MONOTONIC SOLUTIONS OF NONLINEAR INTEGRAL EQUATIONS OF FRACTIONAL ORDER

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ABSTRACT. In the first part of this paper, we use Schauder’s fixed point theorem and prove an existence result of continuous and monotone solutions over bounded intervals of Urysohn’s type nonlinear integral equation of fractional kind. In the second part of this work, we extend the previous existence result to the case of a quadratic nonlinear integral equation of fractional order defined over $\mathbb{R}_+$, the half line of positive real numbers. The main ingredients used in the proof of the second existence result are the Tychonoff fixed point theorem associated with a measure of noncompactness over $C(\mathbb{R}_+)$. Some examples are provided to illustrate the different results of this work.

1 Introduction In the first part of this paper, we fix two positive real number $T > 0$, $0 \leq \alpha < 1$ and we develop an approach for the existence of monotonic and continuous solutions over $[0, T]$ of the following Urysohn’s type nonlinear integral equation of fractional kind,

$$x(t) = a(t) + \int_0^t \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}} \, ds, \quad t \in [0, T],$$

where $x(\cdot)$ is an unknown function and $a(\cdot) \in C([0, 1])$. Without loss of generality, we will restrict ourselves to nondecreasing solutions of (1.1), the case of decreasing solutions can be done in a similar way. We should mention that most of the existence results of problem (1.1) have been achieved under the condition that $a(\cdot)$ is nondecreasing on $[0, T]$ as well as other conditions on the function $u(t, s, x)$. In this work, we give an

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existence proof that requires weaker conditions on $a(\cdot)$ as well as on the function $u(t, s, x)$. In order to solve (1.1), many different methods have been applied in the literature. Most of these methods use the notion of a measure of noncompactness in Banach spaces combined with the Darbo’s fixed point theorem; see [2, 3, 5, 11].

In the second part of this paper, we are going to investigate the following quadratic integral equation of fractional order

$$x(t) = a(t) + \frac{f(t, x(t))}{\Gamma(\alpha)} \int_{0}^{t} \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}} ds,$$

where $t \in \mathbb{R}_+$, $\alpha \in (0, 1)$ and $\Gamma(\alpha)$ denotes the well-known gamma function. The Tychonoff fixed point theorem in locally convex spaces is the main tool for the existence of $C(\mathbb{R}_+)$-solutions of problem (1.2). We should mention that the theory of quadratic integral equations with fractional order has been extensively studied in the literature. This is due to a wide range of real applications of these equations; see, for example, [4, 7, 12]. Many authors have investigated the existence of solutions of such integral equations on bounded or unbounded intervals; see [2, 3, 5, 11]. The aim of the present paper is to give new existence results for problems (1.1) and (1.2).

2 Notation and some auxiliary results

In this section, we list some definitions and results that will be frequently used in this work. The following Schauder’s fixed point theorem is the main ingredient in the existence proof of Section 3.

Theorem 2.1 (Schauder’s fixed point theorem). Let $K$ be a closed convex subset of a Banach space $E$. If $T : K \to K$ is continuous and $A = T(K)$ is relatively compact, then $T$ has a fixed point in $K$.

Next, let $C(\mathbb{R}_+)$ be the space of continuous functions $x : \mathbb{R}_+ \to \mathbb{R}$ and let $I_n = [0, n]$, $n \in \mathbb{N}$. The space $C(\mathbb{R}_+)$ is the locally convex Fréchet space of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$ with the metric

$$d(x, y) = \sup \left\{ 2^{-n} \frac{\|x - y\|_n}{1 + \|x - y\|_n} \right\},$$

where

$$\|x\|_n := \sup \{|x(t)| : t \in I_n\}.$$
The convergence in $C(\mathbb{R}_+)$ is the uniform convergence in the compact intervals, i.e. $x_j$ converge to $x$ in $C(\mathbb{R}_+)$ if and only if $\|x_j - x\|_n$ converge to 0 in $(C(I_n), \|\cdot\|_n)$, $\forall n \in \mathbb{N}$.

By Arzelà-Ascoli theorem, a set $M \subset C(\mathbb{R}_+)$ is compact if and only if for each $n \in \mathbb{N}$, $M$ is a compact set in the Banach space $(C(I_n), \|\cdot\|_n)$; see [8].

In what follows, we present some basic facts concerning measures of noncompactness in $C(\mathbb{R}_+)$ introduced in [1]. If $X$ is a nonempty subset of $C(\mathbb{R}_+)$, then we denote by $\overline{X}$ and $\text{conv}(X)$ the closure and the closed convex closure of $X$, respectively. Next, let us denote by $\mathcal{M}_{C(\mathbb{R}_+)}$ the family of nonempty and bounded subsets of $C(\mathbb{R}_+)$ and by $\mathcal{N}_{C(\mathbb{R}_+)}$ its subfamily consisting of relatively compact subsets. Now, we recall the definition of measure of noncompactness which will be used in our further investigations.

**Definition 2.2.** [1] A function $\mu : \mathcal{M}_{C(\mathbb{R}_+)} \to \mathbb{R}_+$ is said to be a measure of noncompactness in $C(\mathbb{R}_+)$ if it satisfies the following conditions:

1. The family $\ker \mu = \{X \in \mathcal{M}_{C(\mathbb{R}_+)} : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathcal{N}_{C(\mathbb{R}_+)}$.
2. $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
3. $\mu(\overline{X}) = \mu(\text{conv}(X)) = \mu(X)$.
4. $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda \mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
5. If $\{X_n\}_{n}$ is a sequence of nonempty, bounded, closed subsets of $C(\mathbb{R}_+)$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and $\lim_{n \to \infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Let $X \subset \mathcal{M}_{C(\mathbb{R}_+)}$ and fix $T > 0$, $\epsilon > 0$. By $w^T(x, \epsilon)$, we denote the modulus of continuity of the function $x$, that is

$$w^T(x, \epsilon) = \sup \{|x(t) - x(s)| : t, s \in [0, T], |t - s| \leq \epsilon\}.$$ 

Further, we put

$$w^T(X, \epsilon) = \sup \{w^T(x, \epsilon) : x \in X\},$$

$$w^T_0(X) = \lim_{\epsilon \to 0} w^T(X, \epsilon),$$

$$w_0(X) = \lim_{T \to \infty} w^T_0(X).$$

It is shown in [1] that $w_0$ is a measure of noncompactness. The following lemma borrowed from [2] will be used in our existence proof.
Lemma 2.3. Let \( g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) be a continuous function. Then the function

\[
m(t) = \frac{1}{\Gamma(\alpha)} \int_0^t g(t, s) \frac{1}{(t-s)^{1-\alpha}}
\]
is continuous on the interval \( \mathbb{R}_+ \).

Finally, to obtain our existence result we need the following fixed point theorem.

Theorem 2.4 (Schauder Tychono \[6\]). Let \( \Omega \) be a closed convex subset of a locally convex Hausdorff space \( E \). Assume that \( H : \Omega \to \Omega \) is continuous and that \( \overline{H(\Omega)} \) is compact in \( \Omega \). Then \( H \) has a fixed point in \( \Omega \).

3 Existence result by Schauder’s fixed point theorem

In this section, we study the existence of solutions of the quadratic integral equation of fractional order (1.1) by using the Schauder’s fixed point theorem. We should mention that this fixed point theorem is a powerful tool for proving various types of nonlinear integral equations; see, for example, \[9, 10\].

Theorem 3.1. Consider the following nonlinear integral equation

(3.1) \[
x(t) = a(t) + \int_0^t \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}} ds, \quad t \in [0, T], \quad 0 < \alpha < 1.
\]

Assume that

(h\(_1\)) \( a(\cdot) \in C([0,1]) \) and \( a(0) \geq 0 \).

(h\(_2\)) The real valued function \( u(t, s, x) \) is continuous on \([0, T]^2 \times \mathbb{R}_+\) and nondecreasing with respect to its three variables, separately.

(h\(_3\)) If \( 0 \leq t_1 < t_2 \leq T \), then \( a(t_2) - a(t_1) + \frac{u(t_2, t_1, 0)}{\alpha} (t_2^\alpha - t_1^\alpha) \geq 0 \).

(h\(_4\)) There exists a function \( \varphi \in C([0, T]) \), positive over \([0, T]\) and satisfying the following inequality

\[
a(t) + \int_0^t \frac{u(t, s, \varphi(s))}{(t-s)^{1-\alpha}} ds \leq \varphi(t) \quad t \in [0, T].
\]

Then (3.1) has a continuous, positive and nondecreasing solution on \([0, T]\).
Proof. We first consider the subset \( \Omega_T \) of \( C([0,T]) \), given by

\[
\Omega_T = \{ x(\cdot) \in C([0,T]), \ 0 \leq x(t) \leq \varphi(t), \ t \in [0,T] \}.
\]

It is clear that \( \Omega_T \) is a nonempty, closed and convex subset of \( C([0,T]) \). Moreover, \( \Omega_T \) is uniformly bounded by \( \| \varphi \|_\infty = \sup_{t \in [0,T]} |\varphi(t)| \). Consider the integral operator defined on \( C([0,T]) \) by

\[
Fx(t) = a(t) + \int_0^t \frac{u(t, s, x(s))}{(t - s)^{1-\alpha}} \, ds,
\]

then we prove that \( F(\Omega_T) \) is equicontinuous. Let \( t_1, t_2 \in [0,T] \), we may assume that \( t_1 < t_2 \), then we have

\[
|Fx(t_2) - Fx(t_1)| \leq |a(t_2) - a(t_1)| + \left| \int_0^{t_1} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} - \frac{u(t_1, s, x(s))}{(t_1 - s)^{1-\alpha}} \, ds \right|
\]

\[
+ \left| \int_{t_1}^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds \right|
\]

\[
\leq |a(t_2) - a(t_1)| + \left| \int_0^{t_1} \frac{u(t_2, s, x(s)) - u(t_1, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds \right|
\]

\[
+ \left| \int_{t_1}^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds \right|
\]

\[
\leq |a(t_2) - a(t_1)| + \int_0^{t_1} \frac{|u(t_2, s, x(s)) - u(t_1, s, x(s))|}{(t_2 - s)^{1-\alpha}} \, ds
\]

\[
+ \int_{t_1}^{t_2} \frac{1}{(t_2 - s)^{1-\alpha}} - \frac{1}{(t_1 - s)^{1-\alpha}} \, ds
\]

\[
+ \int_{t_1}^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds
\]

\[
\leq |a(t_2) - a(t_1)| + \int_0^{t_1} \frac{|u(t_2, s, x(s)) - u(t_1, s, x(s))|}{(t_2 - s)^{1-\alpha}} \, ds
\]

\[
+ \frac{u(t_1, t_1, \| \varphi \|_\infty)}{\alpha} \int_0^{t_1} \frac{1}{(t_1 - s)^{1-\alpha}} - \frac{1}{(t_2 - s)^{1-\alpha}} \, ds
\]

\[
+ u(t_2, t_2, \| \varphi \|_\infty) \int_{t_1}^{t_2} \frac{1}{(t_2 - s)^{1-\alpha}} \, ds
\]

\[
\leq |a(t_2) - a(t_1)| + \int_0^{t_1} \frac{|u(t_2, s, x(s)) - u(t_1, s, x(s))|}{(t_2 - s)^{1-\alpha}} \, ds
\]

\[
+ \frac{u(t_1, t_1, \| \varphi \|_\infty)}{\alpha} (t_1^{\alpha} - t_2^{\alpha} + (t_2 - t_1)^{\alpha})
\]
Since the function $u(t, s, x)$ is uniformly continuous on $[0, T]^2 \times [0, \|\varphi\|_\infty]$, we have

$$\lim_{t_2 \to t_1} |u(t_2, s, x(s)) - u(t_1, s, x(s))| = 0$$

uniformly in $s \in [0, T]$ and $x(\cdot) \in \Omega_T$. Hence, we have

$$\text{(3.3)} \quad \int_0^{t_2} \frac{|u(t_2, s, x(s)) - u(t_1, s, x(s))|}{(t_2 - s)^{1-\alpha}} \, ds$$

$$\leq \sup_{s \in [0,T], x \in [0,\|\varphi\|_\infty]} \frac{|u(t_2, s, x(s)) - u(t_1, s, x(s))|}{(t_2 - s)^{1-\alpha}} \to 0 \text{ as } t_1 \to t_2.$$ 

By using the continuity of $a(t)$ together with (3.2) and (3.3), one concludes that $\lim_{t_1 \to t_2} |Fx(t_2) - Fx(t_1)| = 0$ independently of $x(\cdot) \in \Omega_T$. Hence, $F(\Omega_T)$ is an equicontinuous subset of $C([0, T])$. Next, we prove that $F(\Omega_T) \subset \Omega_T$. Since $\forall t, s \in [0, T]$, the function $x \to u(t, s, x)$ is nondecreasing, then $\forall x(\cdot) \in \Omega_T$, we have

$$\text{(3.4)} \quad a(t) + \int_0^t \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}} \, ds \leq a(t) + \int_0^t \frac{u(t, s, \varphi(s))}{(t-s)^{1-\alpha}} \, ds$$

$$\leq \varphi(t), \quad t \in [0, T].$$

The last inequality is due to assumption $(h_4)$. Moreover, by using assumptions $(h_1), (h_2)$ and $(h_3)$, one gets

$$\text{(3.5)} \quad Fx(t) = a(t) + \int_0^t \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}} \, ds \geq a(t) + \int_0^t \frac{u(t, 0, 0)}{(t-s)^{1-\alpha}} \, ds$$

$$\geq a(t) + u(t, 0, 0) \frac{t^\alpha}{\alpha}$$

$$= a(0) + \left( a(t) - a(0) + u(t, 0, 0) \frac{t^\alpha}{\alpha} \right) \geq a(0) \geq 0.$$ 

By using (3.4), (3.5) and the continuity of $x(\cdot)$, one concludes that $Fx(\cdot) \in \Omega_T$ whenever $x(\cdot) \in \Omega_T$. Since $\Omega_T$ and consequently $F(\Omega_T)$ is uniformly bounded and since $F(\Omega_T)$ is equicontinuous, then by Arzelà-Ascoli theorem, one concludes that $F(\Omega_T)$ is a relatively compact subset of $C([0, T])$. 
Next, we prove that $F : \Omega_T \to \Omega_T$ is continuous. Let $(x_n(\cdot))_n$ be a sequence in $\Omega_T$ converging to $x(\cdot)$. Since $\Omega_T$ is a closed subset of $C([0, T])$, then $x(\cdot) \in \Omega_T$. The uniform continuity of the function $u(\cdot, \cdot, \cdot)$ on $[0, T]^2 \times [0, \|\varphi\|_\infty]$ implies that $\forall \epsilon > 0, \exists \eta > 0$ such that if $\|x_n(\cdot) - x(\cdot)\|_\infty < \eta$, then we have

$$\sup_{t \in [0,T]} |F x_n(t) - F x(t)| \leq \int_0^t \left( \sup_{t, s \in [0,T]} |u(t, s, x_n(s)) - u(t, s, x(s))| \right) \frac{(t-s)^{1-\alpha}}{\alpha} \, ds$$

$$< \int_0^T \left( \frac{\alpha}{T^{\alpha}} \right) \frac{1}{(t-s)^{1-\alpha}} \, ds = \epsilon.$$  

Hence, $\lim_{n \to +\infty} \|F x_n - F x\|_\infty = 0$.

Till now, we have shown that $F : \Omega_T \to \Omega_T$ is continuous and $F(\Omega_T)$ is a relatively compact subset of $C([0, T])$. By using Schauder’s fixed point theorem, one concludes that the operator $F$ has a fixed point in $\Omega_T$, that is the problem (3.1) has a positive solution in $C([0, T])$. It remains to prove that there exists a continuous and monotonic solution of (3.1). To this end, we denote by $\Gamma_T$ the subset of $\Omega_T$ given by

$$\Gamma_T = \{ x(\cdot) \in \Omega_T, \quad x(\cdot) \text{ is nondecreasing on } [0, T] \}.$$  

Obviously, $\Gamma_T$ is a closed and convex subset of $\Omega_T$ and $F(\Gamma_T)$ is equicontinuous and uniformly bounded. To prove that $F(\Gamma_T) \subset \Gamma_T$, it suffices to consider $x(\cdot) \in \Gamma_T$, $t_1, t_2 \in [0, T]$ with $t_1 < t_2$ and remark that

$$F x(t_2) - F x(t_1)$$

$$= a(t_2) - a(t_1) + \int_0^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds - \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_1 - s)^{1-\alpha}} \, ds$$

$$\geq a(t_2) - a(t_1) + \int_0^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds - \int_0^{t_1} \frac{u(t_2, s, x(s))}{(t_1 - s)^{1-\alpha}} \, ds$$

$$\geq a(t_2) - a(t_1) + \int_0^{t_1} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds + \int_{t_1}^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds.$$

Since

$$\frac{1}{(t_2 - s)^{1-\alpha}} - \frac{1}{(t_1 - s)^{1-\alpha}} < 0,$$
\(x(\cdot)\) is nondecreasing and since \(u(\cdot, \cdots, \cdot)\) is nondecreasing with respect to its variables, then the previous inequality implies
\[
F(t_2) - F(t_1) \\
\geq a(t_2) - a(t_1) + u(t_2, t_1, x(t_1)) \int_{t_1}^{t_2} \frac{1}{(t_2 - s)^{1-\alpha} (t_1 - s)^{1-\alpha}} ds \\
+ u(t_2, t_1, x(t_1)) \int_{t_1}^{t_2} \frac{ds}{(t_2 - s)^{1-\alpha}} \\
\geq a(t_2) - a(t_1) + u(t_2, t_1, x(t_1)) \left(\frac{t_2^\alpha}{\alpha} - \frac{t_1^\alpha}{\alpha}\right) \\
\geq a(t_2) - a(t_1) + u(t_2, t_1, 0) \left(\frac{t_2^\alpha}{\alpha} - \frac{t_1^\alpha}{\alpha}\right) \geq 0.
\]
The last inequality is due to assumption \((h_3)\). The previous analysis shows that \(F : \Gamma_T \rightarrow \Gamma_T\) satisfies the different conditions of Schauder’s fixed point theorem. Hence, \(3.1\) has a solution which is continuous, positive and nondecreasing on \([0, T]\).

**Example 1.** Let \(T, \beta > 0\) be arbitrary positive real numbers and consider the following nonlinear integral equation of fractional order and power law nonlinearity
\[
(3.6) \quad x(t) = a(t) + \int_0^t \frac{1 + s^\beta(x(s))^\eta}{(t - s)^{1-\alpha}} ds, \quad 0 < \alpha, < 1, \quad t \in [0, T],
\]
where \(a(t) \in C([0, T])\) and satisfies the conditions \(a(0) \geq 0\) and the function \(a(t) + (t^\alpha/\alpha)\) is nondecreasing. It is clear that under these conditions, \(3.6\) satisfies conditions \((h_1)\), \((h_2)\) and \((h_3)\) of the previous theorem. It remains to check that condition \((h_4)\) is also satisfied. To this end, we consider the constant function given by \(\varphi(t) = R_T, \forall t \in [0, T]\), where \(R_T\) is a positive real number satisfying the following condition
\[
\|a\|_\infty + \frac{T^\alpha}{\alpha} + R_T^\eta \frac{T^{\alpha + \beta}}{\alpha} \leq R_T.
\]
Note that since
\[
\lim_{R \to +\infty} \|a\|_\infty + \frac{T^\alpha}{\alpha} + R_T^\eta \frac{T^{\alpha + \beta}}{\alpha} - R = -\infty,
\]
the above inequality has always a solution. Hence, condition \((h_4)\) is also satisfied. Consequently, \(x(t)\) has a positive, monotonic and continuous solution on \([0, T]\) no matter how large is \(T\).

**Example 2.** Consider the following nonlinear integral equation of fractional order

\[
x(t) = \frac{\sqrt{1+t^3}}{3} - \frac{\sqrt{t}}{2} + \int_0^t \frac{x^2(s) + 1}{4\sqrt{t-s}} \, ds, \quad t \in [0, 1].
\]

It is clear that \(a(t) = \frac{\sqrt{1+t^3}}{3} - \frac{\sqrt{t}}{2}\) satisfies condition \((h_1)\) and it is not a monotone function on \([0, 1]\). Moreover, the function \(u(t, s, x) = (1+x^2)/4\) satisfies condition \((h_2)\). Also, note that if \(0 \leq t_1 < t_2 \leq 1\), then

\[
a(t_2) - a(t_1) + u(t_2, t_1, 0)(2\sqrt{t_2} - 2\sqrt{t_1}) = \frac{\sqrt{1+t_2}}{3} - \frac{\sqrt{1+t_1}}{3} \geq 0.
\]

Hence, condition \((h_3)\) is satisfied. To check condition \((h_4)\), it suffices to consider the function \(\varphi(t) = \sqrt{1+t}, \ t \in [0, 1]\). Straightforward computations show that

\[
a(t) + \int_0^t \frac{\varphi^2(s) + 1}{4\sqrt{t-s}} - \varphi(t) = \frac{\sqrt{t}}{2} + \frac{t^{3/2}}{3} - \frac{2}{3} \sqrt{1+t} \leq 0, \quad t \in [0, 1].
\]

Consequently, condition \((h_4)\) is also satisfied. By using the previous theorem, one concludes that \(x(t)\) has a continuous and a nondecreasing positive solution on \([0, 1]\).

**4 An existene result by Tychonoff fixed point theorem** In this section we will discuss the existence of solutions of the quadratic integral equation \((1.2)\) in the Fréchet space \(C(\mathbb{R}_+)\). We should mention that the results of this paragraph can be considered as the extension of the existence results given in \([11]\) to the monotonic solutions of quadratic Urysohn’s integral equations of fractional order case. In what follows we will assume that the functions involved in equation \((1.2)\) satisfies the following conditions:

(i) \(a : \mathbb{R}_+ \to \mathbb{R}_+\) is continuous, bounded and nondecreasing on \(\mathbb{R}_+\).

(ii) The function \(f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+\) is continuous and nondecreasing with respect to each of variables \(t, x\) separately.
(iii) There exists a nondecreasing function $k : \mathbb{R} \rightarrow \mathbb{R}_+$, such that

$$|f(t, x) - f(t, y)| \leq k(r)|x - y|$$

for $t \geq 0$ and for all $x, y \in [a_0, r]$.

(iv) $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ is continuous, positive and $u(t, s, x)$ is nondecreasing with respect to each variable $t, s$ and $x$ separately.

(v) There exists a continuous function $g : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and a continuous and nondecreasing function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$|u(t, s, x)| \leq g(t, s)\phi(|x|)$$

for all $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$.

(vi) There exists a positive solution $r_0$ of the following inequality

$$\sup_{t \geq 0} |a(t)| + \frac{\phi(r)}{\Gamma(\alpha)} \left[ rk(r) \sup_{t \geq 0} m(t) + \sup_{t \geq 0} m(t) f(t, 0) \right] \leq r$$

where

$$m(t) = \int_0^t \frac{g(t, s)}{(t-s)^{1-\alpha}} ds,$$

satisfies

$$\frac{k(r_0)\phi(r_0)}{\Gamma(\alpha)} \sup_{t \geq 0} m(t) < 1.$$  

Now we can formulate our existence result.

**Theorem 4.1.** Under assumptions (i)–(vi), equation (1.2) has at least one nondecreasing solution belonging $C(\mathbb{R}_+)$. 

**Proof.** For convenience, we write the operator

$$(Hx)(t) = a(t) + \frac{f(t, x(t))}{\Gamma(\alpha)} \int_0^t \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}} ds,$$

in the form

(4.1) $$(Hx)(t) = a(t) + (Fx)(t)(Ux)(t),$$

where $F$ is the superposition operator generated by the function $f(t, x)$ and $U$ is the operator defined by

$$(Ux)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(t, s, x(s))}{(t-s)^{1-\alpha}} ds.$$
Next, let \( r_0 \) be a number satisfying assumption (vi) and define the set 
\( B_{r_0} \subset C(\mathbb{R}^+) \) by
\[
B_{r_0} = \left\{ x \in C(\mathbb{R}^+) : x(t) \geq a_0, \ t \geq 0 \ \text{and} \ \sup_{t \geq 0} |x(t)| \leq r_0 \right\}.
\]
Note that, in view of our assumptions, \( B_{r_0} \) is nonempty, bounded, closed and convex. Moreover, the operators \( F \) and \( U \) are well defined on \( B_{r_0} \).

Next, we show that the operators \( F \) and \( U \) transform \( B_{r_0} \) into a subset of \( C(\mathbb{R}^+) \). Indeed, take an arbitrary function \( x \in B_{r_0}, T > 0 \) and \( \epsilon > 0 \). Let us take arbitrary \( t_1, t_2 \in [0, T] \) such that \( |t_1 - t_2| \leq \epsilon \), then we have
\[
|\langle Fx \rangle(t_2) - \langle Fx \rangle(t_1)| = |f(t_2, x(t_2)) - f(t_1, x(t_1))| \\
\leq |f(t_2, x(t_2)) - f(t, x(t_1))| \\
+ |f(t, x(t_1)) - f(t_1, x(t_1))| \\
\leq k(r_0)|x(t_2) - x(t_1)| + w^T_{r_0}(f, \epsilon) \\
\leq k(r_0)w^T_{r_0}(x, \epsilon) + w^T_{r_0}(f, \epsilon)
\]

where
\[
w^T_{r_0}(f, \epsilon) = \sup\{|f(t_2, x(t_1)) - f(t_1, x(t_1))| : t_1, t_2 \in [0, T], x \in [-r_0, r_0], |t_2 - t_1| \leq \epsilon\}.
\]

Note that \( w^T_{r_0}(f, \epsilon) \to 0 \) as \( \epsilon \to 0 \), this is a consequence of the uniform continuity of the function \( f \) on \([0, T] \times [-r_0, r_0] \). We infer that \( F \) is continuous on \([0, T] \) for any \( T > 0 \). This implies that \( Fx \) is continuous on \( \mathbb{R}^+ \). Therefore \( F \) transforms \( B_{r_0} \) into a subset of \( C(\mathbb{R}^+) \). We show that the same holds for the operator \( U \). Let \( T > 0 \) and \( t_1, t_2 \in [0, T] \).

Without loss of generality, we may assume that \( t_1 \leq t_2 \). Then we have
\[
|\langle Ux \rangle(t_2) - \langle Ux \rangle(t_1)| \\
= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} ds - \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_1 - s)^{1-\alpha}} ds \right| \\
\leq \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} ds - \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_2 - s)^{1-\alpha}} ds \right| \\
+ \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_2 - s)^{1-\alpha}} ds - \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_1 - s)^{1-\alpha}} ds \right|
\]
\[
+ \frac{1}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds \right| \\
\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{|u(t_2, s, x(s)) - u(t_1, s, x(s))|}{(t_2 - s)^{1-\alpha}} \, ds \\
+ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{|u(t_1, s, x(s))| [(t_1 - s)^{1-\alpha} - (t_2 - s)^{1-\alpha}]}{ds} \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds.
\]

Keeping in mind assumption (v), we obtain

(4.3) \quad |(Ux)(t_2) - (Ux)(t_1)|

\[
\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{w_{\bar{w}_0}(u, \epsilon)}{(t_2 - s)^{1-\alpha}} \, ds \\
+ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} g(t_1, s) \phi(r_0) [(t_1 - s)^{1-\alpha} - (t_2 - s)^{1-\alpha}] \, ds \\
+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{g(t_2, s) \phi(r_0)}{(t_2 - s)^{1-\alpha}} \, ds \\
\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{w_{\bar{w}_0}(u, \epsilon)}{(t_2 - s)^{1-\alpha}} \, ds \\
+ \frac{1}{\Gamma(\alpha)} \phi(r_0) g_T \int_0^{t_1} [(t_1 - s)^{1-\alpha} - (t_2 - s)^{1-\alpha}] \, ds \\
+ \frac{1}{\Gamma(\alpha)} \phi(r_0) g_T \int_{t_1}^{t_2} \frac{ds}{(t_2 - s)^{1-\alpha}} \\
\leq \frac{1}{\Gamma(\alpha + 1)} \left[ w_{\bar{w}_0}(u, \epsilon) T^\alpha + 2\phi(r_0) g_T \epsilon^\alpha \right].
\]

Here,

\[
w_{\bar{w}_0}(u, \epsilon) = \sup \{ |u(t_2, s, x(s)) - u(t_1, s, x(s))| : t_1, t_2, s \in [0, T], |t_1 - t_2| \leq \epsilon, x \in [-r_0, r_0] \}
\]

and

\[
g_T = \sup \{ g(t, s) : (t, s) \in [0, T] \times [0, T] \}.
\]
Observe that by invoking the uniform continuity of the function \( u(t, s, x) \) on \([0, T]^2 \times [-r_0, r_0] \), we obtain

\[
\lim_{\epsilon \to 0} u_{r_0}(u, \epsilon) = 0.
\]

Keeping in mind estimate (4.3), we conclude that the function \( Ux \) is continuous on \([0, T] \) for any arbitrary \( T > 0 \). This yields the continuity of \( Ux \) on \( \mathbb{R}_+ \). Finally, combining the continuity of the functions \( Fx \) and \( Ux \), we deduce that the function \( Hx \) is continuous on \( \mathbb{R}_+ \). Moreover for an arbitrary \( x \in B_{r_0} \) and \( t \geq 0 \), we obtain

\[
|Hx(t)| \leq |a(t)| + \frac{1}{\Gamma(\alpha)} \int_0^t |u(t, s, x(s))| \, ds \\
\leq |a(t)| + \frac{1}{\Gamma(\alpha)} [r_0 k_{r_0} + f(t, 0)] \phi(r_0) \int_0^t \frac{g(s)}{(t-s)^{1-\alpha}} \, ds \\
\leq |a(t)| + \frac{\phi(r_0)}{\Gamma(\alpha)} [r_0 k_{r_0} + f(t, 0)] m(t).
\]

In view of assumption (vi), we deduce that \( \sup_{t \geq 0} |Hx(t)| \leq r_0 \). Moreover, by assumptions (i), (ii) and (iv), it is easy to see that \( (Hx)(t) \geq a(0), \forall t \geq 0 \). The previous analysis shows that \( H \) transforms \( B_{r_0} \) into itself. Now, let us consider the subset \( \Omega \) of \( B_{r_0} \) consisting of all functions from \( B_{r_0} \) which are nondecreasing on \( \mathbb{R}_+ \). Observe that \( \Omega \) is nonempty, bounded, closed and convex. Moreover, \( H \) transforms \( \Omega \) into a subset of \( C(\mathbb{R}_+) \). Let us show that \( H \) transforms \( \Omega \) into itself. Let \( x \in \Omega \) and fix \( t_1, t_2 \in \mathbb{R}_+ \) with \( t_1 \leq t_2 \), then we have

\[
(Hx)(t_2) - (Hx)(t_1) \\
= a(t_2) - a(t_1) + \frac{f(t_2, x(t_2))}{\Gamma(\alpha)} \int_0^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds \\
- f(t_1, x(t_1)) \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_1 - s)^{1-\alpha}} \, ds \\
= a(t_2) - a(t_1) + \frac{f(t_2, x(t_2))}{\Gamma(\alpha)} \int_0^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds \\
- \frac{f(t_1, x(t_1))}{\Gamma(\alpha)} \int_0^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds.
\]
\[
\begin{align*}
+ & \frac{f(t_1, x(t_1))}{\Gamma(\alpha)} \int_0^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds \\
- & \frac{f(t_1, x(t_1))}{\Gamma(\alpha)} \int_0^{t_1} \frac{u(t_2, s, x(s))}{(t_1 - s)^{1-\alpha}} \, ds \\
= & a(t_2) - a(t_1) + \left[ \frac{f(t_2, x(t_2)) - f(t_1, x(t_1))}{\Gamma(\alpha)} \right] \int_0^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds \\
& + \frac{f(t_1, x(t_1))}{\Gamma(\alpha)} \left[ \int_0^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds - \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_1 - s)^{1-\alpha}} \, ds \right].
\end{align*}
\]

In view of assumptions (ii) and (iv), we have
\[
(4.4) \quad \left[ \frac{f(t_2, x(t_2)) - f(t_1, x(t_1))}{\Gamma(\alpha)} \right] \int_0^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds \geq 0.
\]

It remains to show that
\[
(4.5) \quad \int_0^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds - \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_1 - s)^{1-\alpha}} \, ds \geq 0.
\]

This is done as follows. Since
\[
\begin{align*}
\int_0^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds - \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_1 - s)^{1-\alpha}} \, ds \\
= & \int_0^{t_1} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds - \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_1 - s)^{1-\alpha}} \, ds \\
& + \int_{t_1}^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds,
\end{align*}
\]

taking into account the assumption (iv), we obtain
\[
(4.6) \quad \int_0^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds - \int_0^{t_1} \frac{u(t_1, s, x(s))}{(t_1 - s)^{1-\alpha}} \, ds \\
\geq & \int_0^{t_1} u(t_2, s, x(s)) \left[ (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] \, ds \\
& + \int_{t_1}^{t_2} \frac{u(t_2, s, x(s))}{(t_2 - s)^{1-\alpha}} \, ds \\
\geq & u(t_2, t_1, x(t_1)) \int_0^{t_1} \left[ (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] \, ds
\]
\[
\begin{align*}
  &+ u(t_2, t_1, x(t_1)) \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \\
  &\ge u(t_2, t_1, x(t_1)) \left[ \int_0^{t_2} (t_2 - s)^{\alpha-1} ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} ds \right] \\
  &\ge \frac{u(t_2, t_1, x(t_1))}{\alpha} (t_2^\alpha - t_1^\alpha) \ge 0.
\end{align*}
\]

This implies that \((Hx)(t_2) - (Hx)(t_1) \ge 0\). Therefore \(Hx\) is nondecreasing on \(\mathbb{R}_+\) and \(H(\Omega) \subset \Omega\). Now, let us take a nonempty set \(X \subset \Omega\). Fix \(T > 0\), \(\epsilon > 0\) and choose \(x \in X\) and \(t_1, t_2 \in [0, T]\) such that \(|t_1 - t_2| < \epsilon\). Then, by using (4.2) and (4.3), we obtain

\[
\begin{align*}
  |(Hx)(t_2) - (Hx)(t_1)| &\le |a(t_2) - a(t_1)| + (Fx)(t_2) - (Ux)(t_2) - (Ux)(t_1)| \\
  &+ (Ux)(t_2) - (Fx)(t_2) - (Fx)(t_1)| \\
  &\le w^T(a, \epsilon) + [r_0 k(r_0) + f(t_2, 0)] \\
  &\times \frac{1}{\Gamma(\alpha + 1)} \left[ w_0^T(u, \epsilon) + 2\phi(r_0)\epsilon^\alpha \right] \\
  &+ \frac{\phi(r_0)}{\Gamma(\alpha)} m(t_2)[k(r_0)w^T(x, \epsilon) + w^T(f, \epsilon)].
\end{align*}
\]

Now, using the uniform continuity of the functions \(f\) and \(u\) on \([0, T] \times [-r_0, r_0]\) and \([0, T] \times [0, T] \times [-r_0, r_0]\), respectively, we get

\[
w_0^T(Hx, \epsilon) \le \frac{k(r_0)\phi(r_0)}{\Gamma(\alpha)} \sup_{t \le T} m(t)w_0^T(x, \epsilon)
\]

and

\[
w_0(HX) \le \frac{k(r_0)\phi(r_0)}{\Gamma(\alpha)} \sup_{t \ge 0} m(t)w_0(X, \epsilon).
\]

Next, we show that \(H\) is continuous on the set \(\Omega\). Let \((x_n)_n \subset \Omega\) be a sequence converging to \(x\) and fix \(T > 0\). We show that \(|Hx_n - Hx|\) converges uniformly to 0 on \([0, T]\). Since

\[
|Hx_n(t) - Hx(t)|
\]
\[ \leq \frac{|f(t, x_n(t)) - f(t, x(t))|}{\Gamma(\alpha)} \int_0^T \frac{|u(t, s, x_n(s))|}{(t-s)^{1-\alpha}} ds \\
+ \frac{|f(t, x_n(t))|}{\Gamma(\alpha)} \int_0^T \frac{|u(t, s, x_n(s)) - u(t, s, x(s))|}{(t-s)^{1-\alpha}} ds \\
\leq k(r_0)|x_n(t) - x(t)| \frac{\phi(r_0)}{\Gamma(\alpha)} m(t) \\
+ \frac{(r_0 k(r_0) + f(t, 0))}{\Gamma(\alpha)} \int_0^T \frac{|u(t, s, x_n(s)) - u(t, s, x(s))|}{(t-s)^{1-\alpha}} ds \]

and since \( \sup_{t \in [0, T]} m(t) < +\infty \),

\[ \lim_{n \to \infty} \left[ k(r_0)|x_n(t) - x(t)| \frac{\phi(r_0)}{\Gamma(\alpha)} m(t) \right] = 0 \]

uniformly on \([0, T]\). On the other hand, from the continuity of \(u\)

\[ |u(t, s, x_n(s)) - u(t, s, x(s))| \leq w_0^T(u, \sup_{s \in [0, T]} |x_n(s) - x(s)|) \]

where

\[ w_0^T(u, \epsilon) = \sup \{ |u(t, s, x) - u(t, s, y)| : t, s \in [0, T], |x - y| \leq \epsilon \} . \]

From the fact that

\[ \sup_{t \in [0, T]} (r_0 k(r_0) + f(t, 0)) < \infty, \]

one concludes that

\[ \lim_{n \to \infty} |H x_n(t) - H x(t)| = 0 \]

uniformly on \([0, T]\). Since \(T\) is arbitrary, one sees that \(H : \Omega \to \Omega\), is continuous. To use the Tychonoff fixed point theorem, we construct an appropriate subset \(Q\) of \(\Omega\) as follows. Let \((\Omega_n)_n\) be a sequence of subsets of \(\Omega\) defined by

\[
\begin{cases} 
\Omega_1 = \overline{\text{conv}(H(\Omega))} \\
\Omega_n = \overline{\text{conv}(H(\Omega_{n-1}))} \text{ for } n \geq 1.
\end{cases}
\]
Observe that, $\Omega_n \subset \Omega_{n-1}$ for all $n \geq 1$. Moreover, straightforward computations show that $w_0(\Omega) \leq w_0(B_{r_0})$. Moreover, we have $w_0(B_{r_0}) = 2r_0$; see \[\text{11}\]. Let $\bar{Q} = \bigcap_{n \geq 1} \Omega_n$. It is clear that $\bar{Q}$ is nonempty, closed and convex and $H(\bar{Q}) \subset \bar{Q}$. It follows from (vi) and the definition of $w_0$ that

$$w_0(\bar{Q}) = w_0\left( \bigcap_{n \geq 1} \Omega_n \right) = \lim_{n \to \infty} w_0(\Omega_n) = \lim_{n \to \infty} w_0(H(\Omega_{n-1}))$$

$$\leq \lim_{n \to \infty} k^n w_0(\Omega) = 0,$$

where

$$k = \frac{k(r_0)\phi(r_0)}{\Gamma(n)} \sup_{t \geq 0} m(t) < 1.$$

This proves that $\bar{Q} \subset \mathfrak{M}_{C(\mathbb{R}^+)}$. That is $H(\bar{Q})$ is compact in $C(\mathbb{R}^+)$. Finally, the Tychonoff fixed point theorem implies that $H$ has a fixed point in $\bar{Q}$. This completes the proof of the previous theorem.

\[\square\]

REFERENCES


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