PRICING TIMER OPTIONS UNDER FAST MEAN-REVERTING STOCHASTIC VOLATILITY

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ABSTRACT. Timer options are derivative securities whose expiration date is random, depending on the realized volatility of the underlying asset. In this paper, we consider the pricing of timer options in a stochastic volatility model, where the volatility of the underlying asset follows an ergodic diffusion process, running on a fast time scale. An asymptotic approximation to the option price is derived, and numerical examples are provided to assess its accuracy.

1 Introduction Timer options, also sometimes referred to as mileage options, are derivative securities whose expiration date is random, depending on the realized volatility of the underlying asset. When volatility is high, the options mature more rapidly, whilst in low volatility periods, the options take longer to mature. In particular, the exercise date is defined to be the time at which the accumulated variance of the underlying asset has reached a given level. Timer call options were introduced for sale by Société Générale Corporate and Investment Banking in April, 2007. Aside from providing another vehicle for trading volatility, it has been argued that timer options provide investors with greater flexibility, and prevent them from overpaying for options due to the discrepancy between realized and implied volatility (see [24]).

Timer options were in fact proposed in the academic literature before they were introduced in the market. In particular, Bick [4] studied timer options, emphasizing connections with portfolio insurance. However, since then there has been relatively little attention paid to these securities. Recently, Li [22, 23] studied the pricing and hedging of timer options under the Heston stochastic volatility model, and derived a semi-analytic formula for the option price in this case. Bernard and Cui [2]

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studied methods for pricing timer options based on time-change arguments. They derive an analytic formula for the option price in stochastic volatility diffusion models when the Brownian motions driving the stock and volatility are independent, and present a simulation algorithm in the case of correlated driving Brownian motions. Carr and Lee [7] consider the problem of hedging timer options when the asset process is a continuous semimartingale.

In this paper, we consider the pricing of timer options under a stochastic volatility model for the stock price, in which the volatility process is ergodic and runs on a fast time scale. In particular, under the real-world measure $\mathbb{P}$, the stock price satisfies the following stochastic differential equation.

\begin{align}
    dS_t &= \mu S_t \, dt + f(Y_t) S_t \, dW_t^1, \\
    dY_t &= \frac{1}{\varepsilon} b(Y_t) \, dt + \frac{1}{\sqrt{\varepsilon}} g(Y_t) (\rho \, dW_t^1 + \sqrt{1 - \rho^2} \, dW_t^2),
\end{align}

where $\mu \in \mathbb{R}$, $\rho \in (-1, 1)$, $W = (W^1, W^2)$ is a standard two-dimensional Brownian motion, and $b$ and $g$ are such that $Y_t$ is an ergodic diffusion process on an interval $J$ satisfying strong mixing conditions. We take $J = \mathbb{R}$, although extensions to other cases, such as when $Y$ is a CIR process on $(0, \infty)$ are straightforward. The parameter $\varepsilon$ is small, and governs the time scale of the volatility process. An approximate pricing formula is derived based on an asymptotic expansion in this parameter. The correlation parameter $\rho$ is typically negative, indicating a ‘leverage effect’ whereby volatility typically increases when the stock price declines. The function $f : J \to \mathbb{R}_+$ transforms the value of the ergodic diffusion $Y_t$ to the volatility coefficient of the stock.

The timer option has payoff $\max(S_{\tau} - K, 0)$ at the random time $\tau$, where:

\begin{align}
    \tau &= \inf \{t > 0, \ I_t = B \},
\end{align}

where $B$ is the prescribed variance budget, and $I$ is the cumulative realized variance:

\begin{align}
    I_t &= \int_0^t f^2(Y_s) \, ds.
\end{align}

Models of the form (1) were originally applied to pricing plain vanilla equity options in stochastic volatility models by Fouque et al. [14] (see
Applying tools from singular perturbation theory on the pricing partial differential equation, they derive an asymptotic approximation to the price of a European call option of the form:

\[
 u \approx u_0 - (T - t) \left( V_2 s^2 \frac{\partial^2 u_0}{\partial s^2} + V_3 s^3 \frac{\partial^3 u_0}{\partial s^3} \right),
\]

where

\[
 u_0 = C_{BS}(s, K, T - t, r, \sigma),
\]

where \( s \) is the current stock price, \( K \) is the strike price of the option, \( T - t \) is the time to expiration, \( r \) is the continuously compounded risk-free interest rate, \( \sigma \) is the effective volatility, i.e., the square-root of the mean of \( f^2 \) under the stationary distribution of \( Y \), \( V_2 \) and \( V_3 \) are constants that are both \( O(\sqrt{\varepsilon}) \), and \( C_{BS} \) is given by the usual Black-Scholes option pricing formula

\[
 C_{BS}(s, K, T - t, r, \sigma) = sN(d_1) - Ke^{-r(T-t)}N(d_2),
\]

where

\[
 d_1 = \frac{\log(s/K) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma \sqrt{T - t}},
\]

\[
 d_2 = d_1 - \sigma \sqrt{T - t},
\]

and

\[
 N(z) = \int_{-\infty}^{z} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}.
\]

Formula (4) uses the long run average volatility of the stock, which may be estimated, for example by time series methods, as well as the two constants \( V_2 \) and \( V_3 \), which may be calibrated to the observed implied volatility surface. An alternative approximation using only two parameters, which may be calibrated directly to observed option prices is presented in [12, 18] (see also Section 3.3 below).

The analogue of formula (4) for timer options is the main result of this paper, and is derived in Section 3. Crucially, this formula uses the same parameters as (4) and can therefore be calibrated using observed prices of European options. The validity of the asymptotic expansion (4) depends upon the smallness of the parameter \( \varepsilon \) (equivalently, the speed
of the volatility time-scale), and is discussed in [16] for the case where \( f \) and \( 1/f \) are bounded. Yamamoto and Takahashi [25] derive the next term (the \( O(\varepsilon) \) term) for the expansion (4), and demonstrate its increased accuracy. Validity of higher order versions of the approximation is discussed in [8]. Methods for estimating \( \varepsilon \) are also discussed by Fouque et al. [13, 14], and the authors estimate \( \varepsilon \approx 1/200 \) (fast mean reversion) for the S&P 500 index in the case where \( Y \) is an OU process and \( f(y) = e^{y} \). Boswijk [5] finds a slower volatility time-scale \( \varepsilon \approx 1/5.074 \) using data on the Amsterdam Exchange Index (AEX). Fouque et al. [17] find evidence of two volatility time-scales, and propose a model with an additional volatility factor, again deriving pricing expansions in this case.

The ansatz (1) together with a singular perturbation expansion has been used to price other options as well. American options are considered in [15]. Fouque and Han [10, 11] study Asian and compound options, respectively, while Ilhan et al. [20] consider barrier, lookback and passport options. Applications to interest rate derivatives are studied in [9].

The remainder of the paper is structured as follows. The second section presents the derivation of the partial differential equation for timer option pricing. The third section presents the asymptotic analysis, including the approximate pricing formula for the timer call, and discusses its calibration and application in practice. The fourth section presents a numerical example illustrating the use of the approximation and assessing its accuracy. The fifth section concludes.

2 Derivation of the pricing equation In this section, we derive the partial differential equation for pricing timer options based on the model (1), which will be the subject of the asymptotic analysis of the next section. The derivation follows closely those in [13, 22].

A (nonunique) risk-neutral measure \( Q \) is specified by selecting a sufficiently regular, adapted process \( \lambda \) (see, e.g., [21]) and such that the process \( \tilde{W} \) defined by

\[
(9) \quad \tilde{W}_t = W_t + \int_0^t \lambda_s \, ds
\]

is a standard \( Q \)-Brownian motion, and \( e^{-rt}S_t \) is a \( Q \)-martingale, where \( r \) is the deterministic, continuously compounded risk-free interest rate.
A simple calculation shows that this leads to the specification
\begin{equation}
\lambda_1^t = \frac{\mu - r}{f(Y_t)}, \quad \lambda_2^t = \gamma_t,
\end{equation}
where \( \gamma_t \) is any sufficiently regular adapted process, commonly referred to as the market price of volatility risk. In order to facilitate the asymptotic analysis to follow, we assume that \( \gamma_t \) takes the form \( \gamma_t = \gamma(Y_t) \).

Under the chosen risk-neutral measure \( Q \) the stochastic differential equation (1) for the stock price then assumes the form
\begin{equation}
\begin{aligned}
dS_t &= rS_t dt + f(Y_t)S_t d\overline{W}_t^1 \\
dY_t &= \left( \frac{1}{\varepsilon} b(Y_t) - \frac{1}{\varepsilon} g(Y_t)A(Y_t) \right) dt \\
&\quad + \frac{1}{\varepsilon} g(Y_t) \left( \rho d\overline{W}_t^1 + \sqrt{1 - \rho^2} d\overline{W}_t^2 \right),
\end{aligned}
\end{equation}
where
\begin{equation}
A(y) = \rho \frac{\mu - r}{f(y)} + \gamma(y) \sqrt{1 - \rho^2}.
\end{equation}

The price of the timer call at time \( t \wedge \tau \) is
\begin{equation}
C_{t \wedge \tau} = \mathbb{E}_Q^\mathcal{F}_{t \wedge \tau} \left[ e^{-(\tau - t \wedge \tau)} \max(S_{t \wedge \tau} - K, 0) \right].
\end{equation}
The process \( (S_t, Y_t, I_t) \) is Markov, and \( \tau \) is the first time that this Markov process hits the plane \( \{ I = B \} \). An application of the Feynman-Kac formula leads to the following partial differential equation for the option value \( C_{t \wedge \tau} = u(t \wedge \tau; S_{t \wedge \tau}, V_{t \wedge \tau}, I_{t \wedge \tau}) \):
\begin{equation}
\begin{aligned}
\frac{\partial u}{\partial t} + rs \frac{\partial u}{\partial s} + \left( \frac{b(y)}{\varepsilon} - \frac{g(y)A(y)}{\sqrt{\varepsilon}} \right) \frac{\partial u}{\partial y} + f^2(y) \frac{\partial^2 u}{\partial x^2} \\
&\quad + \frac{f^2(y)\rho^2}{2} \frac{\partial^4 u}{\partial s^4} + \frac{g^2(y)\rho^2}{2e} \frac{\partial^2 u}{\partial y^2} + \frac{\rho sf(y)g(y)}{\sqrt{\varepsilon}} \frac{\partial^2 u}{\partial s \partial y} - ru = 0
\end{aligned}
\end{equation}
with \( (t, s, y, x) \in (0, \infty) \times (0, \infty) \times J \times (0, B) \) and boundary condition \( u(t, s, v, B) = \max(s - K, 0) \). As noted by Li [22], given the value of the realized variance, the option pricing problem is independent of the current time variable \( t \), so that we have
\begin{equation}
\frac{\partial u}{\partial t} = 0,
\end{equation}
and the partial differential equation for $u$, therefore, becomes

$$
(16) \qquad rs \frac{\partial u}{\partial s} + \left( \frac{b(y)}{\varepsilon} - \frac{g(y)\Lambda(y)}{\sqrt{\varepsilon}} \right) \frac{\partial u}{\partial y} + f^2(y) \frac{\partial u}{\partial x} \\
+ \frac{f^2(y)s^2}{2} \frac{\partial^2 u}{\partial s^2} + \frac{g^2(y)}{2\varepsilon} \frac{\partial^2 u}{\partial y^2} + \frac{\rho s f(y)g(y)}{\sqrt{\varepsilon}} \frac{\partial^2 u}{\partial s \partial y} - ru = 0
$$

with $(s, y, x) \in (0, \infty) \times J \times (0, B)$. Observe that the above equation is parabolic, with $x$ serving the role of the (backward) time variable, $s$ and $y$ being spatial variables, and with terminal condition $u(s, y, B) = \max(s - K, 0)$.

3 Asymptotic analysis

In this section, we carry out the asymptotic analysis of the timer option pricing equation (16). The analysis is based on a perturbation expansion in the parameter $\varepsilon$, which may be interpreted as governing the time-scale of the volatility process (see [13] for a detailed analysis and interpretation). For small values of $\varepsilon$, the volatility process is running at a very fast speed, and hence may be regarded as converging rapidly (in the time-scale of the stock price process) to its stationary distribution. The analysis follows closely that carried out for European call options in [14].

The partial differential equation (16) can be written as

$$
(17) \qquad L_2 u + \frac{1}{\sqrt{\varepsilon}} L_1 u + \frac{1}{\varepsilon} L_0 u = 0,
$$

where

$$
(18) \qquad L_0 u = b(y) \frac{\partial u}{\partial y} + \frac{g^2(y)}{2} \frac{\partial^2 u}{\partial y^2},
$$

$$
(19) \qquad L_1 u = \rho s f(y)g(y) \frac{\partial^2 u}{\partial s \partial y} - g(y)\Lambda(y) \frac{\partial u}{\partial y},
$$

$$
(20) \qquad L_2 = f^2(y) \frac{\partial u}{\partial x} + rs \frac{\partial u}{\partial s} + \frac{s^2 f^2(y)}{2} \frac{\partial^2 u}{\partial s^2} - ru = 0.
$$

Note that $L_2$ is almost the Black-Scholes operator (with $x$ playing the role of the time variable, i.e., in ‘volatility time’). Furthermore, $L_0$ acts in the $y$ variable, and is the generator of the ergodic diffusion $\tilde{Y}_t$:

$$
(21) \qquad d\tilde{Y}_t = b(\tilde{Y}_t) dt + g(\tilde{Y}_t) dZ_t,
$$
where $Z$ is a standard Brownian motion (note that this can also be interpreted as setting $\varepsilon = 1$ in the specification of $Y$).

Under additional assumptions on $L_0$ (it is sufficient that $L_0$ has a spectral gap in $L^2(\varphi(y)dy)$; for a rigorous discussion, see [3]; for conditions ensuring the existence of a spectral gap, see [19]), we have the following.

1. There is a unique stationary density $\varphi(y)$ such that $L_0^*\varphi = 0$, where

$$L_0^*f = \frac{1}{2} \frac{d^2}{dy^2}(g^2(y)f(y)) - \frac{d}{dy}(b(y)f(y)).$$

2. For a given function $h \in L^2(\varphi(y)dy)$, if there is a solution $p$ to the Poisson equation $L_0 p = h$, then $\langle h \rangle = 0$ where $\langle h \rangle$ denotes the average of $h$ under the stationary distribution

$$\langle h \rangle = \int h(y)\varphi(y) dy.$$ (Formally, this follows by integration by parts: $\langle h \rangle = \int h\varphi = \int (L_0^* p) \varphi = \int p(L_0^* \varphi) = 0$.)

3. The only solutions$^1$ to $L_0 p = 0$ are constants. (Formally, by Itô’s lemma: $p(\tilde{Y}_t) = p(\tilde{Y}_0) + \int_0^t L_0 p(\tilde{Y}_u) du + \int_0^t p(L_1 \tilde{Y}_u) g(\tilde{Y}_u) dW_u$, so that conditioning on $Y_0 = x$ and taking expectations gives $p(x) = E_x[p(\tilde{Y}_t)]$. Also, for $y \neq x$, $p(y) = E_y[p(\tilde{Y}_t)]$. Letting $t \to \infty$ and using ergodicity gives $p(x) = p(y) = \langle p \rangle$).

In what follows, the above properties will be referred to as Generator Properties 1, 2 and 3, respectively.

Assume that there exists a solution to equation (16) of the form

$$u = u_0 + \sqrt{\varepsilon} u_1 + \varepsilon u_2 + \varepsilon^{3/2} u_3 + \cdots.$$ (24)

Substitute this into (17) and collect terms to get

$$0 = \frac{1}{\varepsilon}(L_0 u_0) + \frac{1}{\sqrt{\varepsilon}}(L_1 u_0 + L_0 u_1) + (L_0 u_2 + L_1 u_1 + L_2 u_0) + \sqrt{\varepsilon}(L_0 u_3 + L_1 u_2 + L_2 u_1) + \cdots.$$ (25)

$^1$Subject to growth conditions, e.g., $p \in L^2(\varphi(y)dy)$. 

Matching terms of order $1/\varepsilon$ in (25) yields $L_0u_0 = 0$. Under growth assumptions on $u_0$, Generator Property 3 implies that $u_0$ does not depend on $y$, $u_0 = u_0(s,x)$. Matching terms of order $1/\sqrt{\varepsilon}$ in (25) then yields $L_1u_0 + L_0u_1 = 0$. However, $L_1$ takes derivatives with respect to $y$ (see (19)), and therefore $L_1u_0 = 0$. This implies that $L_0u_1 = 0$, and using the same reasoning as for $u_0$, we have that (making growth assumptions on $u_1$) $u_1$ does not depend on $y$, i.e., $u_1 = u_1(s,x)$.

3.1 The first approximation ($u_0$) Matching terms of order 1 in (25) yields

$$L_0u_2 + L_1u_1 + L_2u_0 = 0.$$  

Since $L_1$ takes a derivative with respect to $y$, and $u_1$ does not depend on $y$, this simplifies to $L_0u_2 + L_2u_0 = 0$. Generator Property 2 then implies that $\langle L_2u_0 \rangle = 0$. However, $u_0$ is independent of $y$, so the only place where $y$ appears in the expression $L_2u_0$ is in the coefficients of the operator $L_2$. Thus we have $\langle L_2u_0 \rangle = \langle L_2 \rangle u_0$, where

$$\langle L_2 \rangle = \langle f^2 \rangle \frac{\partial}{\partial x} + rs \frac{\partial}{\partial s} + \frac{s^2 \langle f^2 \rangle}{2} \frac{\partial^2}{\partial s^2} - r,$$

and the condition from Generator Property 2 becomes

$$\langle L_2 \rangle u_0 = \sigma^2 \frac{\partial u_0}{\partial x} + rs \frac{\partial u_0}{\partial s} + \frac{s^2 \sigma^2}{2} \frac{\partial^2 u_0}{\partial s^2} - ru_0 = 0,$$

where $\sigma^2$ is the mean of $f^2$ under the stationary distribution of $Y$:

$$\sigma^2 = \langle f^2 \rangle = \int f^2(y)\varphi(y)\,dy.$$

Note that (28) is essentially the Black-Scholes equation with $x$ playing the role of the time variable.\footnote{Indeed, it can be regarded as the Black-Scholes equation with interest rate $r/\sigma^2$ and volatility 1.} Matching boundary conditions in (25) gives the terminal condition $u_0(s,B) = \max(s - K,0)$. It is easy to see that the solution to the PDE is

$$u_0(s,x) = C_{BS}(s,K,(B-x)/\sigma^2,r,\sigma),$$

i.e., the Black-Scholes formula (5) with the volatility $\sigma$ set to $\bar{\sigma}$, and the time to expiration set equal to $(B-x)/\sigma^2$, the remaining variance budget divided by the rate of variance accumulation under the stationary distribution of $Y$. Observe that this is exactly equal to the price of the timer option in the event that volatility is a constant, $\sigma$, for in this case $\bar{\sigma}^2 = \sigma^2$ and $\tau = B/\sigma^2$.\footnote{Indeed, it can be regarded as the Black-Scholes equation with interest rate $r/\sigma^2$ and volatility 1.}
3.2 Derivation of the first correction \((u_1)\) Now that \(u_0\) is known, we can use the relationship \(L_0u_2 + L_2u_0 = 0\) together with zero boundary conditions for \(u_2\) (obtained by matching boundary conditions in (25)) in order to derive the second order term in the expansion \(u_2\). We already know by Generator Property 2 that \(\langle L_2u_0 \rangle = \langle L_2 \rangle u_0 = 0\). So, we have

\[
L_0u_2 = (\langle L_2 \rangle - L_2)u_0
= (\sigma^2 - f^2(y)) \frac{\partial u_0}{\partial x} + \frac{s^2}{2}(\sigma^2 - f^2(y)) \frac{\partial^2 u_0}{\partial s^2}.
\]

Denote by \(\psi(y)\) a solution of the equation

\[
L_0\psi(y) = f^2(y) - \sigma^2.
\]

Then

\[
u_2(s, y, x) = -\psi(y) \left( \frac{\partial u_0}{\partial x}(s, x) + \frac{s^2}{2} \frac{\partial^2 u_0}{\partial s^2}(s, x) \right) + c(s, x),
\]

where \(c(s, x)\) is a constant that may depend on \(s\) and \(x\).

Matching terms of order \(\sqrt{\varepsilon}\) in (25) yields

\[
L_0u_3 + L_1u_2 + L_2u_1 = 0.
\]

Since this is a Poisson equation for \(u_3\), Generator Property 2 yields that \(\langle L_1u_2 + L_2u_1 \rangle = 0\). Since \(u_1\) does not depend on \(y\), we have

\[
\langle L_2 \rangle u_1 = -\langle L_1u_2 \rangle.
\]

Noting that \(c\) is independent of \(y\) (and hence \(L_1c = 0\)), and using (33) we can compute

\[
-\langle L_1u_2 \rangle = C_1 \left( \frac{\partial^2 u_0}{\partial s \partial x} + s^2 \frac{\partial^2 u_0}{\partial s^2} + \frac{s^3}{2} \frac{\partial^3 u_0}{\partial s^3} \right) - C_2 \left( \frac{\partial u_0}{\partial x} + \frac{s^2}{2} \frac{\partial^2 u_0}{\partial s^2} \right),
\]

where \(C_1\) and \(C_2\) are constants determined by averaging under the stationary distribution of \(Y\):

\[
C_1 = \rho(fg\psi'), \quad C_2 = \langle g\Lambda\psi' \rangle.
\]
Note that the terms in (36) involving $u_0$ can all be calculated analytically based on the formula (30) and the known derivatives of the Black-Scholes formula (the ‘Greeks’).

Both $u_0$ and $\partial_x u_0$ solve the homogeneous equation $\langle L_2 \rangle p = 0$. Furthermore, in each term in (36) the power of $s$ is equal to the degree of the derivative (in $s$) that it multiplies. This implies (for example, by changing coordinates to $z = \log(s)$; see [13]) that the solution to (35) with the terminal condition $u_1(s, B) = 0$ is

$$
(38) \quad u_1(s, x) = -\frac{(B - x)}{\sigma^2} \left[ C_1 \left( s \frac{\partial^2 u_0}{\partial s \partial x} + s^2 \frac{\partial^2 u_0}{\partial s^2} + \frac{s^3}{2} \frac{\partial^3 u_0}{\partial s^3} \right) 
- C_2 \left( \frac{\partial u_0}{\partial x} + \frac{s^2}{2} \frac{\partial^2 u_0}{\partial s^2} \right) \right].
$$

We have thus derived the approximation to the timer option price (taking only the first two terms in the expansion (25)) to be

$$
(39) \quad u \approx u_0(s, x) - \sqrt{\varepsilon} \cdot \frac{B - x}{\sigma^2} \left[ C_1 \left( s \frac{\partial^2 u_0}{\partial s \partial x} + s^2 \frac{\partial^2 u_0}{\partial s^2} + \frac{s^3}{2} \frac{\partial^3 u_0}{\partial s^3} \right) 
- C_2 \left( \frac{\partial u_0}{\partial x} + \frac{s^2}{2} \frac{\partial^2 u_0}{\partial s^2} \right) \right],
$$

where $u_0$ and its derivatives may be evaluated based on the Black-Scholes formula (30). The formula (39) is very similar to the one derived in [14] for European options under fast mean-reverting stochastic volatility, and the differences are intuitive. In [14] the zero-order approximation $u_0$ is given by the Black-Scholes formula evaluated with a constant time to expiration $(T - t)$ where $t$ is the current time and $T$ is the expiration date of the option, and the first order correction is also proportional to $T - t$. We see that in our formula (39) the time to expiration of the European option has been replaced by the scaled remaining variance budget $(B - x)/\sigma^2$. Secondly, our formula has extra terms in the correction, corresponding to the partial derivative $\partial_x u_0$ and the mixed partial derivative $\partial^2_{sx} u_0$, which reflects the fact that in the European option problem the time variable is deterministic, whereas in the timer option problem, we may regard time as running on a ‘stochastic clock’ governed by the volatility process $f(Y_t)$ (for much more on this interpretation, and its use in deriving numerical methods for timer option pricing; see [2]).
3.3 Implementation in practice  There are at least two distinct ways in which the pricing formula (39) may be used. One approach would be to attempt to estimate the parameters of the stochastic volatility model using a statistical method, and the market price of risk using observable prices of derivative securities. This is a difficult statistical exercise. However, were one to apply this approach, techniques similar to those in [25] could be applied to derive the higher order expansion of the timer option price (i.e., to solve for $u_2$ in (25)), resulting in greater accuracy of the analytical approximation.

The alternative would be to try to match the parameters in the formula (39) using prices of options that can be observed in the market. The timer approximation formula (39) involves three parameters that are not directly observable in the market: the effective volatility $\tilde{\sigma}$ (square-root of the mean of $f^2$ under the real-world stationary distribution of $Y$), and the two constants $\tilde{C}_1 = \sqrt{\tilde{\sigma}}\cdot C_1$ and $\tilde{C}_2 = \sqrt{\tilde{\sigma}}\cdot C_2$. The effective volatility $\tilde{\sigma}$ may be estimated using statistical methods based on the time series of stock returns. To estimate $\tilde{C}_1$ and $\tilde{C}_2$, recall the approximation formula for European call options of Fouque et al. [14], given by (4). The coefficients $V_2$ and $V_3$ in this approximation are given by

$$V_2 = \sqrt{\tilde{\sigma}}\left(\rho(fg\psi') - \frac{1}{2}\langle\Lambda g\psi'\rangle\right), \quad V_3 = \frac{1}{2}\sqrt{\tilde{\sigma}}\rho(fg\psi'),$$

from which we see that $\tilde{C}_1 = 2V_3$ and $\tilde{C}_2 = 4V_3 - 2V_2$. One may then follow the algorithm presented in [13] to estimate these parameters based on the implied volatility surface.

Alternatively, following Fouque et al. [18] (see also [12] for more details), we can introduce the adjusted effective volatility

$$\sigma^* = \sqrt{\sigma^2 + \tilde{C}_2}$$

and the alternative approximation

$$u \approx u^*_0 + \tilde{C}_1 u^*_1,$$

where

$$u^*_0 = C_{BS}(s, K, (B - x)/(\sigma^*)^2, r, \sigma^*)$$

and

$$u^*_1 = \frac{B - x}{(\sigma^*)^2} \left( s \frac{\partial^2 u^*_0}{\partial s \partial x} + s^2 \frac{\partial^2 u^*_0}{\partial s^2} + s^3 \frac{\partial^3 u^*_0}{\partial s^3} \right).$$
The advantage of (42) is that it only has two parameters ($\sigma^*$ and $\tilde{C}_1$) rather than three, and it does not require the statistical estimation of the effective volatility $\sigma$. Following the derivation in [12] yields that the order of the accuracy of the approximation (42) is the same as that for (39). Furthermore, the parameters appearing in the timer option approximation are the same as those appearing in the European option approximation in [12, 18]. In particular, this implies that the constants used in the timer option approximation may be derived from those for the approximation to European option prices. Thus, the coefficients of the timer option formula may be calibrated based on the observed prices of European call options on the same underlying.

4 Examples

In this section, we present numerical examples illustrating the accuracy and use of the asymptotic approximation (39) in a model where the coefficients $\tilde{C}_1$ and $\tilde{C}_2$ are analytically tractable functions of the coefficients of the stochastic volatility model (1). We stress that in practice it is difficult to estimate all of the coefficients of the stochastic volatility model, and that the approximation formula (39) (or (42)) would rather be used by calibrating the parameters to the observed prices of European options, as described above.

We consider an exponential-OU stochastic volatility model, with driving volatility process (under the real-world measure)

$$dY_t = \frac{1}{\varepsilon} (\beta - Y_t) dt + \sigma (\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2)$$

with $f(y) = e^y$. With this specification, Fouque et al. [13] show that

$$\psi'(y) = \frac{2}{\sigma^2 \varphi(y)} \int_{-\infty}^{y} (f^2(z) - \sigma^2) \varphi(z) dz,$$

where $\varphi$ is the density of the stationary distribution (under $\mathbb{P}$) of $Y$ (in this case, normal with mean $\beta$ and variance $\sigma^2/2$). $L_0$ is self-adjoint in $L^2(\varphi(y)dy)$ and has a spectral gap. We set the parameters to the values used in [16]:

$$\varepsilon = \frac{1}{200}, \quad \beta = \log(0.1), \quad \sigma = 1.0,$$

$$\rho = -0.2, \quad \mu = 0.2, \quad r = 0.04,$$
and take the market price of volatility risk $\gamma = 0$. Simple integrations yield

$$
\sigma^2 = \langle e^{2\nu} \rangle = \exp(2\beta + \sigma^2) = 0.02718,
$$

$$
C_1 = \rho(fg\psi) = -\frac{2\rho}{\sigma} \exp \left( 3\beta + \frac{5\sigma^2}{4} \right) \cdot (\exp(\sigma^2) - 1) = 0.002399,
$$

$$
C_2 = \langle g\Lambda\psi' \rangle = -\frac{2\rho(\mu - r)}{\sigma} \exp \left( \beta + \frac{\sigma^2}{4} \right) \cdot (\exp(\sigma^2) - 1) = 0.01412.
$$

We compare the performance of the zero order approximation $u_0$ and the approximation (39) to prices generated by a Monte-Carlo simulation. We use Euler’s method to simulate the stochastic differential equations (11), with $S_0 = 100$, $Y_0 = \beta$, and use $10^5$ simulations with time-step of $10^{-4}$.

Timer call options with a variety of strikes and variance budgets are considered. In particular, we consider three variance budgets, $B = \sigma^2 \cdot T$ for $T = 1/12, 1/4, 1/2$, corresponding to options with approximately one month, three months and six months to expiration, respectively. For each choice of variance budget $B$ (equivalently, for each choice of $T$), options with strike prices equal to $S_0 + \frac{\sigma}{\sqrt{T}} \cdot \sqrt{T}$ are priced for $\kappa = 0, \pm 1, \ldots, \pm 3$. The results are shown in Table 4. For prices computed with Monte-Carlo simulation, the numbers in parentheses indicate the standard errors. For prices computed using the first approximation $u_0$, given by (30), or by the approximation with the first correction $u_0 + \sqrt{\varepsilon} u_1$, given by (39), the numbers in parentheses indicate the relative error compared to the Monte-Carlo price, given by $(P_A - P_{MC})/P_{MC}$, where $P_A$ is the price computed using the analytical approximation, and $P_{MC}$ is the price computed using Monte-Carlo simulation.

The method generally performs quite well. The first approximation $u_0$ is accurate for in the money options, where relative errors are typically on the order of 1%, but becomes somewhat less accurate for options that are out of the money. The corrected approximation $u_0 + \sqrt{\varepsilon} u_1$ performs similarly to the first approximation when the options are in the money, and exhibits significantly improved accuracy when the options are out of the money. Generally, the approximation works better for long times to expiration (higher variance budgets), and for options that are nearer the money, which is in accordance with the observations of Fouque et al. [13], and is not surprising given the fact that the method relies on averaging against the stationary distribution of volatility, and a longer life of the option allows the volatility process more time to converge to its stationary state.
<table>
<thead>
<tr>
<th>Option Parameters</th>
<th>Monte-Carlo Price</th>
<th>$u_0$</th>
<th>$u_0 + \sqrt{\varepsilon t_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T = 1/12$, $\kappa = -3$</td>
<td>7.5906 (0.0146)</td>
<td>7.5482 (−0.56%)</td>
<td>7.5419 (−0.64%)</td>
</tr>
<tr>
<td>$T = 1/12$, $\kappa = -2$</td>
<td>5.4420 (0.0136)</td>
<td>5.3979 (−0.81%)</td>
<td>5.3973 (−0.82%)</td>
</tr>
<tr>
<td>$T = 1/12$, $\kappa = -1$</td>
<td>3.5698 (0.0118)</td>
<td>3.5276 (−1.18%)</td>
<td>3.5337 (−1.01%)</td>
</tr>
<tr>
<td>$T = 1/12$, $\kappa = 0$</td>
<td>2.1022 (0.0095)</td>
<td>2.0664 (−1.70%)</td>
<td>2.0775 (−1.17%)</td>
</tr>
<tr>
<td>$T = 1/12$, $\kappa = 1$</td>
<td>1.0954 (0.0070)</td>
<td>1.0683 (−2.47%)</td>
<td>1.0803 (−1.38%)</td>
</tr>
<tr>
<td>$T = 1/12$, $\kappa = 2$</td>
<td>0.5022 (0.0047)</td>
<td>0.4822 (−3.98%)</td>
<td>0.4918 (−2.97%)</td>
</tr>
<tr>
<td>$T = 1/12$, $\kappa = 3$</td>
<td>0.2021 (0.0029)</td>
<td>0.1889 (−6.54%)</td>
<td>0.1948 (−3.62%)</td>
</tr>
<tr>
<td>$T = 1/4$, $\kappa = -3$</td>
<td>13.3809 (0.0254)</td>
<td>13.3702 (−0.08%)</td>
<td>13.3462 (0.26%)</td>
</tr>
<tr>
<td>$T = 1/4$, $\kappa = -2$</td>
<td>9.6500 (0.0239)</td>
<td>9.6298 (−0.21%)</td>
<td>9.6154 (−0.36%)</td>
</tr>
<tr>
<td>$T = 1/4$, $\kappa = -1$</td>
<td>6.3877 (0.0210)</td>
<td>6.3610 (−0.42%)</td>
<td>6.3599 (−0.44%)</td>
</tr>
<tr>
<td>$T = 1/4$, $\kappa = 0$</td>
<td>3.8237 (0.0171)</td>
<td>3.7930 (−0.80%)</td>
<td>3.8034 (−0.53%)</td>
</tr>
<tr>
<td>$T = 1/4$, $\kappa = 1$</td>
<td>2.0505 (0.0128)</td>
<td>2.0198 (−1.50%)</td>
<td>2.0355 (−0.73%)</td>
</tr>
<tr>
<td>$T = 1/4$, $\kappa = 2$</td>
<td>0.9773 (0.0088)</td>
<td>0.9557 (−2.22%)</td>
<td>0.9702 (−0.73%)</td>
</tr>
<tr>
<td>$T = 1/2$, $\kappa = 3$</td>
<td>0.4128 (0.0056)</td>
<td>0.4016 (−2.71%)</td>
<td>0.4117 (−0.27%)</td>
</tr>
<tr>
<td>$T = 1/2$, $\kappa = -3$</td>
<td>19.2452 (0.0364)</td>
<td>19.2632 (0.09%)</td>
<td>19.2134 (−0.17%)</td>
</tr>
<tr>
<td>$T = 1/2$, $\kappa = -2$</td>
<td>13.9722 (0.0344)</td>
<td>13.9733 (0.01%)</td>
<td>13.9358 (−0.26%)</td>
</tr>
<tr>
<td>$T = 1/2$, $\kappa = -1$</td>
<td>9.3467 (0.0306)</td>
<td>9.3289 (−0.19%)</td>
<td>9.3114 (−0.38%)</td>
</tr>
<tr>
<td>$T = 1/2$, $\kappa = 0$</td>
<td>5.6917 (0.0253)</td>
<td>5.6598 (−0.56%)</td>
<td>5.6622 (−0.52%)</td>
</tr>
<tr>
<td>$T = 1/2$, $\kappa = 1$</td>
<td>3.1401 (0.0193)</td>
<td>3.1002 (−1.27%)</td>
<td>3.1145 (−0.82%)</td>
</tr>
<tr>
<td>$T = 1/2$, $\kappa = 2$</td>
<td>1.5650 (0.0136)</td>
<td>1.5321 (−2.10%)</td>
<td>1.5489 (−1.03%)</td>
</tr>
<tr>
<td>$T = 1/2$, $\kappa = 3$</td>
<td>0.7046 (0.0090)</td>
<td>0.6857 (−2.69%)</td>
<td>0.6990 (−0.80%)</td>
</tr>
</tbody>
</table>

**Table 1:** Timer option prices under fast mean-reverting volatility.

A key requirement for the accuracy of the method is that $\varepsilon$ be small, i.e., that the strength of volatility mean reversion be very high (or, equivalently, that volatility is running on a very fast time scale). The results in Table 4 use $\varepsilon = 1/200$, which is the value estimated based on S&P index returns by Fouque et al. [14]. As mentioned above, Boswijk [5] estimated $\varepsilon = 1/5.074$ based on return data for the AEX. The results presented in Table 2 price the same timer options as in Table 4, using $\varepsilon = 1/5.074$ rather than $\varepsilon = 1/200$. As noted by Yamamoto and Takahashi [25], the approximation performs much less well in this regime. The same basic pattern for the accuracy with respect to the money-ness occurs; the magnitudes of the errors of the approximations $u_0$ and $u_0 + \sqrt{\varepsilon t_1}$ are similar when the option is in the money. For out of the money options, the higher order approximation does better, although
neither method is particularly accurate. Finally, note that there is little relation between the variance budget (equivalently the expected time to expiration) and the accuracy. This is likely because for this relatively low level of mean reversion, the volatility process has not had enough time to converge to its stationary state, even for the options with the larger variance budgets.

<table>
<thead>
<tr>
<th>Option Parameters</th>
<th>Monte-Carlo Price</th>
<th>( u_0 )</th>
<th>( u_0 + \sqrt{v_0}u_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T = 1/12, \kappa = -3 )</td>
<td>8.0065 (0.0148)</td>
<td>7.5482 (−5.72%)</td>
<td>7.5090 (−6.21%)</td>
</tr>
<tr>
<td>( T = 1/12, \kappa = -2 )</td>
<td>5.8402 (0.0139)</td>
<td>5.3979 (−7.57%)</td>
<td>5.3939 (−7.64%)</td>
</tr>
<tr>
<td>( T = 1/12, \kappa = -1 )</td>
<td>3.9220 (0.0123)</td>
<td>3.5276 (−10.06%)</td>
<td>3.5659 (−9.08%)</td>
</tr>
<tr>
<td>( T = 1/12, \kappa = 0 )</td>
<td>2.3828 (0.0101)</td>
<td>2.0664 (−13.28%)</td>
<td>2.1357 (−10.37%)</td>
</tr>
<tr>
<td>( T = 1/12, \kappa = 1 )</td>
<td>1.2880 (0.0123)</td>
<td>1.0683 (−13.28%)</td>
<td>1.1437 (−11.20%)</td>
</tr>
<tr>
<td>( T = 1/2, \kappa = 0 )</td>
<td>14.1005 (0.0261)</td>
<td>13.3702 (−5.18%)</td>
<td>13.2194 (−6.25%)</td>
</tr>
<tr>
<td>( T = 1/2, \kappa = -2 )</td>
<td>10.3578 (0.0247)</td>
<td>9.6298 (−7.03%)</td>
<td>9.5393 (−7.90%)</td>
</tr>
<tr>
<td>( T = 1/2, \kappa = -1 )</td>
<td>7.0408 (0.0222)</td>
<td>6.3610 (−9.65%)</td>
<td>6.3540 (−9.75%)</td>
</tr>
<tr>
<td>( T = 1/4, \kappa = 0 )</td>
<td>4.3693 (0.0185)</td>
<td>3.7930 (−13.19%)</td>
<td>3.8586 (−11.69%)</td>
</tr>
<tr>
<td>( T = 1/4, \kappa = -2 )</td>
<td>1.2377 (0.0101)</td>
<td>0.9557 (−22.79%)</td>
<td>1.0468 (−15.43%)</td>
</tr>
<tr>
<td>( T = 1/4, \kappa = -1 )</td>
<td>0.5637 (0.0067)</td>
<td>0.4016 (−28.76%)</td>
<td>0.4648 (−17.54%)</td>
</tr>
<tr>
<td>( T = 1/4, \kappa = 1 )</td>
<td>2.4506 (0.0142)</td>
<td>2.0198 (−17.58%)</td>
<td>2.1185 (−13.55%)</td>
</tr>
<tr>
<td>( T = 1/4, \kappa = 2 )</td>
<td>1.0683 (0.0021)</td>
<td>0.9557 (−22.79%)</td>
<td>1.0468 (−15.43%)</td>
</tr>
<tr>
<td>( T = 2, \kappa = 0 )</td>
<td>20.1023 (0.0372)</td>
<td>19.2632 (−4.17%)</td>
<td>18.9506 (−5.73%)</td>
</tr>
<tr>
<td>( T = 2, \kappa = -2 )</td>
<td>14.8371 (0.0555)</td>
<td>13.9733 (−5.82%)</td>
<td>13.7379 (−7.41%)</td>
</tr>
<tr>
<td>( T = 2, \kappa = -1 )</td>
<td>10.1685 (0.0320)</td>
<td>9.3289 (−8.26%)</td>
<td>9.2191 (−9.34%)</td>
</tr>
<tr>
<td>( T = 2, \kappa = 1 )</td>
<td>6.4019 (0.0270)</td>
<td>5.6598 (−11.59%)</td>
<td>5.6746 (−11.36%)</td>
</tr>
<tr>
<td>( T = 2, \kappa = 2 )</td>
<td>3.6799 (0.0211)</td>
<td>3.1002 (−15.75%)</td>
<td>3.1900 (−13.31%)</td>
</tr>
<tr>
<td>( T = 2, \kappa = 3 )</td>
<td>1.9314 (0.0154)</td>
<td>1.5321 (−20.68%)</td>
<td>1.6374 (−15.22%)</td>
</tr>
<tr>
<td>( T = 2, \kappa = 3 )</td>
<td>0.9298 (0.0106)</td>
<td>0.6857 (−26.26%)</td>
<td>0.7694 (−17.25%)</td>
</tr>
</tbody>
</table>

TABLE 2: Timer option prices under slow mean-reverting volatility.

As mentioned above, as was done in [25] for European call options, more accurate higher order approximations may be derived for timer options; however these require more detailed knowledge of the parameters and form of the volatility process \( Y \), and thus rely on the solution of a difficult statistical problem in practice.
5 Conclusion  Timer options are options with a random maturity time that depends on the realized level of the volatility of the underlying asset. In this paper, we examined the pricing of timer options under a stochastic volatility model, with the volatility satisfying an ergodic diffusion process running on a fast time scale. Based on singular perturbation techniques that have been employed successfully in other option pricing problems, an asymptotic expansion for the timer option price was derived. The expansion is straightforward to implement, using only the Black-Scholes option pricing function and its derivatives, and can be easily calibrated to the implied volatility surface. Numerical experiments were also presented, illustrating the accuracy of the method when pricing options of varying moneyness and (expected) time to expiration.

REFERENCES


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