LIMIT CYCLES OF DIFFERENTIAL SYSTEMS
VIA THE AVERAGING METHODS

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ABSTRACT. In this article we first recall the averaging methods which are useful in studying limit cycle bifurcations from simple differential systems under small perturbations. We then review some recent results on limit cycle bifurcations obtained by using the averaging methods. Finally we present a new result by applying the averaging method to study the Hopf bifurcation of some smooth differential systems in $\mathbb{R}^n$, we show that the function in the number of limit cycles bifurcated from one singularity of the systems can be an exponential function in the dimension of the system with exponent $n - 2$, and that the function can also be a power function in the degree of the system with its nonlinear part a polynomial. As we know this last phenomena is first found using the averaging method, which is an extension of Theorem 1 of [29].

1 Introduction and the averaging methods

Studying the existence of periodic orbits of differential systems via the averaging methods has a long history (see for instance Marsden and McCracken [33], Chow and Hale [6], Sanders and Verhulst [35], Luo et al. [31], Verhulst [37], Buică and Llibre [2], Buică, Françoise and Llibre [1] and the references therein). The problem on the number, distribution and stability of limit cycles for a differential system is difficult (see for instance Ye et al. [38], Zhang et al. [39], Li [20], Christopher and Li [7] and the references therein). We will show that the averaging methods are useful tools for investigating the number of limit cycles for some differential systems. Add also this method can be used to obtain the shape, stability and the approximate expression of limit cycles.

In the rest of this section we will recall the basic theory on the averaging methods. Of course, the averaging methods have different presentations depending on the problems which will be studied. Here we

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restrict to those for which the number, shape and stability of limit cycles can be obtained. The first one is to present the first-order averaging method as it was obtained by Buica and Llibre in [2], which does not need differentiability of the vector field. The conditions imposing the existence of a simple isolated zero of the averaged function are given in terms of the Brouwer degree.

**Theorem 1.** (First order averaging method) Consider the differential system

\begin{equation}
\dot{x}(t) = \varepsilon f(t, x) + \varepsilon^2 g(t, x, \varepsilon),
\end{equation}

where $f : \mathbb{R} \times D \to \mathbb{R}^n$, $g : \mathbb{R} \times D \times (-\varepsilon f, \varepsilon f) \to \mathbb{R}^n$ are continuous functions, $T$-periodic in the first variable, and $D$ is an open subset of $\mathbb{R}^n$. Define $f^0 : D \to \mathbb{R}^n$ by

\begin{equation}
f^0(z) = \frac{1}{T} \int_0^T f(s, z)ds,
\end{equation}

and assume that

(i) $f$ and $g$ are locally Lipschitz with respect to $x$;
(ii) there exists $b \in D$ such that $f^0(b) = 0$ and a neighborhood $V \subset D$ of $b$ in which $f^0(z) \neq 0$ for all $z \in V \setminus \{b\}$ and $d_B(f^0, V, b) \neq 0$ (where $d_B(f^0, V, b)$ denotes the Brouwer degree of $f^0$ in the neighborhood $V$ of $b$).

Then, for $|\varepsilon| > 0$ sufficiently small, there exists an isolated $T$-periodic solution $(\cdot; \varepsilon)$ of system (1) such that $\phi(b, 0) = b$.

Buică et al. [3] provided the asymptotic stability of the limit cycles obtained from averaging theory with $f$ having $C^1$ differentiability.

**Theorem 2.** Assume that the functions $f$ and $g$ of (1) are $C^1$ and Lipschitz respectively in a neighborhood of the limit cycle $\phi(\cdot, \varepsilon)$ given in Theorem 1 by the simple zero $b$ of $f^0$, then for $|\varepsilon| > 0$ sufficiently small if the eigenvalues of the Jacobian matrix of $f^0$ at $b$ all have negative (respectively positive) real part, then the limit cycle $\phi(\cdot, \varepsilon)$ is asymptotically stable (respectively unstable).

Furthermore, if the functions $f$ and $g$ of (1) have better regularity on their smoothness then we can get more information on the limit cycle $\phi(\cdot, \varepsilon)$ which are obtained in Theorem 1.
Theorem 3. Assume that the functions \( f \) and \( g \) of (1) are \( C^2 \) and \( C^1 \), respectively. If there exists an \( b \in D \) such that \( f^0(b) = 0 \) and \( \det(D_z f^0(b)) \neq 0 \), where \( D_z f^0 \) denotes the Jacobian matrix of \( f^0(z) \) with respect to \( z \), then for \( |\varepsilon| > 0 \) sufficiently small the following hold.

(i) there exists a \( T \)-periodic limit cycle \( \phi(\cdot, \varepsilon) \) of (1) such that \( \phi(\cdot, \varepsilon) \to b \) as \( \varepsilon \to 0 \);

(ii) the stability of the limit cycle \( \phi(\cdot, \varepsilon) \) given in (i) is the same as that of the singularity \( b \) of the averaged system \( \dot{z} = \varepsilon f^0(z) \). In fact the singularity \( b \) has the stability behavior of the Poincaré map associated to the limit cycle \( \phi(\cdot, \varepsilon) \).

A proof of this result can be found in [37] or in [13]. This shows that using the averaging methods we obtain not only the existence of limit cycles but also their stability and approximate expressions.

We now recall the second and third order averaging methods (see, e.g., [2]).

Theorem 4 (Second order averaging method). Consider the differential system

\[
\dot{x}(t) = \varepsilon f_1(t, x) + \varepsilon^2 f_2(t, x) + \varepsilon^3 g(t, x, \varepsilon),
\]

where \( f_1, f_2 : \mathbb{R} \times D \to \mathbb{R}^n \), \( g : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n \) are continuous functions, \( T \)-periodic in the first variable, and \( D \) is an open subset of \( \mathbb{R}^n \). Assume that

(i) \( f_1(t, \cdot) \in C^1(D) \) for all \( t \in \mathbb{R} \), \( f_2, g \) and \( D_x f_1 \) are locally Lipschitz with respect to \( x \), and \( g \) is differentiable with respect to \( \varepsilon \). Define \( f_1^0, f_2^0 : D \to \mathbb{R}^n \) by

\[
\begin{align*}
   f_1^0(z) &= \frac{1}{T} \int_0^T f_1(s, z) ds, \\
   f_2^0(z) &= \frac{1}{T} \int_0^T \left( D_x f_1(s, z) \cdot \int_0^s f_1(t, z) dt + f_2(s, z) \right) ds;
\end{align*}
\]

(ii) for \( V \subset D \) an open and bounded set and for \( \varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\} \), there exists \( a_\varepsilon \in V \) such that \( f_1^0(a_\varepsilon) + \varepsilon f_2^0(a_\varepsilon) = 0 \) and \( d_B(f_1^0 + \varepsilon f_2^0, V, 0) \neq 0 \).

Then, for \( |\varepsilon| > 0 \) sufficiently small, there exists a \( T \)-periodic solution \( \phi(\cdot, \varepsilon) \) of system (3).
Theorem 5 (Third order averaging method). Consider the differential system

\[ \dot{x}(t) = \varepsilon f_1(t, x) + \varepsilon^2 f_2(t, x) + \varepsilon^3 f_3(t, x) + \varepsilon^4 g(t, x, \varepsilon), \]

where \( f_1, f_2, f_3 : \mathbb{R} \times D \to \mathbb{R}^n, g : \mathbb{R} \times D \times (-\varepsilon, \varepsilon) \to \mathbb{R}^n \) are continuous functions, \( T \)-periodic in the first variable, and \( D \) is an open subset of \( \mathbb{R}^n \). Assume that

(i) \( f_1(t, \cdot) \in C^2(D), f_2(t, \cdot) \in C^4(D) \) for all \( t \in \mathbb{R}, f_3, g, D_{xx}f_1, D_{xx}f_3 \)

are locally Lipschitz with respect to \( x \), and \( g \) is twice differentiable with respect to \( \varepsilon \), where \( D_{xx}f_1 \) is the Hessian matrix of \( f_1 \) with respect to \( x \). Define \( f_0^1, f_0^2, f_0^3 : D \to \mathbb{R}^n \) with \( f_0^1 \) and \( f_0^2 \) as in (4) and (5), respectively, and \( f_0^3 \) as

\[ f_0^3(z) = \frac{1}{T} \int_0^T \left( \frac{1}{2} D_{zz}f_1(s, z)(y_1(s, z))^2 + D_{z}f_1(s, z)y_2(s, z) \right. \]

\[ + D_{z}f_2(s, z)y_1(s, z) + f_3(s, z) \) \] ds,

where

\[ y_1(s, z) = \int_0^s f_1(t, z) dt, \]

\[ y_2(s, z) = \int_0^s \left( D_{z}f_1(t, z) \int_0^t f_1(r, z)dr + f_2(t, z) \right) dt; \]

(ii) for \( V \subset D \) an open and bounded set and for \( \varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\} \),

there exists \( a_\varepsilon \in V \) such that \( f_0^1(a_\varepsilon) + \varepsilon f_0^2(a_\varepsilon) + \varepsilon^2 f_0^3(a_\varepsilon) = 0 \) and

\( d_B(f_0^1, \varepsilon f_0^2 + \varepsilon^2 f_0^3, V, 0) \neq 0. \)

Then, for \( |\varepsilon| > 0 \) sufficiently small, there exists a \( T \)-periodic solution \( \phi(\cdot, \varepsilon) \) of system (6).

For the averaging methods of order higher than 3 we can get similar results than the previous ones, but the expression is more complicated, and will not be presented here. The above averaging methods ask the terms in the right hand side of equations are all \( o(1) \) in terms of the small parameter \( \varepsilon \). If it is not the case we will have the following developed averaging methods to deal with them.

We consider the following differential system

\[ \dot{x}(t) = f_0(t, x) + \varepsilon f_1(t, x) + \varepsilon^2 g(t, x, \varepsilon), \]
where $\varepsilon$ is a small parameter, $f_0, f_1 : \mathbb{R} \times D \rightarrow \mathbb{R}^n$ and $g : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \rightarrow \mathbb{R}^n$ are $C^2$ functions, $T$-periodic in the first variable, and $D$ is an open subset of $\mathbb{R}^n$. Assume that the unperturbed system of (7) 

$$(8) \quad \dot{x}(t) = f_0(t, x),$$

has a solution, denoted by $x_0(t, z)$, satisfying $x_0(0, z) = z$. Write the linearization of (8) along the solution $x_0(t, z)$ in 

$$(9) \quad \dot{y} = P(t, z)y$$

with 

$$P(t, z) = D_x f_0(t, x_0(t, z)).$$

Let $Y(\cdot, z)$ be some fundamental matrix solution of (9).

**Theorem 6.** Let $\beta_0 : V \rightarrow \mathbb{R}^{n-k}$ be a $C^2$ function, where $V \subset \mathbb{R}^k$ is open and bounded. Assume that

(i) $Z = \{ z_\alpha = (\alpha, \beta_0(\alpha)), \alpha \in \overline{V} \} \subset D$ and that for each $z_\alpha \in Z$, the unique solution $x_\alpha$ of (8) with $x(0) = z_\alpha$ is $T$-periodic;

(ii) for each $z_\alpha \in Z$, there exists a fundamental matrix solution of (9), denoted by $Y_\alpha(t) = Y(t, z_\alpha)$, such that the matrix $Y_\alpha^{-1}(0) - Y_\alpha^{-1}(T)$ has in the upper right corner the null $k \times (n - k)$ matrix, while in the lower right corner has the $(n - k) \times (n - k)$ matrix $\Delta_\alpha$ with $\det(\Delta_\alpha) \neq 0$.

We consider the function $F_1 : V \rightarrow \mathbb{R}^k$ given by 

$$(10) \quad F_1(\alpha) = \mathcal{P} \int_0^T Y_\alpha^{-1}(t) f_1(t, x_\alpha(t)) \, dt,$$

where $\mathcal{P}$ is the projection from $\mathbb{R}^k \times \mathbb{R}^{n-k}$ to its first $k$ coordinates $\mathbb{R}^k$.

If there exists a $b \in V$ satisfying $F_1(b) = 0$ and $\det(D_\alpha F_1(b)) \neq 0$, then there exists a $T$-periodic solution $\phi(\cdot, \varepsilon)$ of system (7) such that $\phi(0, \varepsilon) \rightarrow z_\alpha$ as $\varepsilon \rightarrow 0$.

This result was presented by Malkin (1956) and Roseau (1966) (see, e.g., Francoise [12]). For a new and short proof, see [1]. Here we use the presentation of [1]. This last theorem has some very nice corollaries (see also [1]). The first one is on perturbation of isochronous systems.
Corollary 7 (Perturbations of isochronous systems). Assume that there exists an open subset $V$ with $V \subset \mathbb{D}$ such that for each $z \in V$, the solutions $x(\cdot, z)$ of (8) is $T$-periodic. Let $F_1 : V \to \mathbb{R}^n$ be given by

$$F_1(z) = \int_0^T Y^{-1}(t, z)F_1(t, x(t, z)) \, dt.$$  

If there exists $b \in V$ such that $F_1(b) = 0$ and $\det(D_z F_1(b)) \neq 0$, then there exists a $T$-periodic solution $\phi(\cdot, \varepsilon)$ of system (7) satisfying $\phi(0, \varepsilon) \to b$ as $\varepsilon \to 0$.

The second one is on the perturbation of linear systems.

Corollary 8 (Perturbation of linear systems). Consider system (7) with $f_0(t, x) = P(t)x + q(t)$. Assume that system

$$\dot{y} = P(t)y,$$

has $k$ $T$-periodic linearly independent solutions. Denote by $Y(t)$ some fundamental matrix solution of this linear system, and assume that

(i) $\mathcal{P} \int_0^T Y^{-1}(s)q(s) \, ds = 0$, where $\mathcal{P}$ is the projection from $\mathbb{R}^k \times \mathbb{R}^{n-k}$ to $\mathbb{R}^k$;

(ii) $\det(\Delta) \neq 0$, where $\Delta$ is the $(n-k) \times (n-k)$ matrix from the lower right corner of the matrix $Y^{-1}(0) - Y^{-1}(T)$.

Then the following statements hold.

(a) there exists $\beta_0 : \mathbb{R}^k \to \mathbb{R}^{n-k}$ such that, for all $\alpha \in \mathbb{R}^k$, $z_\alpha = (\alpha, \beta_0(\alpha))$ satisfies

$$\left(Y^{-1}(T) - Y^{-1}(0)\right) z = \int_0^T Y^{-1}(s)q(s) \, ds$$

and the unique solution $x_\alpha(t)$ of $\dot{x} = P(t)x + q(t)$ with $x(0) = z_\alpha$ is $T$-periodic.

(b) Let $F_1 : \mathbb{R}^k \to \mathbb{R}^k$ be the function defined by

$$F_1(\alpha) = \mathcal{P} \int_0^T Y^{-1}(t)f_1(t, x_\alpha(t)) \, dt.$$  

If there exists $b \in V$ such that $F_1(b) = 0$ and $\det(D_\alpha F_1(b)) \neq 0$, then there exists a $T$-periodic solution $\phi(\cdot, \varepsilon)$ of system (7) such that $\phi(0, \varepsilon) \to z_b$ as $\varepsilon \to 0$. 
Based on perturbation of linear periodic differential systems via averaging method, there had been also Han’s work [14], who established conditions for the existence of a unique periodic solution and gave estimates for its supremum and for the values of the small parameter for which this solution exists. We will not present it here.

Finally we consider a special case of systems (7) with

\[ f_0(t, x) = (f_1^1(t, u), f_2^1(t, u, v)), \]
\[ f_1(t, x) = (f_1^1(t, u, v), f_2^1(t, u, v)) \]
\[ g(t, x, \varepsilon) = (g_1^1(t, u, v, \varepsilon), g_2^1(t, u, v, \varepsilon)), \]
where $f_0^1, f_1^1, g_1^1, g_2^1 \in \mathbb{R}^k$ and $f_0^2, f_2^1, g_2 \in \mathbb{R}^{n-k}$.

**Corollary 9** (Perturbation of a special case). Assume that there exists an open subset $V$ with $V \subset PD$ such that for each $\alpha \in V$, the unique solution $u_\alpha(t)$ of the system $\dot{u} = f_0^1(t, u)$ satisfying $u(0) = \alpha$ is $T$-periodic, and the system $\dot{v} = f_2^1(t, u_\alpha(t), v)$ has a unique $T$-periodic solution, denoted $v_\alpha(t)$.

Let $U_\alpha(t)$ and $V_\alpha(t)$ be respectively fundamental matrix solutions of the systems $\dot{u} = D_u f_0^1(t, u_\alpha(t))u$ and $\dot{v} = D_v f_2^1(t, u_\alpha(t), v_\alpha(t))$. Define

\[ F_1(\alpha) = \int_0^T U_\alpha^{-1}(t)f_1(t, u_\alpha(t), v_\alpha(t)) \, dt. \]

Assume that $\Delta_\alpha = V_\alpha^{-1}(0) - V_\alpha^{-1}(T)$ has $\det(\Delta_\alpha) \neq 0$, and that there exists $b \in V$ such that $F_1(\alpha) = 0$ and $\det(D_\alpha F_1(b)) \neq 0$. Then there exists a $T$-periodic solution $\phi(\cdot, \varepsilon)$ of system (7) such that $\phi(0, \varepsilon) \rightarrow (b, v_\alpha(0))$ as $\varepsilon \rightarrow 0$.

There are also some other forms of averaging method for studying the existence of invariant tori, subharmonic bifurcations and so on, which are beyond our main concern of this paper. We refer the interesting readers to [16, 40] and the reference therein.

At the end of this section we present the Bezout’s theorem, which gives the maximum number of zeros of a system of polynomial functions (see, e.g., [36]). We will use it in Section 3.

**Theorem 10** (Bezout’s theorem). Let $P_i$ be polynomials in the variables $(x_1, \ldots, x_n) \in \mathbb{R}^n$ of degree $d_i$ for $i = 1, \ldots, n$. Consider the following polynomial system

\[ P_i(x_1, \ldots, x_n) = 0, \quad i = 1, \ldots, n. \]

If the number of solutions of this system is finite, then it is bounded by $d_1 \cdots d_n$. 

2 Review: recent results on limit cycle bifurcations via averaging method

In this section we review some recent results on limit cycle bifurcations using the averaging methods.

2.1 Application of the averaging methods

We first recall some results on the bifurcation of limit cycles of differential systems via averaging methods. Consider a smooth differential system of the form

\[ \dot{x} = Ax + F(x, \varepsilon), \]

where \( F(x, 0) = O(|x|^2), x \in \mathbb{R}^n, n \geq 3 \) and

\[ A = \text{diag} \left( \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}, 0 \right), \quad b \neq 0. \]

For \( |\varepsilon| \neq 0 \) small the system may have a limit cycle near the origin. Furthermore, if \( F(x, 0) = 0 \), then the system when \( \varepsilon \neq 0 \) may have a limit cycle near a closed curve of the form

\[ L_h : x_1^2 + x_2^2 = h (> 0), \quad x_j = 0 \text{ for } j = 3, \ldots, n. \]

For the limit cycle bifurcation near either the origin or \( L_h \) a general theory together with a vector bifurcation function via Liapunov-Schmidt method was established in Theorems 1–4 and Remark 2 by Han in [15], see also Theorems 6.9–6.12 and Remark 4 of [18] by Han and Zhu. In the case of \( n = 3 \) by introducing a suitable rescaling of variables transform the original system to the normal form system and then using the averaging method the condition for the existence of one or two limit cycles was given in [18] and [17]. For \( n \geq 3 \) following the same idea one can also introduce the averaging method to study the bifurcation of limit cycles. Here we only generally mentioned the results on the existence of limit cycles via averaging methods, but did not present them carefully (for details we refer the readers to the original papers). Because in this paper we are mainly interested in the number of limit cycles bifurcated using the averaging methods.

We now study the bifurcation of limit cycles from a class of \( C^{m+1} \) smooth differential systems in \( \mathbb{R}^n \) with \( n \geq 3 \) and \( m \geq 2 \). Assume that these systems have a singularity at the origin with the linear part having eigenvalues \( \varepsilon a \pm bi \) and \( \varepsilon c_k \) for \( k = 3, \ldots, n \), where \( \varepsilon > 0 \) is a
small parameter. Such systems can be written in

\[
\begin{align*}
\dot{x} &= \varepsilon ax - by + \sum_{i_1 + \ldots + i_n = m} a_{i_1, \ldots, i_n} x^{i_1} y^{i_2} z_3^{i_3} \ldots z_n^{i_n} + A, \\
\dot{y} &= bx + \varepsilon ay + \sum_{i_1 + \ldots + i_n = m} b_{i_1, \ldots, i_n} x^{i_1} y^{i_2} z_3^{i_3} \ldots z_n^{i_n} + B, \\
\dot{z}_k &= \varepsilon c_k z_k + \sum_{i_1 + \ldots + i_n = m} c_{i_1, \ldots, i_n}^{(k)} x^{i_1} y^{i_2} z_3^{i_3} \ldots z_n^{i_n} + L_k, \\
k &= 3, \ldots, n,
\end{align*}
\]

where \(a_{i_1, \ldots, i_n}, b_{i_1, \ldots, i_n}, c_{i_1, \ldots, i_n}^{(k)}, a, b\) and \(c_k\) are real parameters and \(b \neq 0\).
The functions \(A, B\) and \(L_k\) are the Lagrange expressions of the error function of \((m + 1)\)th order in the expansion of the functions of the system in Taylor series.

The first result was given in Theorem 1 of [29] by Llibre and Zhang for system (11) with \(m = 2\), where the limit cycles came from the Hopf bifurcation via the first order averaging method.

**Theorem 11.** For \(m = 2\) there exist \(C^3\) systems (11) for which \(l \in \{0, 1, \ldots, 2^{n-3}\}\) limit cycles bifurcate from the origin at \(\varepsilon = 0\), i.e. for \(\varepsilon\) sufficiently small the system has exactly \(l\) limit cycles in a neighborhood of the origin and these limit cycles tend to the origin when \(\varepsilon \to 0\).

In the next section we will generalize this result to all natural number \(m \geq 3\). The following result is an immediate consequence of the proof of Theorem 11.

**Corollary 12.** There exist quadratic polynomial differential systems (11) (i.e. with \(m = 2\) and \(A = B = L_k = 0\)) for which \(l \in \{0, 1, \ldots, 2^{n-3}\}\) limit cycles bifurcate from the origin at \(\varepsilon = 0\), i.e. for \(\varepsilon\) sufficiently small the system has exactly \(l\) limit cycles in a neighborhood of the origin and these limit cycles tend to the origin when \(\varepsilon \to 0\).

For lower dimensional systems (11) we can get more detail information on the limit cycles bifurcated from the Hopf zero. For three dimensional system of form (11) restricted to quadratic polynomial differential systems, Buzzi, Llibre and de Silva [4] obtained some sufficient conditions for the existence or not of one limit cycle and its kind of stability. We now recall only the results in the four dimensional case from Theorem 3 of Llibre and Zhang [29] because if we set the fourth equation vanishes we have a three dimensional differential system, and then the results of
Write system (11) as a four dimensional one in the form

\[
\begin{align*}
\dot{x} &= \varepsilon ax - by + \sum_{i+j+k+l=2} a_{ijkl}x^i y^j z^k w^l + A, \\
\dot{y} &= bx + \varepsilon ay + \sum_{i+j+k+l=2} b_{ijkl}x^i y^j z^k w^l + B, \\
\dot{z} &= \varepsilon cz + \sum_{i+j+k+l=2} c_{ijkl}x^i y^j z^k w^l + C, \\
\dot{w} &= \varepsilon dw + \sum_{i+j+k+l=2} d_{ijkl}x^i y^j z^k w^l + D,
\end{align*}
\]  

(12)

where \(a_{ijkl}, b_{ijkl}, c_{ijkl}, d_{ijkl}, a, b, c\) and \(d\) are real parameters, \(ab \neq 0\), and \(A, B, C\) and \(D\) are the higher order terms. We assume without loss that \(b > 0\). Set

\[
\begin{align*}
A &= KG_1 - LG_2, \\
B &= 2a(N_2 F_1 - 2N_3 F_2) + cF_1^2 G_2 + dF_1 F_2 G_1, \\
C &= 2a(2aN_3 - dF_1 G_1), \\
D &= 2a(M_1 F_1^2 + 2M_2 F_1 F_2 + M_3 F_2^2) + cKF_1 + dL F_2, \\
E &= -a(2aN_3 F_1 + 2aM_3 F_2 + dL), \\
\Lambda &= dF_2 G_1 - 2aN_2, \\
\Gamma &= -8a^2 c_{0002} N_3^2 + 4ac_0 F_2 N_3^2 - 2ac_{0011} N_3 A - c_{0002}(\Lambda^2 + \Delta)/2, \\
\Phi &= 2c_{0011} C F_1 (2aB + CF_2) - 2cBCF_1^2 \\
&\quad - 2c_{0020} C^2 F_1^2 - 2c_{0002}(2aB + CF_2)^2, \\
\Psi &= 4a ((cF_2 - 2ac_{0020}) \Lambda^2 + 2ac_{0011} F_2 M - 2ac_{0002} F_2^2 f^2),
\end{align*}
\]

(13)

where

\[
\begin{align*}
F_1 &= a_{001} + b_{010}, & F_2 &= a_{100} + b_{010}, & G_1 &= c_{020} + c_{200}, \\
G_2 &= d_{020} + d_{200}, & K &= d_{002} F_1^2 - d_{001} F_1 F_2 + d_{002} F_2^2, \\
L &= c_{002} F_1^2 - c_{001} F_1 F_2 + c_{002} F_2^2, \\
M_1 &= c_{002} d_{001} - c_{001} d_{002}, & M_2 &= c_{002} d_{002} - c_{002} d_{002}.
\end{align*}
\]
\[ \begin{align*}
M_3 &= c_{0011}d_{0002} - c_{0002}d_{0011}, \\
N_1 &= d_{0020}G_1 - c_{0020}G_2,
\end{align*} \]
\[ \begin{align*}
N_2 &= d_{0011}G_1 - c_{0011}G_2, \\
N_3 &= d_{0002}G_1 - c_{0002}G_2.
\end{align*} \]

Assume that
\[ \begin{align*}
F_1^2 + F_2^2 &\neq 0, \\
G_1^2 + G_2^2 &\neq 0.
\end{align*} \]

We have the following results which come from the Hopf bifurcation of system (11) at the origin.

**Theorem 13.** For a C³ system (12) with \( G_1 \neq 0 \) the following statements hold.

(a) (a.1) For \( F_1 \neq 0 \) and \( A \neq 0 \), if \( B^2 - 4AC > 0 \) and \((4AE - DB + D\sqrt{B^2 - 4AC})F_1 > 0 \) (resp. \((4AE - DB - D\sqrt{B^2 - 4AC})F_1 > 0 \)), system (12) has a limit cycle \( \Gamma_{1e} \) (resp. \( \Gamma_{2e} \)) tending to a singular point as \( \varepsilon \downarrow 0 \). Moreover for suitable choice of the parameters system (12) can have the two limit cycles \( \Gamma_{1e} \) and \( \Gamma_{2e} \). In these last case both limit cycles tend to different singular points when \( \varepsilon \downarrow 0 \).

(a.2) For \( F_1 \neq 0 \) and \( A = 0 \), if \( B \neq 0 \) and \( \Phi G_1 > 0 \), system (12) has a limit cycle \( \Gamma_{1e} \) tending to the origin as \( \varepsilon \downarrow 0 \).

(a.3) For \( F_1 = 0 \), \( F_2 \neq 0 \) and \( N_2 \neq 0 \), if \( \Delta > 0 \) and \((\Gamma - (2c_{0011}N_2 + c_{0002}A)\sqrt{\Delta})G_1 > 0 \) (resp. \((\Gamma + (2c_{0011}N_2 + c_{0002}A)\sqrt{\Delta})G_1 > 0 \)), system (12) has a limit cycle \( \Gamma_{3e} \) (resp. \( \Gamma_{4e} \)) tending to the origin as \( \varepsilon \downarrow 0 \). Moreover for suitable choice of the parameters system (12) can have the two limit cycles \( \Gamma_{3e} \) and \( \Gamma_{4e} \). In these last case both limit cycles tend to different singular points when \( \varepsilon \downarrow 0 \).

(a.4) For \( F_1 = 0 \), \( F_2 = 0 \) and \( N_2 = 0 \), if \( \Lambda \neq 0 \) and \( \Psi G_1 > 0 \), system (12) has a limit cycle \( \Gamma_{5e} \) tending to the origin as \( \varepsilon \downarrow 0 \).

(b) For \( \varepsilon > 0 \) sufficiently small the limit cycle \( \Gamma_{1e}, \Gamma_{2e}, \Gamma_{1e}, \Gamma_{3e}, \Gamma_{4e} \) or \( \Gamma_{5e} \) of statement (a) is given respectively by the graph:

\[ \begin{align*}
r(\theta) &= \varepsilon \sqrt{\frac{4AE - DB + D\sqrt{B^2 - 4AC}}{(F_1A)^2}} + O(\varepsilon^2), \\
z(\theta) &= \varepsilon \left( B - \sqrt{B^2 - 4AC} \right)/(2A) + O(\varepsilon^2), \\
w(\theta) &= -\varepsilon \left( 4aA + F_2(B - \sqrt{B^2 - 4AC}) \right)/(2AF_1) + O(\varepsilon^2);\end{align*} \]
\[
    r(\theta) = \varepsilon \sqrt{\left( 4AE - DB - D\sqrt{B^2 - 4AC} \right) / \left( F_1 A^2 \right)} + O(\varepsilon^2),
\]
\[
    z(\theta) = \varepsilon \left( B + \sqrt{B^2 - 4AC} \right) / \left( 2A \right) + O(\varepsilon^2),
\]
\[
    w(\theta) = -\varepsilon \left( 4aA + F_2 \left( B + \sqrt{B^2 - 4AC} \right) \right) / \left( 2AF_1 \right) + O(\varepsilon^2);
\]
\[
    r(\theta) = \varepsilon \sqrt{\Phi / \left( G_1 B^2 F_1^2 \right)} + O(\varepsilon^2),
\]
\[
    z(\theta) = \varepsilon C/B + O(\varepsilon^2),
\]
\[
    w(\theta) = -\varepsilon \left( 2aB + CF_2 \right) / \left( BF_1 \right) + O(\varepsilon^2);
\]
\[
    r(\theta) = \varepsilon \sqrt{\left( \Gamma - (2ac_{0011}N_2 + c_{0002}\lambda)\sqrt{\Delta} \right) / \left( G_1 F_2^2 N_2^2 \right)} + O(\varepsilon^2),
\]
\[
    z(\theta) = -\varepsilon 2a/F_2 + O(\varepsilon^2),
\]
\[
    w(\theta) = -\varepsilon \left( \Lambda + \sqrt{\Delta} \right) / \left( 2N_2 F_2 \right) + O(\varepsilon^2);
\]
\[
    r(\theta) = \varepsilon \sqrt{\left( \Gamma + (2ac_{0011}N_2 + c_{0002}\lambda)\sqrt{\Delta} \right) / \left( G_1 F_2^2 N_2^2 \right)} + O(\varepsilon^2),
\]
\[
    z(\theta) = -\varepsilon 2a/F_2 + O(\varepsilon^2),
\]
\[
    w(\theta) = -\varepsilon \left( \Lambda - \sqrt{\Delta} \right) / \left( 2N_2 F_2 \right) + O(\varepsilon^2);
\]

or
\[
    r(\theta) = \varepsilon \sqrt{\Psi / \left( G_1 F_2^2 \Lambda \right)} + O(\varepsilon^2),
\]
\[
    z(\theta) = -\varepsilon 2a/F_2 + O(\varepsilon^2),
\]
\[
    w(\theta) = -\varepsilon 2af / \left( \Lambda + O(\varepsilon^2) \right),
\]

where \( \theta \in \mathbb{S}^1 \) and the coordinates \((r, z, w)\) and \( \theta \) come from \((x, y, z, w)\) via the cylindrical coordinate transformation.

(c) For the results of this section we need that system (12) be at least \( C^4 \).

(c.1) For \( F_1 > 0 \) and \( \Lambda \neq 0 \), the limit cycle \( \Gamma_{1\varepsilon} \) (resp. \( \Gamma_{2\varepsilon} \)) has at least one-dimensional stable (resp. unstable) manifold,
and consequently the limit cycle is not a global repeller (resp. attractor). For \( F_1 < 0 \) and \( A \neq 0 \) the one-dimensional invariant manifold of the limit cycle has converse stability than for \( F_1 > 0 \).

(c.2) For \( F_1 > 0 \), \( A = 0 \) and \( B \neq 0 \), the limit cycle \( \Gamma_{1e} \) has at least one-dimensional stable (resp. unstable) manifold provided that \( B > 0 \) (resp. \( B < 0 \)). For \( F_1 < 0 \) the one-dimensional invariant manifold of the limit cycle has a different stability than for \( F_1 > 0 \).

(c.3) For \( F_1 = 0 \), \( F_2 > 0 \) and \( N_2 \neq 0 \) the limit cycle \( \Gamma_{3e} \) (resp. \( \Gamma_{4e} \)) has at least one-dimensional stable (resp. unstable) manifold. For \( F_1 = 0 \), \( F_2 < 0 \) and \( N_2 \neq 0 \), the one-dimensional invariant manifold of the limit cycles has a different stability than for \( F_2 > 0 \).

(c.4) For \( F_1 = 0 \), \( F_2 \neq 0 \), \( N_2 = 0 \) and \( \Lambda \neq 0 \), the limit cycle \( \Gamma_{3e} \) has at least one-dimensional stable (resp. unstable) manifold provided that \( \Delta > 0 \) (resp. \( \Delta < 0 \)).

We have seen from Theorem 13 that the averaging methods provide not only the existence of limit cycles but also their stability and asymptotic expressions. Theorem 13 has some applications in forth order differential equations

\[
\frac{d^4x}{dt^4} + p \frac{d^3x}{dt^3} + q \frac{d^2x}{dt^2} + k \frac{dx}{dt} + lx = f \left( x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \frac{d^3x}{dt^3} \right),
\]

and also in the simplified system of immune response without influence of damaged organ and time delay given in [32] by Marchuk

\[
\begin{align*}
\frac{dX}{dt} &= (\beta - \gamma Z)X, \\
\frac{dY}{dt} &= \alpha XZ - \mu_1 (Y - \delta), \\
\frac{dZ}{dt} &= \rho Y - (\mu_2 + \eta \gamma X)Z, \\
\frac{dW}{dt} &= \sigma X - \mu_3 W.
\end{align*}
\]

**Theorem 14 ([29]).** There is an open set in the parameter spaces for which system (14) has at least one limit cycle.

Until now we consider only the application of averaging methods to higher dimensional systems. We now turn to two dimensional systems, especially the Liénard differential systems, which were presented by Liénard in 1928, and are of the forms

\[\ddot{x} + f(x)\dot{x} + g(x) = 0.\]
This equation looks simpler in form, but in fact is very complicated in dynamics. We consider the cases that \( f \) and \( g \) are polynomials of degrees \( n \) and \( m \), respectively. Denote by \( H(m; n) \) the Hilbert number, i.e., the maximum number of limit cycles that the Liénard differential equation with \( m \) and \( n \) prescribed can have.

On the Hilbert number of Liénard systems, it is well known that \( H(1; 1) = 0 \) and \( H(1; 2) = 1 \) by Lins, de Melo and Pugh [21] in 1977; \( H(2; 1) = 1 \) by Coppel [8] in 1988; \( H(3; 1) = 1 \) by Dumortier and Rousseau [11] in 1990 and Dumortier and Li [9] in 1996; and \( H(2; 2) = 1 \) first by Li [19] in 1986 and then reproved by Dumortier and Li [10] in 1997 and by Luo et al. [31, Theorem 3.3.7] in 1997. For other cases the Hilbert number is unknown. Llibre et al. [24] applied the first three order averaging methods to the Liénard systems, and obtained some nice results.

**Theorem 15** ([24]). For the polynomial Liénard differential systems

\[
\dot{x} = y, \quad \dot{y} = -x - \epsilon (f(x)y + g(x)),
\]

if \( f(x) \) and \( g(x) \) are polynomials of degree \( n \) and \( m \) respectively, then for \( |\epsilon| > 0 \) sufficiently small, the maximum number of limit cycles of the polynomial Liénard differential systems (15) bifurcating from the periodic orbits of the linear center \( \dot{x} = y, \dot{y} = -x \), using the first order averaging theory, is \( [n/2] \), where \( [\cdot] \) denotes the integer part function.

**Theorem 16** ([24]). For the polynomial Liénard differential systems

\[
\dot{x} = y, \quad \dot{y} = -x - \epsilon (f_1(x)y + g_1(x)) - \epsilon^2 (f_2(x)y + g_2(x)),
\]

with \( f_i \) and \( g_i \) polynomials of degree \( n \) and \( m \) respectively, if \( |\epsilon| > 0 \) is sufficiently small, then the maximum number of limit cycles of the polynomial Liénard differential systems (16) bifurcating from the periodic orbits of the linear center \( \dot{x} = y, \dot{y} = -x \), using the second order averaging theory, is \( \max\{\lfloor (n - 1)/2 \rfloor + \lfloor m/2 \rfloor, \lfloor n/2 \rfloor \} \).

**Theorem 17** ([24]). For the polynomial Liénard differential systems

\[
\dot{x} = y,
\]

\[
\dot{y} = -x - \epsilon (f_1(x)y + g_1(x)) - \epsilon^2 (f_2(x)y + g_2(x))
\]

\[
- \epsilon^3 (f_3(x)y + g_3(x)),
\]

with \( f_i \) and \( g_i \) polynomials of degree \( n \) and \( m \) respectively, if \( |\epsilon| > 0 \) is sufficiently small, then the maximum number of limit cycles of the
polynomial Liénard differential systems (17) bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, using the third order averaging theory, is $\lfloor (n + m - 1)/2 \rfloor$.

The works on the number of limit cycles for Liénard differential systems using the averaging methods of order higher than 3 are in the progress. But the calculations are extremely tedious and complicated.

2.2 Application of the developed averaging methods

We now consider the application of the developed averaging methods, i.e., Theorem 6 and its corollaries, which deal with the cases in which some terms on the right hand side of differential systems have no small parameter as their coefficients.

As the first application we consider the polynomial differential system

\begin{align*}
\dot{x} &= -y + \varepsilon P(x, y, z), \\
\dot{y} &= x + \varepsilon Q(x, y, z) + \varepsilon \cos t, \\
\dot{z} &= \alpha z + \varepsilon R(x, y, z),
\end{align*}

(18)

with $P, Q, R$ polynomials in the variables $x, y, z$ of degree $n$. When $\alpha$ and $\varepsilon$ are zero the unperturbed system has all $\mathbb{R}^3$ except the $z$-axis filled of periodic orbits. This case was studied by Cima, Llibre and Teixeira [5] in 2008 without the perturbation due to $\varepsilon \cos t$.

**Theorem 18 ([5]).** Assume that $\alpha = 0$. For $\varepsilon \neq 0$ sufficiently small, system (18) without $\varepsilon \cos t$ has at most $n(n-1)/2$ limit cycles bifurcating from the periodic orbits of the linear system $\dot{x} = -y, \dot{y} = x, \dot{z} = 0$. Moreover, there are such perturbed systems having $n(n-1)/2$ limit cycles.

When $\alpha \neq 0$ and $\varepsilon = 0$ the unperturbed system (18) only has the plane $z = 0$ except the origin filled of periodic orbits, that is, system (18) with $\varepsilon = 0$ restricted to the plane $z = 0$ has a global center at the origin. This is a nondegenerate center.

**Theorem 19 ([27]).** Assume that $\alpha \neq 0$. For convenient polynomials $P$, $Q$ and $R$, system (18) with $\varepsilon \neq 0$ sufficiently small has at least $m \in \{1, 2, \ldots, 2[(n - 1)/2] + 1\}$ limit cycles bifurcating from the periodic orbits of the linear center contained in $z = 0$ when $\varepsilon = 0$.

Now we consider the case that the unperturbed system has a degenerate global center.
Theorem 20 ([27]). For the polynomial differential system in $\mathbb{R}^3$

\begin{align}
\dot{x} &= -y(3x^2 + y^2) + \varepsilon P(x, y, z), \\
\dot{y} &= x(x^2 - y^2) + \varepsilon Q(x, y, z), \\
\dot{z} &= z(x^2 + y^2) + \varepsilon R(x, y, z),
\end{align}

with $\max\{\deg P, \deg Q, \deg R\} = n$, which when $\varepsilon = 0$ restricted to the plane $z = 0$ has a degenerate global center at the origin. For convenient polynomials $P$, $Q$ and $R$, system (19) with $\varepsilon \neq 0$ sufficiently small has at least $m \in \{1, 2, \ldots, [(n-1)/2]\}$ limit cycles bifurcating from the periodic orbits of the center contained in $z = 0$ when $\varepsilon = 0$.

The usefulness of Theorem 6 is also shown by the following result, in which Llibre and Zhang [30] provided the first analytic proof on the existence of a Hopf-zero bifurcation for the Michelson system [34]

\begin{align}
\dot{x} &= y, \quad \dot{y} = z, \quad \dot{z} = c^2 - y - \frac{x^2}{2},
\end{align}

at $c = 0$.

Theorem 21 ([30]). For $c \geq 0$ sufficiently small the Michelson system has a Hopf-zero bifurcation at the origin for $c = 0$. Moreover the bifurcated periodic orbit satisfies $x(t) = -2c\cos t + o(c)$, $y(t) = 2c\sin t + o(c)$ and $z(t) = 2c\cot t + o(c)$ for $c > 0$ sufficiently small.

We should mention that the averaging method not only provides the existence of periodic orbits of the Michelson system but also estimates the shape of the periodic orbit in function of $c > 0$ sufficiently small. On the existence of periodic orbit of Michelson system there have appeared lots of works, such as Michelson (1986) using a series in sinus for all small $c$, Kent and Elgin (1991) using the continuation code AUTO, Lamb, Teixeira and Webster (2005) using the Liapunov-Schmidt reduction method and so on.

The developed averaging method (Theorem 6) can also be applied to higher order differential equations for obtaining the existence of their periodic solutions.

Theorem 22 ([28]). (a) The third-order differential equation

\[ \ddot{x} - \mu \dot{x} + \dot{x} - \mu x = \varepsilon \left( \sin x - x + \frac{1}{2} \cos t \right), \]
for $\varepsilon \neq 0$ sufficiently small, has at least two limit cycles.

(b) The third-order differential equation

$$\ddot{x} - \dot{x} + x = \varepsilon \left( \sum_{i+j+k=0, i,j,k \geq 0} a_{ijk} \dot{x}^i \dot{x}^j \dot{x}^k + \cos t \right),$$

when $\varepsilon \neq 0$ sufficiently small, has at least $m \in \{1, 2, \ldots, 2[(n-1)/2] + 1\}$ limit cycles choosing conveniently the coefficients $a_{ijk}$.

(c) For the third-order differential equation

$$\ddot{x} - \dot{x} + x = \varepsilon \cos(x + t),$$

and for all $m \in \mathbb{N}$, there exists $\varepsilon_m > 0$ such that if $\varepsilon \in [-\varepsilon_m, \varepsilon_m] \setminus \{0\}$, the equation has at least $m$ limit cycles.

Except the above mentioned results, there are also some other recent results which will not be listed here. We refer the readers to [22, 23, 25, 26].

### 3 A new result: limit cycle bifurcations of higher dimensional systems

In this section we extend Theorem 11 for $m = 2$ to general $m \geq 2$. The main results are the following.

**Theorem 23.** For systems (11) with $a_{ck} \neq 0$, $k = 3, \ldots, n$, the following statements hold.

(a) If $m$ is even, there exists system (11) for which $l \in \{0, 1, \ldots, m^{n-2}/2\}$ limit cycles bifurcate from the origin at $\varepsilon = 0$, that is, for $\varepsilon$ sufficiently small the system has exactly $l$ limit cycles in a neighborhood of the origin and these limit cycles tend to the origin when $\varepsilon \searrow 0$.

(b) If $m$ is odd, there exists system (11) for which $l \in \{0, 1, \ldots, m^{n-2}\}$ limit cycles bifurcate from the origin at $\varepsilon = 0$.

We remark that Statement (a) is for arbitrary even number $m$, and coincides with the result of [29] when $m = 2$. For $m$ odd statement (b) shows that the number of limit cycles can exponentially increase in terms of the dimension $n$ of the system with exponent $n - 2$.

From the proof of Theorem 23 we can easily get the following.

**Corollary 24.** For $n \geq 3$ and $m \geq 2$, there exist real polynomial differential systems of degree $m$ in $\mathbb{R}^n$, consisting of the linear part of (11) and homogeneous polynomials of degree $m$, which can have $l$ limit cycles with either $l = m^{n-2}/2$ for $m$ even or $l = m^{n-2}$ for $m$ odd.
As we know it is the first result showing the phenomena that the function in the number of limit cycles bifurcated from one singularity of higher dimensional polynomial differential systems is a power function in the degree of the systems.

Proof of Theorem 23. For using the averaging method we need to transfer system (11) to the form of system (1). Taking the cylindrical transformation $x = r \cos \theta$, $y = r \sin \theta$, $z_i = z_i$, $i = 3, \ldots, n$, system (11) becomes

$$\dot{r} = \varepsilon ar + \sum_{i_1 + \cdots + i_n = m} A_{i_1, \ldots, i_n}(\theta)(r \cos \theta)^{i_1}(r \sin \theta)^{i_2} z_3^{i_3} \cdots z_n^{i_n} + O(m + 1),$$

$$\dot{\theta} = b + \frac{1}{r} \sum_{i_1 + \cdots + i_n = m} B_{i_1, \ldots, i_n}(\theta)(r \cos \theta)^{i_1}(r \sin \theta)^{i_2} z_3^{i_3} \cdots z_n^{i_n} + O(m + 1),$$

$$\dot{z}_k = \varepsilon c_k z_k + \sum_{i_1 + \cdots + i_n = m} c^{(k)}_{i_1, \ldots, i_n}(r \cos \theta)^{i_1}(r \sin \theta)^{i_2} z_3^{i_3} \cdots z_n^{i_n} + O(m + 1),$$

for $k = 3, \ldots, n$, where

$$A_{i_1, \ldots, i_n}(\theta) = a_{i_1, \ldots, i_n} \cos \theta + b_{i_1, \ldots, i_n} \sin \theta,$$

$$B_{i_1, \ldots, i_n}(\theta) = b_{i_1, \ldots, i_n} \cos \theta - a_{i_1, \ldots, i_n} \sin \theta,$$

and $O(m + 1) = O((r, z_3, \ldots, z_n)^{m+1})$ denotes the terms having orders larger than $m$. Taking $a_{00\ldots i_n} = b_{00\ldots i_n} = 0$, it is easy to show that in a suitable neighborhood of $(r, z_3, \ldots, z_n) = (0, 0, \ldots, 0)$, we have $\dot{\theta} \neq 0$.

Treating $\theta$ as the independent variable we get from system (21) the
following:

\[
\begin{align*}
\frac{dr}{dt} &= r(\epsilon a r + \sum_{i_1 + \cdots + i_m = m} A_{i_1, \ldots, i_m}(\theta)(r \cos \theta)^{i_1}(r \sin \theta)^{i_2} z_3^{i_3} \cdots z_m^{i_m} + O(m + 1)) \\
&= br + \sum_{i_1 + \cdots + i_m = m} B_{i_1, \ldots, i_m}(\theta)(r \cos \theta)^{i_1}(r \sin \theta)^{i_2} z_3^{i_3} \cdots z_m^{i_m} + O(m + 1),
\end{align*}
\]

(22)

\[
\begin{align*}
\frac{dz_k}{dt} &= r(\epsilon c_k z_k + \sum_{i_1 + \cdots + i_m = m} c_k^{(i_1, \ldots, i_m)}(r \cos \theta)^{i_1}(r \sin \theta)^{i_2} z_3^{i_3} \cdots z_m^{i_m} + O(m + 1)) \\
&= br + \sum_{i_1 + \cdots + i_m = m} B_{i_1, \ldots, i_m}(\theta)(r \cos \theta)^{i_1}(r \sin \theta)^{i_2} z_3^{i_3} \cdots z_m^{i_m} + O(m + 1).
\end{align*}
\]

(23)

For \( \epsilon > 0 \), set

\[
(r, z_3, \ldots, z_n) = (\rho \epsilon^{\frac{1}{1+m}}, \eta_3 \epsilon^{\frac{1}{1+m}}, \ldots, \eta_n \epsilon^{\frac{1}{1+m}}).
\]

We have

\[
\frac{d\rho}{d\theta} = \epsilon f_1(\theta, \rho, \eta_3, \ldots, \eta_n) + \epsilon^2 g_1(\theta, \rho, \eta_3, \ldots, \eta_n, \epsilon),
\]

(24)

\[
\frac{d\eta_k}{d\theta} = \epsilon f_k(\theta, \rho, \eta_3, \ldots, \eta_n) + \epsilon^2 g_k(\theta, \rho, \eta_3, \ldots, \eta_n, \epsilon),
\]

where

\[
\begin{align*}
&f_1 = \frac{(\alpha \rho + \sum_{i_1 + \cdots + i_m = m} A_{i_1, \ldots, i_m}(\theta)(\rho \cos \theta)^{i_1}(\rho \sin \theta)^{i_2} \eta_3^{i_3} \cdots \eta_n^{i_m})}{b}, \\
&f_k = \frac{(c_k \eta_k + \sum_{i_1 + \cdots + i_m = m} c_k^{(i_1, \ldots, i_m)}(\rho \cos \theta)^{i_1}(\rho \sin \theta)^{i_2} \eta_3^{i_3} \cdots \eta_n^{i_m})}{b}.
\end{align*}
\]

The averaged system of (23) is

\[
\dot{y} = \epsilon f^0(y),
\]

(25)

where \( y = (\rho, \eta_3, \ldots, \eta_n) \) and \( f^0(y) = (f_1^0(y), f_3^0(y), \ldots, f_n^0(y)) \) with

\[
f_i^0(y) = \frac{1}{2\pi} \int_0^{2\pi} f_i(\theta, \rho, \eta_3, \ldots, \eta_n) d\theta, \quad \text{for } i = 1, 3, \ldots, n.
\]
In order for calculating the isolated zeros of \( f_0(y) \), we need to get the explicit expression of \( f_0^1(y), f_0^3(y), \ldots, f_0^n(y) \). For this we will use the obvious fact: \( \int_0^{2\pi} \cos^i \theta \sin^j \theta d\theta = 0 \) if and only if either \( i \) or \( j \) is odd.

Proof of Statement (a). Since \( m \) is even, we have

\[
 f_0^0(y) = \frac{a \rho}{b} + \frac{1}{2b\pi} \times \int_0^{2\pi} (m-2)/2 \sum_{i,j,k} A_{i_1,\ldots,i_n}^* (\theta) \rho^{i_1+i_2+i_3} \eta_3^{i_3} \ldots \eta_n^{i_n} d\theta, 
\]

\[
 f_k^0(y) = \frac{c_k \rho}{b} + \frac{1}{2b\pi} \times \int_0^{2\pi} m/2 \sum_{i,j,k} c_{i_1,\ldots,i_n}^{(k)} (\rho \cos \theta)^{i_1} (\rho \sin \theta)^{i_2+i_3} \eta_3^{i_3} \ldots \eta_n^{i_n} d\theta, 
\]

for \( k = 3, \ldots, n \), where

\[
 A_{i_1,\ldots,i_n}^* (\theta) = a_{i_1,\ldots,i_n} \cos^{i_1+1} \theta \sin^{i_2} \theta + b_{i_1,\ldots,i_n} \sin^{i_2+1} \theta \cos^{i_1} \theta. 
\]

By the averaging theory we need to calculate the nondegenerate singularities of the averaged system (25) in the region \( \rho > 0 \). It is equivalent to compute the isolated simple roots of the algebraic equations \( f_0^0(y) = 0 \) with \( \rho > 0 \).

We claim that there is a system (11) for which the corresponding system of algebraic equations

\[
 f_1^0(y) = 0, \quad f_3^0(y) = 0, \quad \ldots, \quad f_n^0(y) = 0, 
\]

has exactly \( m^{n-2}/2 \) isolated simple roots.

We now prove the claim. In the expression of \( f_1^0(y) \) we choose \( a_{i_1,\ldots,i_n} = b_{i_1,\ldots,i_n} = 0 \) for \( (i_1, \ldots, i_n) \neq (0, \ldots, 0) \) and set

\[
 A_j = \frac{1}{2\pi} \int_0^{2\pi} A_{i_1,\ldots,i_n}^* (\theta) d\theta. 
\]

Then \( f_1^0(y) = 0 \) can be simplified to

\[
 a_\rho + A_1 \rho \eta_3^{m-1} + A_3 \rho^3 \eta_3^{m-3} + \cdots + A_{m-1} \rho^{m-1} \eta_3 = 0. 
\]
In the expression of \( f^0_3(y) \) we choose \( c^{(3)}_{i_1,\ldots,i_n} = 0 \) for \( (i_1,\ldots,i_n) \neq (0,\ldots,0) \) and set

\[
B_l = \frac{1}{2\pi} \int_0^{2\pi} \sum_{i_1+i_2=1 \atop i_3=m-1} c^{(3)}_{i_1,i_2,i_3,0,\ldots,0} (\cos \theta)^{i_1} (\sin \theta)^{i_2} d\theta.
\]

Then \( f^0_3(y) = 0 \) is simply

\[
c_3\eta_3 + B_0\eta_3^m + B_2\rho^2\eta_3^{m-2} + \cdots + B_m \rho^m = 0.
\]

Since \( a \neq 0 \), it follows from (27) and (29) that

\[
\frac{B_0\eta_3^m + \cdots + B_m \rho^m}{A_1 \rho^{m-1} + \cdots + A_{m-1} \rho^{m-1} \eta_3} = \frac{c_3 \eta_3}{a \rho},
\]

Consequently, we have

\[
\left( B_0 - \frac{c_3 A_1}{\rho} \right) \left( \frac{\eta_3}{\rho} \right)^m + \cdots + \left( B_{m-2} - \frac{c_3 A_{m-1}}{\rho} \right) \left( \frac{\eta_3}{\rho} \right)^2 + B_m = 0.
\]

Since \( B_{2k} \) and \( A_{2k+1} \) for \( k = 0,\ldots,(m-2)/2 \) can be chosen arbitrarily, that is, the coefficients of equation (30) in \( \eta_3/\rho \) can be arbitrary real numbers, so for the suitable choice of the values of \( B_{2k} \) and \( A_{2k+1} \) equation (30) can have exactly \( m/2 \) simple nonzero real roots \( (\eta_3/\rho)^2 \).

Rewriting equation (27) in

\[
\eta_3^{m-1} = \frac{-a}{A_1 + A_3 \left( \frac{\rho}{\eta_3} \right)^2 + \cdots + A_{m-1} \left( \frac{\rho}{\eta_3} \right)^m}.
\]

Since \( m \) is even, for each root \( \eta_3/\rho \) of equation (30) we have a unique real solution

\[
\eta_3 = \left( \frac{-a}{A_1 + A_3 \left( \frac{\rho}{\eta_3} \right)^2 + \cdots + A_{m-1} \left( \frac{\rho}{\eta_3} \right)^m} \right)^{1/m-1}.
\]
Thus, we get from equations (27) and (29) exactly $m/2$ simple roots, denoted by $(\rho_i, \eta_{3i})$ with $\rho_i > 0$ for $i = 1, 2, \ldots, m/2$.

In the expression of $f_k^l(y)$ for $k = 4, \ldots, n$, we choose $c^{(k)}_{i_1, \ldots, i_n} = 0$ for either $(i_1, i_2) \neq (0, 0)$ or $(i_{m+1}, \ldots, i_n) \neq (0, \ldots, 0)$. Then equations $f_k^l(y) = 0$ can be simplified to

$$c_k \eta_k + \sum_{i_1 + \cdots + i_k = m} c^{(k)}_{i_0, 0, i_2, \ldots, i_k} \eta_0^{i_0} \eta_1^{i_2} \cdots \eta_k^{i_k} = 0. \quad (31)$$

Equation (31) with $k = 4$ is

$$c_4 \eta_4 + c_{000m} \eta_4^m + \sum_{i=0}^{m-1} c^{(4)}_{000(i-1)m} \eta_4^{m-1} \eta_3 + \cdots + c^{(4)}_{000m} \eta_4^m = 0. \quad (32)$$

It can be treated as an algebraic equation in $\eta_4$ of degree $m$. Substituting each solution $\eta_{3i}$, $i = 1, \ldots, m/2$, of (27) and (29) into (32), then equation (32) can have exactly $m$ different solutions, denoted by $\eta_{4ij}$ for $j = 1, 2, \ldots, m$, provided the suitable choice of the values of $c^{(4)}_{i0, 0, i_2, \ldots, i_5, 0, \ldots, 0}$.

Substituting the $m^2/2$ different solutions $(\eta_{3i}, \eta_{4ij})$ for $i = 1, \ldots, m/2$ and $j = 1, \ldots, m$ into (31) with $k = 5$. By choosing the suitable values of the coefficients $c^{(5)}_{0000i_8, i_2, \ldots, i_5, 0, \ldots, 0}$ for each one of $(\eta_{3i}, \eta_{4ij})$'s equation (31) with $k = 5$ can have $m$ different solutions, denoted by $\eta_{5ij}$, $s = 1, 2, \ldots, m$. So equations (27), (29) and (31) with $k = 4$ and 5 have $m^3/2$ different simple solutions $(\rho_i, \eta_{3i}, \eta_{4ij}, \eta_{5ij})$ for $i = 1, \ldots, m/2$ and $j, s = 1, \ldots, m$.

By induction we can prove that equations (27), (29) and (31) can have $m^{n-2}/2$ different solutions $y_q = (\rho_q, \eta_{3q}, \eta_{4q}, \ldots, \eta_{nq})$ with $\rho_q > 0$ for $q = 1, \ldots, m^{n-2}/2$. By the Bezout's Theorem \[36\] (see Theorem 10) the $m^{n-2}/2$ are the maximum number of the solutions that these systems can have taking into account the equation (30). It follows that each solution $y_q$ for $q = 1, \ldots, m^{n-2}/2$ is simple. This proves the claim.

Since all these $m^{n-2}/2$ roots are simple, it follows that the determinant of system (25) at these points does not vanish. Hence, by Theorem 3 system (23) can have $m^{n-2}/2$ limit cycles. Through the transformation $(r, z_3, \ldots, z_n) = (\rho e^{-\frac{\eta_1}{\eta_3}}, \eta_3 e^{-\frac{\eta_1}{\eta_3}}, \ldots, \eta_n e^{-\frac{\eta_1}{\eta_3}})$, system (22) and consequently system (11) has $m^{n-2}/2$ limit cycles. Furthermore, these $m^{n-2}/2$ limit cycles tend to zero when $\varepsilon \to 0$.

Working in a similar way to the proof of the above claim, we can prove that equations (27), (29) and (31) can have exactly $l \in \{0, 1, \ldots, m^{n-2}/2 - 1\}$ simple roots provided the suitable choice of the values of the coefficients. Then we get that system (11) can have $l \in \{0, 1, \ldots, m^{n-2}/2 - 1\}$
limit cycles, which tend to zero when \( \varepsilon \to 0 \). This proves statement (a) of Theorem 23.

**Proof of Statement (b).** Since \( m \) is odd, we have

\[
f^0_1(y) = \frac{ap}{b} + \frac{1}{2b_0} \times \int_0^{2\pi} \left( \sum_{k=0}^{(m-1)/2} \sum_{i_1 + \cdots + i_n = m-2k-1} A_{i_1, \ldots, i_n}(\theta) \rho^{i_1+i_2} \eta_3^{i_3} \cdots \eta_n^{i_n} \right) d\theta,
\]

\[
f^0_k(y) = \frac{ck\eta_k}{b} + \frac{1}{2b_0} \times \int_0^{2\pi} \left( \sum_{k=0}^{(m-1)/2} \sum_{i_1 + \cdots + i_n = m-2k} c_{i_1, \ldots, i_n}^{(k)} (\rho \cos \theta)^{i_1} (\rho \sin \theta)^{i_2} \eta_3^{i_3} \cdots \eta_n^{i_n} \right) d\theta,
\]

for \( k = 3, \ldots, n \), where

\[ A_{i_1, \ldots, i_n}(\theta) = a_{i_1, \ldots, i_n} \cos^{i_1+1} \theta \sin^{i_2} \theta + b_{i_1, \ldots, i_n} \sin^{i_2+1} \theta \cos^{i_1} \theta. \]

We claim that there is a system (11) for which the system of algebraic equations

\[ f^0_1(y) = 0, \quad f^0_3(y) = 0, \quad \ldots, \quad f^0_n(y) = 0, \]

has exactly \( mn^{n-2} \) simple roots. For proving the claim we choose \( a_{i_1, \ldots, i_n} = b_{i_1, \ldots, i_n} = 0 \) if \( (i_1, \ldots, i_n) \neq (0, \ldots, 0) \). Then \( f^0_1(y) = 0 \) can be written in

\[
a \rho + A_1 \rho \eta_3^{m-1} + A_3 \rho^2 \eta_3^{m-3} + \cdots + A_{m-2} \rho^{m-2} \eta_3^2 + A_m \rho^m = 0,
\]

where \( A_j \)'s were defined in (26).

Similarly, we choose \( c_{i_1, \ldots, i_n}^{(3)} = (0, \ldots, 0) \) if \( (i_1, \ldots, i_n) \neq (0, \ldots, 0) \). Then equation \( f^0_3(y) = 0 \) can be simplified to

\[
c_3 \eta_3 + B_0 \eta_3^{m} + B_2 \rho^2 \eta_3^{m-2} + \cdots + B_{m-1} \rho^{m-1} \eta_3 = 0,
\]

where \( B_j \)'s were defined in (28).

From (33) and (34) we get

\[
\left( B_0 - \frac{c_3 A_1}{a} \right) \left( \frac{\eta_3}{\rho} \right)^{m-1} + \left( B_2 - \frac{c_3 A_3}{a} \right) \left( \frac{\eta_3}{\rho} \right)^{m-3} + \cdots + \left( B_{m-1} - \frac{c_3 A_{m}}{a} \right) \eta_3 = 0.
\]
Since $c_3, A_i$’s and $B_j$’s can be chosen arbitrarily, we choose suitable values of these parameters satisfying $A_j/a < 0$ for $j = 1, 3, \ldots, m$ such that equation (35) can have exactly $(m - 1)/2 + 1$ roots, one is $\eta_3 = 0$ and the others are nonzero given in $(\eta_3/\rho)^2$.

For $\eta_3 = 0$, equation (33) has a unique positive solution, denoted by $\rho_0$. For each nonzero root of (35) we get from equation (33) that

$$\eta_3^{m-1} = \frac{-a}{A_1 + \ldots + A_m - 2(\frac{a}{\rho_3})^{m-3} + A_m(\frac{a}{\rho_3})^{m-1}}.$$ 

Since $m$ is odd, we have two solutions

$$\eta_3 = \pm \left( \frac{-a}{A_1 + \ldots + A_m - 2(\frac{a}{\rho_3})^{m-3} + A_m(\frac{a}{\rho_3})^{m-1}} \right)^{\frac{1}{m-1}}.$$ 

Recall that $a/A_j < 0$ for $j = 1, 3, \ldots, m$. Hence, each positive real root $(\eta_3/\rho)^2$ of (35) corresponds to two solutions $(\rho, \eta_3)$ of equations (33) and (34). Thus equations (33) and (34) can have the maximum number, saying $m$, of real roots, denoted by $(\rho_i, \eta_{3i})$ with $\rho_i > 0, i = 1, \ldots, m$.

For $k = 4, \ldots, n$, we set $c_k(i_1, \ldots, i_n) = 0$ for either $(i_1, i_2) \neq (0, 0)$ or $(i_{k+1}, \ldots, i_n) \neq (0, \ldots, 0)$. Then equation $f_k^0(y) = 0$ can be written in

$$c_k \eta_k + \sum_{i_3+\ldots+i_k=m} c_{0,0,i_3,\ldots,i_k,0,\ldots,0}(i_3^{i_3} \cdots i_k^{i_k}) \eta_k = 0.$$ 

Working in a similar way to the proof of statement (a) we can prove that the system of algebraic equations (33), (34) and (36) can have the maximum number, saying $m^{n-2}$, of simple real roots $(\rho, \eta_3, \ldots, \eta_n)$ with $\rho > 0$. This proves the claim.

Using the last claim and working in a similar way to the proof of statement (a) we can finish the proof of statement (b). This completes the proof of Theorem 23.

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