*}
\text{sgn} \left( \Re \left( z_1'(\tau_{\text{crit}}) \right) \right) = \text{sgn} \left( \mathcal{F}_1'(\omega_1) \right) = \text{sgn} \left( 4\omega_1^4 + 2\kappa_1 \omega_1 \right) = \text{sgn} \left( 2\omega_1^2 + \kappa_1 \right) = \text{sgn} \left( \sqrt{\kappa_1^2 - 4\zeta_1} \right) = 1,
\end{equation*}

which completes the proof.

\begin{remark}
Clearly, for all parameter values $\zeta_2 > 0$ therefore in case of $\tau_1 = 0$, $\tau_2 > 0$ instability occurs only if $\mathcal{F}_2$ has positive root(s), i.e., if $\kappa_2 < -2\sqrt{\zeta_2}$ holds. Comparing the formula for $\kappa_2$ and the formula for

\begin{equation}
\kappa_2^2 - 4\zeta_2 = (a - b)^2 \left( -4T^2[(a + b)^2 - a^2] + \frac{1}{(a + b - a)^4} \times [a^2 + 2(a + b)^3 - a^2(2a + b) + 2(a + b)^2 \gamma + b\gamma^2 - a(2a + b)(a + b + \gamma) + \gamma^2] \right),
\end{equation}

it seems hopeless to find such parameters for which (27) holds. Therefore, this delay per se causes most likely no instability (cf. Figure 3) which means biologically that the delay in birth causes no changes in the qualitative behaviour of the system.

\begin{remark}
\end{remark}
The stability of the bifurcating periodic solution depends on the nonlinearity of the system. In the remainder of this section, we always assume that the assumptions of the last theorem hold with $\tau_0 := \tau_{crit1}$, $\omega_0 := \omega_1$ and study the stability of these periodic solutions that is we determine whether the Hopf bifurcation is supercritical (stable limit cycle exists around the unstable equilibrium) or subcritical (unstable limit cycle exists around the stable equilibrium).

Moving the interior equilibrium $(S, I)$ to the origin by the coordinate transformation $x_1 := S - \bar{S}$, $x_2 := I - \bar{I}$ and separating the linear terms from the nonlinear terms, we get system (3) in the form

$$\dot{x} = Ax + Bx(t - \tau) + f(x, x(t - \tau))$$
where \( x := (x_1, x_2) \) and

\[
A := \begin{bmatrix}
    r(1 - T) & -\Gamma/T \\
    0 & -\Gamma
\end{bmatrix}, \quad B := \begin{bmatrix} 0 & 0 \\
rT & \Gamma
\end{bmatrix},
\]

\[
f(x, x(-\tau)) := \begin{bmatrix}
    -\beta x_1 x_2 \\
    \beta x_1 (-\tau) x_2 (-\tau)
\end{bmatrix}.
\]

We shall investigate the stability of the periodic solutions of (28) at the critical value \( \tau_0 \). Our aim is to reduce this investigation for the ordinary differential equation of the form

\[
\begin{bmatrix}
    \dot{\xi} \\
    \dot{\eta}
\end{bmatrix} = \begin{bmatrix} 0 & \omega_0 \\
-\omega_0 & 0 \end{bmatrix} \begin{bmatrix}
    \xi \\
    \eta
\end{bmatrix} + \begin{bmatrix}
    F(\xi, \eta) \\
    G(\xi, \eta)
\end{bmatrix}
\]

with \( F(0, 0) = G(0, 0) = 0, \quad F'(0, 0) = G'(0, 0) = 0 \) and to apply the following result (cf. [2]).

**Lemma 4.4.** The trivial solution of (29) is attractive or is a repellor if \( \delta < 0 \) or \( \delta > 0 \), respectively, where

\[
\delta := \frac{1}{8\omega_0} \cdot \left[ (F_{\xi\xi}(0, 0) + F_{\eta\eta}(0, 0))(G_{\xi\xi}(0, 0)
\right.
\]

\[
- F_{\xi\eta}(0, 0) - G_{\eta\eta}(0, 0)) + (G_{\xi\xi}(0, 0) + G_{\eta\eta}(0, 0))
\]

\[
\cdot (F_{\xi\xi}(0, 0) - F_{\eta\eta}(0, 0) + G_{\xi\eta}(0, 0))
\]

\[
+ \frac{1}{8} \cdot [3F_{\xi\xi}(0, 0) + F_{\xi\eta}(0, 0) + G_{\xi\eta}(0, 0) + 3G_{\eta\eta}(0, 0)].
\]

This reduction includes four steps as follows.

*The first step* is to transform (28) into the operator differential equation

\[
\dot{x}_t = \mathcal{L}_\mu x_t + N_\mu(x_t)
\]

where

\[
x_t(\vartheta) := x(t + \vartheta), \quad x : [-\tau, 0] \to \mathbb{R}^2, \quad \vartheta \in [-\tau, 0], \quad \mu := \tau - \tau_0 \quad (\mu \in \mathbb{R})
\]

and the linear operator \( \mathcal{L}_\mu \) assumes the form

\[
(\mathcal{L}_\mu u)(\vartheta) := \begin{cases}
    u'(\vartheta), & \vartheta \in [-\tau, 0), \\
    Au(0) + Bu(-\tau), & \vartheta = 0
\end{cases} \quad (u \in \mathfrak{B})
\]
while the nonlinear operator $\mathcal{N}$ can be written as

$$
(\mathcal{N}_\mu(u))(\vartheta) := \begin{cases} 
0, & \vartheta \in [-\tau, 0), \\
 f(u(0), u(-\tau)), & \vartheta = 0 
\end{cases} (u \in \mathfrak{B}).
$$

Here, dot still refers to differentiation with respect to time $t$, while prime stands for differentiation with respect to $\vartheta$, $\mathfrak{B}$ denotes the Banach space of continuously differentiable mappings from $[-\tau, 0]$ into $\mathbb{K}^2$ ($\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$) and the dependence on the bifurcation parameter $\mu$ is also emphasized. Equation (31) is just the trivial equation $\dot{x}_t = x_t'$ when $\vartheta \in [-\tau, 0)$, and it becomes (28) when $\vartheta = 0$.

The second step is to determine the normalized eigenfunctions of the linear operator $\mathcal{L}_0$ in order to describe the centre manifold of (31) later. For this purpose, it is useful to define the adjoint operator $\mathcal{L}^*_\mu$ of $\mathcal{L}_\mu$ acting on the adjoint space $\mathfrak{B}^* := C^1([0, \tau], \mathbb{K}^2)$ as follows

$$
\mathcal{L}^*_\mu v(\sigma) = \begin{cases} 
-v'(\sigma), & \sigma \in (0, \tau], \\
 A^T v(0) + B^T v(\tau), & \sigma = 0 
\end{cases} (v \in \mathfrak{B}^*)
$$

and the bilinear form

$$
\langle \psi, \varphi \rangle := \psi^*(0)\varphi(0) + \int_{-\tau}^{0} \psi^*(\xi + \tau)B\varphi(\xi) \, d\xi
$$

for $\psi \in C([0, \tau], \mathbb{K}^2)$ and $\varphi \in C([-\tau, 0], \mathbb{K}^2)$ (cf. [34]) where $^*$ denotes either adjoint operator or transposed conjugate vector.

Clearly, the operator $\mathcal{L}_0$ has the same characteristic roots as the linear part of the delay-differential equation (28):

$$
\text{Ker}(\mathcal{L}_0 - z\mathbb{I}) \neq \{0\} \iff \det(A + Be^{-z\tau} - z\mathbb{I}) = 0,
$$

and the corresponding two characteristic exponents are also the same: $\lambda_{1,2}(\tau_0) = \pm i\omega_0$ (cf. [31]). Now, it is easy to calculate the eigenfunction $p \in \mathfrak{B}$ of $\mathcal{L}_0$ corresponding to the eigenvalue $i\omega_0$ and the eigenfunction $q \in \mathfrak{B}^*$ of $\mathcal{L}^*_0$ corresponding to the eigenvalue $-i\omega_0$. These eigenfunctions satisfy the boundary value problems

$$
(\mathcal{L}_0 p)(\vartheta) = i\omega_0 p(\vartheta), \quad \vartheta \in [-\tau_0, 0],
$$

$$
(\mathcal{L}^*_0 q)(\sigma) = -i\omega_0 q(\sigma), \quad \sigma \in [0, \tau_0],
$$

(34)
that is, 

\[
\begin{aligned}
& p'(\vartheta) = \nu \omega_0 p(\vartheta), \quad \vartheta \in [-\tau_0, 0), \\
& (A - \nu \omega_0 I) p(0) + B p(-\tau_0) = 0, \quad \vartheta = 0,
\end{aligned}
\]

respectively

\[
\begin{aligned}
& q'(\sigma) = \nu \omega_0 q(\sigma), \quad \sigma \in (0, \tau_0], \\
& (A^T + \nu \omega_0 I) q(0) + B^T q(\tau_0) = 0, \quad \sigma = 0.
\end{aligned}
\]

The solutions of these BVPs are

\[
p(\vartheta) = p(0) \exp(\nu \omega_0 \vartheta), \quad \vartheta \in [-\tau_0, 0) \quad \text{with} \quad p(0) = (1, \pi)
\]

and

\[
q(\sigma) = q(0) \exp(\nu \omega_0 \sigma), \quad \sigma \in (0, \tau_0] \quad \text{with} \quad q(0) = \rho(1, \kappa),
\]

respectively, where

\[
\pi := \frac{r T (1 - T) - \nu \omega_0 T}{\Gamma}, \quad \kappa := \frac{r (1 - T) + \nu \omega_0}{r T \exp(\nu \omega_0 \tau_0)}
\]

\[
\rho := \frac{1}{1 + \kappa \pi + \kappa \tau_0 \exp(\nu \omega_0 \tau_0)[r T + \pi \Gamma]}
\]

Here, we are taking into account the orthonormality condition \(\langle q, p \rangle = 1\). Of course, \(\nu \omega_0\) is a simple eigenvalue of \(L_0\):

\[
-\nu \omega_0 \langle q, \overline{p} \rangle = \langle q, -\nu \omega_0 \overline{p} \rangle = \langle q, L_0 \overline{p} \rangle = \langle L_0^* q, \overline{p} \rangle \\
= (-\nu \omega_0 q, \overline{p}) = \nu \omega_0 \langle q, \overline{p} \rangle
\]

(cf. [24]).

The third step is to compute a two-dimensional invariant manifold, the centre manifold. For such a purpose, the phase space \(\mathcal{C}\) is decomposed as \(\mathcal{C} = \mathcal{P} \oplus \mathcal{Q}\), where \(\mathcal{P}\) is a two-dimensional subspace spanned by eigenfunctions of operator \(L_0\): \(\mathcal{P} = \text{span} \{ \mathcal{R}(p), \mathcal{S}(q) \} \) and \(\mathcal{Q}\) is the complementary space of \(\mathcal{P}\). \(\mathcal{P}\) and \(\mathcal{Q}\) are invariant under the flow associated with the linear part of equation (28). The long-term behaviour of
solutions to the equation (28) is well approximated by the flow on this 
manifold (cf. \[21\]). The centre manifold for equation (31) is given by 

\[ M_f := \{ \phi \in \mathcal{C} : \phi = \Phi z + h(z, f), \ z \text{ in a neighbourhood of zero in } \mathbb{R}^2 \}. \]

The flow on this centre manifold is

\[ x_t = \Phi z + h(z, f) \]

where \( h \in \Omega \) and \( z \) satisfies the ordinary differential equation

\[ (37) \quad \dot{z} = Jz + \Psi(0)f(\Phi z) \]

with

\[ \Phi := (\Re(p), \Im(p)), \quad J = \Phi^{-1}\Phi' \quad \text{and} \quad \Psi := (\Re(q), \Im(q)) \]

(cf. \[8, 9\]). A straightforward calculation shows that

\[ \Phi = \begin{bmatrix} \cos(\omega_0 \theta) & \sin(\omega_0 \theta) \\ \Re(\pi) \cos(\omega_0 \theta) + \Im(\pi) \sin(\omega_0 \theta) & \Re(\pi) \sin(\omega_0 \theta) - \Im(\pi) \cos(\omega_0 \theta) \end{bmatrix} \]

and

\[ J = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix}, \quad \Psi(0) = \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix}, \]

where

\[ \psi_{11} := \Gamma r^2 \left[ T - T(T - 2)(T - 1)^2 + \omega_0^2 T \right] / \delta, \]

\[ \psi_{12} := -\Gamma r^2 \omega_0 \left[ \omega_0^2 + r^2 (T - 2)(T - 1) \right] / \delta, \]

\[ \psi_{21} := T \left( \Gamma - T(T - 2)(T - 1)^2 + (\omega_0^2 + r^2 (T - 2)(T - 1)) \times (\omega_0^2 + r^2 (T - 1)^2) T \right) \cos(\omega_0 \tau_0) \]

\[ + \Gamma \omega_0 r (T - (\omega_0^2 + r^2 T \tau_0(T - 1)^2)) \sin(\omega_0 \tau_0) / \delta T^2, \]

\[ \psi_{22} := \Gamma r^2 \left( \Gamma - (\omega_0^2 + r^2 (T - 1)^2) T \right) \cos(\omega_0 \tau_0) \]

\[ + (\Gamma r^2 (T - 1) - (\omega_0^2 + r^2 (T - 2)(T - 1)) \]

\[ \times (\omega_0^2 + r^2 (T - 1)^2) T \sin(\omega_0 \tau_0) / \delta T^2 \]

with
\[
\delta := \frac{\omega_0^2}{\tau_0} [\omega_0^2 + r^2(T - 3)(T - 1)]^2 T^2 \tau_0^2 \\
+ r^2 \left[ \Gamma - T (r^2(T - 2)(T - 1)^2 + \omega_0^2 T) \right]^2.
\]

The nonlinear function \(f\) in equation (37) is then given as

\[
(38) \quad f(\Phi z) = \left[ f_{111} \xi^2 + f_{112} \xi \eta + f_{122} \eta^2 + f_{1111} \xi^4 + f_{1112} \xi^2 \eta + f_{1222} \eta^4 \right] \\
+ \left[ f_{211} \xi^3 + f_{212} \xi^2 \eta + f_{222} \eta^3 + f_{2111} \xi^3 \eta + f_{2112} \xi^2 \eta^2 + f_{2222} \eta^3 \right]
\]

where \(z = [\xi, \eta]^T\) and

\[
f_{111} = -\beta \Re(\pi), \quad f_{112} = \beta \Im(\pi), \\
f_{122} = f_{111} = f_{112} = f_{222} = 0, \\
f_{211} = \beta \cos(\omega_0 \tau_0) \left[ \Im(\pi) \cos(\omega_0 \tau_0) - \Re(\pi) \sin(\omega_0 \tau_0) \right], \\
f_{212} = \frac{\beta}{2} \left\{ \Re(\pi) - \Im(\pi) - (\Re(\pi) + \Im(\pi)) \left[ \cos(2\omega_0 \tau_0) + \sin(2\omega_0 \tau_0) \right] \right\}, \\
f_{222} = \beta \sin(\omega_0 \tau_0) \left[ \Im(\pi) \cos(\omega_0 \tau_0) + \Re(\pi) \sin(\omega_0 \tau_0) \right], \\
f_{1111} = f_{1112} = f_{2222} = 0.
\]

Substituting (38) into (37) yields the dynamical system

\[
(39) \quad \begin{align*}
\dot{\xi} &= \omega_0 \eta + \left( \psi_{11} f_{111} + \psi_{12} f_{112} \right) \xi^2 \\
&\quad + \left( \psi_{11} f_{112} + \psi_{12} f_{122} \right) \xi \eta + \psi_{12} f_{222} \eta^2, \\
\dot{\eta} &= -\omega_0 \xi + \left( \psi_{21} f_{111} + \psi_{22} f_{112} \right) \xi^2 \\
&\quad + \left( \psi_{21} f_{112} + \psi_{22} f_{222} \right) \xi \eta + \psi_{22} f_{222} \eta^2.
\end{align*}
\]

Using formula (30) we can calculate the Poincaré-Lyapunov constant \(\delta\) from (39)

\[
\delta = \frac{1}{8\omega_0} \left[ (\psi_{11} f_{111}^2 + \psi_{12} f_{112}^2 + \psi_{22} f_{222}^2) \\
- (\psi_{21} f_{111}^2 + \psi_{22} f_{112}^2 - \psi_{11} f_{112}^2 - \psi_{12} f_{222}^2) \\
+ (\psi_{21} f_{111}^2 + \psi_{22} f_{112}^2 + \psi_{22} f_{222}^2) \\
- (\psi_{11} f_{111}^2 + \psi_{12} f_{112}^2 + \psi_{21} f_{112}^2 + \psi_{22} f_{222}^2) \right].
\]

The sign of \(\delta\) determines the direction and stability of the Hopf bifurcation which is subcritical or supercritical if \(\delta > 0\) or \(\delta < 0\), respectively.
Example 4.1. Set $a = 0.0600$, $b := 0.0080$, $\beta := 0.005600$, $\gamma := 0.020000$ and $\alpha = 0.320000$. The unique endemic equilibrium of (3) is $(S, T) = (62.1429, 12.0576)$. We can see that if $\tau_2 = 0$ and $\tau_1 < \tau_{s_1}$ then there is no stability switch, i.e., $(S, T)$ remains stable. The polynomial $F_1$ has one positive root: $\omega_0(= \omega_1) = 0.106600$. The critical value of the delay $\tau_1$ at which Hopf bifurcation takes place is $\tau_0(= \tau_{crit_1}) = 1.659400$ (cf. Figure 4) which is supercritical because $\delta = -0.000022$.

![Figure 4: Time evolution of system (3) with $\tau_2 = 0$, $\tau < \tau_{s_1}$ and $\tau > \tau_{crit_1}$](image)

4.2.2 The case of the multiple delay  To obtain the value of the delay at which stability switch may occur one needs to find a positive solution $\omega_3$ of the equation $F_3(y) = 0$ and then calculate the value of $\tau_{crit_3}$ from $\Delta_{(S, T)}(\omega_3, \tau) = 0$.

Clearly, if $z = \omega_3 (\omega > 0)$ is a root, then
\( p(\omega) + q(\omega)e^{-i\omega \tau} + r(\omega)e^{-2i\omega \tau} \)

\[
= -\omega^2 + A\omega + B + (C + D + (E + F)\omega)e^{-i\omega \tau} + Ge^{-2i\omega \tau} = 0
\]

holds. Multiplying with the factor \( e^{i\omega \tau} \) and separating into real and imaginary parts, we obtain

\[
\Re(\Delta(\xi, \eta)(i\omega, \tau)) = (B + G - \omega^2)\cos(\omega\tau) - A\omega \sin(\omega\tau) + C + D = 0
\]

and

\[
\Im(\Delta(\xi, \eta)(i\omega, \tau)) = (B - G - \omega^2)\sin(\omega\tau) + A\omega \cos(\omega\tau) + (E + F)\omega = 0.
\]

Solving (40) (say) for \( \tau \) we get the formula

\[
\tau_{\text{crit}} = \frac{1}{\omega^3} \left[ \sin^{-1} \left( -\frac{C + D}{\sqrt{(B + G - \omega^2)^2 + A^2\omega^4}} \right) - \varphi \right]
\]

where \( \tan(\varphi) = -A\omega^3 / (B + G - \omega^2) \) and \( F_\omega(\omega_3) = 0 \).

Let us denote the root of \( \Delta(\xi, \eta)(\cdot; \tau) \) that assumes the value \( i\omega_3 \) at \( \tau_{\text{crit}} \) by \( z_3(\tau) \) and the characteristic function \( \Delta(\xi, \eta)(\cdot; \tau) \) as a function of the parameter \( \tau \) by

\[ \mathfrak{F}(\mu, \tau) := p(z) + q(z)e^{-z\tau} + r(z)e^{-2z\tau}. \]

We are going to determine the derivative of the implicit function \( z_3 \) at \( \tau_0 \):

\[
z_3'(\tau_{\text{crit}}) = -\frac{\partial}{\partial \tau} \mathfrak{F}(\omega_3, \tau_{\text{crit}}) = -\frac{-zq(z)e^{-z\tau} - 2zr(z)e^{-2z\tau}}{p'(z) + [q'(z) - \tau q(z)]e^{-z\tau} + [r'(z) - 2\tau r(z)]e^{-2z\tau}} \bigg|_{z=\omega_3, \tau=\tau_{\text{crit}}}
\]

\[
= \frac{zq(z) + 2zr(z)e^{-z\tau}}{p'(z)e^{z\tau} + [q'(z) - \tau q(z)] + [r'(z) - 2\tau r(z)]e^{z\tau}} \bigg|_{z=\omega_3, \tau=\tau_{\text{crit}}}
\]
Thus,

\[ z_3'(\tau_{crit}) = \{ \omega_3 (C + D + (E + F) \omega_3) - 2\omega_3 G \exp(-i\omega_3 \tau_{crit}) \} \]
\[ \times \{ (2\omega_3 + A) \exp(i\omega_3 \tau_{crit}) + E + F - \tau_{crit} (C + D + (E + F) \omega_3) \]
\[ - 2\tau_{crit} G \exp(-i\omega_3 \tau_{crit}) \}^{-1} \]
\[ = \frac{\mathfrak{A}_{en}(\omega_3, \tau_{crit}) + \mathfrak{B}_{en}(\omega_3, \tau_{crit})}{\mathfrak{A}_{den}(\omega_3, \tau_{crit}) + \mathfrak{B}_{den}(\omega_3, \tau_{crit})} \]

where

\[ \mathfrak{A}_{en}(\omega_3, \tau_{crit}) := \omega_3 [2G \sin(\omega_3 \tau_{crit}) - (E + F) \omega_3] \]
\[ \mathfrak{B}_{en}(\omega_3, \tau_{crit}) := \omega_3 [C + D + 2G \cos(\omega_3 \tau_{crit})] \]
\[ \mathfrak{A}_{den}(\omega_3, \tau_{crit}) := E + F - \tau_{crit} (C + D) - 2\omega_3 \sin(\omega_3 \tau_{crit}) + (A - 2\tau_{crit} G) \cos(\omega_3 \tau_{crit}) \]
\[ \mathfrak{B}_{den}(\omega_3, \tau_{crit}) := 2\omega_3 \cos(\omega_3 \tau_{crit}) - \tau_{crit} (E + F) \omega_3 + (2\tau_{crit} G + A) \sin(\omega_3 \tau_{crit}) \]

Therefore, Hopf bifurcation occurs if

\[ \text{sgn}(\Re(z_3'(\tau_{crit}))) = \text{sgn}(\mathfrak{A}_{en}(\omega_3, \tau_{crit}) \mathfrak{A}_{den}(\omega_3, \tau_{crit})) \]
\[ + \mathfrak{B}_{en}(\omega_3, \tau_{crit}) \mathfrak{B}_{den}(\omega_3, \tau_{crit}) \]
\[ = \text{sgn}(- (E + F)^2 \omega_3 + [2(C + D) - A(E + F)] \omega_3 \cos(\omega_3 \tau_{crit}) + 4G \omega_3 \cos(2\omega_3 \tau_{crit}) + [A(C + D) + 2(E + F)] \times (G + \omega_3^2) + 4AG \cos(\omega_3 \tau_{crit})] \sin(\omega_3 \tau_{crit})) \]
\[ = \pm 1 \]

holds.

**Example 4.2.** Set \( a = 0.0500, b := 0.0040, \beta := 0.0056, \gamma := 0.0200 \) and \( \alpha = 0.4200 \). The unique endemic equilibrium is \((S, T) = (81.0714, 9.9714)\). The polynomial \( F_3 \) has two positive roots: \( \tilde{\omega}_3 := 0.0352 \) and \( \omega_3 := \)
0.1204 with possible critical values of the delay: \( \tau_{\text{crit}3} := 18.1173 \) and \( \tau_{\text{crit}3} := 0.7953 \). Then the critical value of the delay at which Hopf bifurcation takes place is \( \tau_{\text{crit}3} \), because \( \text{sgn}(\Re(z_0'(\tau_{\text{crit}3}))) = 1 \) and \( \text{sgn}(\Re(z_0'((\tau)))) = -1 \). In the second case, the endemic equilibrium becomes again stable (c.f. Figure 5).

**FIGURE 5:** Time evolution of system (3) when \( A^2 > 2B \) and \( B^2 > G^2 \) hold and \( F_3 \) has two positive roots, for \( \tau_1 = \tau_2 =: \tau < \tau_{\text{crit}3} \) in the first graph, \( \tau_{\text{crit}3} < \tau < \tau_{\text{crit}3} \) in the middle and \( \tau_{\text{crit}3} < \tau \) in the last graph.
5 Summary  In this paper, a detailed study of the effect of time delays on the dynamics of a system modelling host-parasite associations was presented. This delays were introduced in both birth and transmission terms. By analyzing the associated characteristic equation we have obtained some sufficient conditions for the stability of the system. Using the delays as the bifurcation parameter, we have shown that a Hopf bifurcation occurs when this parameter passes through a critical value; i.e., a family of periodic orbits bifurcates from the endemic equilibrium. In case of a single delay, when the one in the birth term is fixed to zero and the other one in the transmission term is used as a parameter, the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits were also discussed.

Acknowledgments  The authors thank professor Péter L. Simon for useful discussions on the topics investigated in this paper. The work of S. Aly was partially supported by the Hungarian Scholarship Board.

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