ABSTRACT. Over the past decade, the consensus of multiagent systems has received an increasing attention in various fields, such as mathematics, physics, biology, and engineering sciences. There are numerous results reported on the consensus of multiagent systems. Now it is necessary to review the recent advances in consensus of multiagent systems. This paper firstly reviews the main mathematical models of multiagent systems, including Boids model, Vicsek model, and Couzin-Levin model. Moreover, this paper reviews the main advances in the consensus of multiagent systems, including the linear local updating rules, nonlinear local updating rules, and leader and asymmetric matrix cases.

1 Introduction  It is well known that a multiagent system is a system composed of multiple interacting intelligent agents [1–19]. Multiagent systems can be used to solve problems which are difficult or impossible for an individual agent or monolithic system to solve [3, 4, 6–9, 13, 15–17]. Some multiagent systems with complex topological structures are typical complex networks. Complex networks are everywhere in the world, such as Internet, the World Wide Web, computer networks, power grids, telephone call graphs, biological neural networks, food webs, cellular and metabolic networks, and so on [1, 2, 5, 10–12, 14, 18, 19].

The study of multiagent systems has a long history. General speaking, it includes the modelling, analysis, design, control, and synchronization of multiagent systems. Over the past decade, consensus (or coordination, synchronization) of multiagent systems has received an increasing attention in various disciplines, including mathematics, physics, computer science, control science, artificial intelligence, and so on [3, 4, 6–9, 13, 15–17]. Consensus is one of typical collective behaviors in
multiagent systems. In fact, consensus is a fundamental nature phenomenon. Intuitively, consensus is a general agreement among all group members via interactions among agents. Consensus usually involves collaboration, rather than compromise.

To characterize the inherent mechanics of consensus in multiagent systems, some mathematical models were introduced, including Boids model [16], Vicsek model [17], Couzin-Levin model [6], and so on. Based on these models, there are numerous results reported on the consensus of multiagent systems over the past decade [3, 6, 7, 9, 13, 15–17]. It should be especially emphasized here that most of the above results are based on the linear local updating rules that govern the interactions among agents and also do not involve the communication constraints in the exchange of information among agents. Recently, there is an interesting advance reported on the consensus of discrete-time multiagent systems with nonlinear local rules and time-varying delays [4]. Moreover, there are some other advances reported on the consensus of continuous-time multiagent systems, such as the consensus of multiagent systems with an active leader and asymmetric adjacency matrix [8].

This paper is organized as follows. Several fundamental mathematical models of multiagent systems are briefly reviewed in Section 2. Section 3 presents some recent advances on the consensus of multiagent systems. Conclusions are finally given in Section 4.

2 Mathematical models of multiagent systems

This section will briefly review some representative mathematical models of multiagent systems, including the well known Boids model [16], Vicsek model [17], and Couzin-Levin model [6, 13].

As we know now, the collective behavior of a school of fish, a flock of birds, or a herd of land animals is a beautiful and familiar part of our natural world. However, it is not easy to understand and repeat this type of complex motion by using intuitive rules. To well discover the dynamic mechanics of the aggregate motion of multiagent systems, it is necessary to construct the suitable mathematical model to further investigate the collective behaviors of multiagent systems [6, 13, 16, 17].

Figure 1 shows a unifying framework for describing the collective behaviors of multiagent systems. Usually, a multiagent system is composed of many independent or dependent agents distributed in a given region. There exist various interactions between agents which obey some simple fundamental local or global rules. It should be especially pointed out that the individual dynamics and the basic rules are two key factors for
determining the collective behaviors of multiagent systems. Figure 2 shows a formation control of airplanes in real-world military manoeuvre (http://slide.news.sina.com.cn/c/slide_1_493_9927.html). Figure 3 shows a typical tuna swarm motion in nature (http://www.princeton.edu/~icouzin/index.htm). In the following, several representative mathematical models are then given to characterize the collective behaviors of multiagent systems.

FIGURE 1: A unifying framework of multiagent systems.

FIGURE 2: Formation control of airplanes.
2.1 Boids model To characterize the aggregate motion of a flock of birds, a herd of land animals, or a school of fish, Reynolds introduced a distributed computational behavioral model [16], called Boids model today.

The three basic local updating rules of Boids model [16] are described by

(i) **Alignment**: Steer to move toward the average heading of local flockmates.
(ii) **Separation**: Steer to avoid crowding local flockmates.
(iii) **Cohesion**: Steer to move toward the average position of local flockmates.

It should be especially pointed out that the above Boids model was based on three dimensional computational geometry of the sort normally used in computer animation or computer aided design [16]. In Boids model, each bird or agent is realized as an independent agent nav-
igating from its local perception of its dynamical environment. Each agent reacts only to flockmates within its certain small neighborhood. Hereafter, the above neighborhood is defined by a distance started from the given agent and an angle measured from the flight direction of the given agent. Moreover, all flockmates outside this local neighborhood are often ignored. To the best of our knowledge, Boids model is the first computational model of collective behaviors of multiagent systems.

2.2 Vicsek model To describing the phase transition in a system of self-driven particles, Vicsek and his colleagues proposed a simple mathematical model, called Vicsek model now [17].

In detail, consider a discrete-time multiagent system with $n$ autonomous agents moving on the plane, labeled from 1 to $n$. Thus $V = \{1, 2, \cdots, n\}$ is the set of all agents. There exist some connections among these $n$ agents. If agent $i$ has access to the information of agent $j$, then $j$ is said to be a neighbor of agent $i$ and the set of all neighbors of agent $i$ at time $t$ is denoted by $N_i(t)$. Also, $i \in N_i(t)$ if and only if agent $i$ has access to the information of itself. The heading of agent $i$ at time $t$ is denoted by $\theta_i(t)$. Suppose that the velocity of each agent is a constant $v$. Then, the dynamics of agent $i$ is characterized by the sequence $\{(x_i(t), y_i(t), \theta_i(t))\}$, where $(x_i(t), y_i(t))$ are the coordinates of agent $i$ at time $t$, $x_i(t), y_i(t) \in \mathbb{R}$, $\theta_i(t) \in [0, 2\pi)$. The dynamics of each agent in $V$ is updated from the following rules [17].

\begin{align}
(1) \quad \theta_i(t + 1) &= \arctan \left( \frac{\sum_{j \in N_i(t)} \sin(\theta_j(t))}{\sum_{j \in N_i(t)} \cos(\theta_j(t))} \right), \\
(2) \quad x_i(t + 1) &= x_i(t) + v \cos(\theta_i(t + 1)), \\
(3) \quad y_i(t + 1) &= y_i(t) + v \sin(\theta_i(t + 1)),
\end{align}

where $t \in \{0, 1, 2, \cdots\}$, and

$$N_i(t) = \left\{ j \in V : \sqrt{(x_i(t) - x_j(t))^2 + (y_i(t) - y_j(t))^2} < r \right\}, \quad r > 0.$$  

When the density of all agents is large enough and the system noise is relative small, all agents tend to move in the same spontaneously selected direction [17].
2.3 Couzin-Levin model  To characterize the collective behaviors of animal groups, Levin and his colleagues introduced a discrete-time mathematical model based on the three fundamental rules of Boids model, called Couzin-Levin model \[6\].

Each individual \(i\) has a position vector \(c_i(t)\), direction vector \(v_i(t)\), and speed \(s_i\) at time \(t\). \(\alpha\) and \(\rho\) are the minimum separate distance and the maximum response distance for the group at all times, respectively. \(r(i, j)\) is the distance between individuals \(i\) and \(j\). \(d_i(t)\) represents a desired direction of travel at time \(t\) and \(d_i(t) = \frac{\hat{d}_i(t)}{|d_i(t)|}\) denotes its unit vector. \(g_i\) is the information direction for individual \(i\) and \(\omega\) is its weight. \(d'_i(t)\) is the final direction of travel after considering information direction for individual \(i\) at time \(t\). The updating rules \[6, 13\] of Couzin-Levin model are described by

(i) If \(r(i, j) < \alpha\), then one has

\[
d_i(t + \Delta t) = -\sum_{j \neq i} \frac{c_j(t) - c_i(t)}{|c_j(t) - c_i(t)|}.
\]

(ii) If \(\alpha < r(i, j) \leq \rho\), then one gets

\[
d_i(t + \Delta t) = \sum_{j \neq i} \frac{c_j(t) - c_i(t)}{|c_j(t) - c_i(t)|} + \sum_{j = 1}^{n} \frac{v_j(t)}{|v_j(t)|}.
\]

(iii) Consider the information direction, one has

\[
d'_i(t + \Delta t) = \frac{\hat{d}_i(t + \Delta t) + \omega g_i}{|\hat{d}_i(t + \Delta t) + \omega g_i|}.
\]

Couzin-Levin model provides some new insights into the mechanisms of effective leadership and decision-making in biological systems \[6\]. Recently, Lü and his colleagues build another similar model merging the locally neighboring reciprocal action and alignment together to investigate the mechanisms of consensus decision-making and its robustness \[13\].

3 Consensus of multiagent systems  Consensus is often defined as unanimous or general agreement between all group members. A general mathematical definition for the consensus of multiagent systems is then given in the following.
Definition 1. If

$$\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0$$

for all $i, j \in V$, then all agents can realize consensus.

Over the last ten years, there are numerous results reported on the consensus of multiagent systems. In the following, only several important recent advances will be briefly reviewed because of the space limit.

3.1 Linear local updating rules According to (1)–(3), the local updating rules of the classical Vicsek model are nonlinear. Therefore, it is very difficult to analyze the consensus of the classical Vicsek model from theory. For simplification, Jadbabaie and his colleagues replaced the nonlinear local updating rules (1) by the following linear local updating rules:

$$\theta_i(t + 1) = \frac{1}{1 + n_i(t)} \left( \theta_i(t) + \sum_{j \in N_i(t)} \theta_j(t) \right),$$

where $N_i(t)$ is the number of neighbors of agent $i$ at time $t$.

Hereafter, all notations are given in the corresponding references and omitted in this paper. The main results are listed in the following.

**Theorem 1** ([9]). Let $\theta(0)$ be fixed and let $\sigma : \{0, 1, 2, \ldots \} \to \mathcal{P}$ be a switching signal satisfying $\sigma(t) \in \mathcal{Q}$, $t \in \{0, 1, \ldots \}$. Then

$$\lim_{t \to \infty} \theta(t) = \theta_{ss} \mathbf{1}$$

where $\theta_{ss}$ is a number depending only on $\theta_0$ and $\sigma$.

**Theorem 2** ([9]). Let $\theta(0)$ be fixed and let $\sigma : \{0, 1, 2, \ldots \} \to \mathcal{P}$ be a switching signal for which there exists an infinite sequence of contiguous, nonempty, bounded, time-intervals $[t_i, t_{i+1})$, $i \geq 0$, starting at $t_0 = 0$, with the property that across each such interval, the agents are linked together. Then

$$\lim_{t \to \infty} \theta(t) = \theta_{ss} \mathbf{1}$$

where $\theta_{ss}$ is a number depending only on $\theta_0$ and $\sigma$. 
In the following, one considers a modified Vicsek model consisting of the same group of agents and one additional agent labeled 0 acting as the group leader. Agent 0 moves at the same constant speed as its followers but with a fixed heading $\theta_0$. And the main results are given as follows.

**Theorem 3** ([9]). Let $\theta(0)$ and $\theta_0$ be fixed and let $\sigma : \{0, 1, 2, \ldots\} \rightarrow \mathcal{P}$ be a switching signal for which there exists an infinite sequence of contiguous, nonempty, bounded, time-intervals $[t_i, t_{i+1})$, $i \geq 0$, starting at $t_0 = 0$, with the property that across each such interval, the $n$-agent group of followers is linked to its leader. Then

$$\lim_{t \to \infty} \theta(t) = \theta_0 \mathbf{1}.$$

The above Theorems 1–3 have provided a theoretical explanation for the collective behavior observed in the simulation studies reported in [17].

### 3.2 Nonlinear local updating rules

It is well known that most of the known results are based on the linear local updating rules that govern the interactions among agents [9, 15]. However, in real-world applications, the local interactions among agents are more likely to be nonlinear local updating rules with time varying delays. In [4], we have further investigated the consensus in a discrete-time multiagent systems with nonlinear local rules and time-varying delays.

The overall updating rule is described by

$$x_i(t + 1) = F \left( \frac{1}{n_i(t)} \sum_{j \in N_i(t)} f \left( x_j \left( t - \tau_j(t) \right) \right) \right),$$

for all $i \in V$. Thus one has the following theorems.

**Theorem 4** ([14]). Suppose that Assumptions (A1)–(A6) hold for the given multiagent system $(V, G(t), (6))$. Then, for any given initial states $x_i(t) \in [a, b]$ with $i \in V$ and $-B < t \leq 0$, the states of all agents reach consensus.

**Theorem 5** ([14]). Suppose that Assumptions (A1)–(A3), (A5) and (A6) hold for a given multiagent system $(V, G(t), (6))$. If there exists a sequence $\{t_k\}$ and an integer $\lambda_m > 0$ such that there exists a directed path from $\Omega_m(t_k)$ to $\Omega_m(t_k)$ and $0 < t_{k+1} - t_k < \lambda_m$ for $\forall k > 0$, then the multiagent system $(V, G(t), (6))$ can reach consensus.
As a typical application of Theorem 4, one considers the classical Vicsek model (1)–(3) with time-varying delays, described by

\[
\begin{align*}
\theta_i(t + 1) &= \arctan \left( \frac{\sum_{j \in N_i(t)} \sin(\theta_j(t - \tau_j(t)))}{\sum_{j \in N_i(t)} \cos(\theta_j(t - \tau_j(t)))} \right), \\
x_i(t + 1) &= x_i(t) + v \cos(\theta_i(t + 1)), \\
y_i(t + 1) &= y_i(t) + v \sin(\theta_i(t + 1)).
\end{align*}
\]

where \( t \in \{0, 1, 2, \cdots \} \). Then one gets the following theorem.

**Theorem 6** ([14]). Suppose that Assumption (A4) holds for the Vicsek model with time-varying delays in the updating rules (7). Also, assume that the graph \( G(\infty) \) is connected. If the initial headings \( \theta_i(t) \in (-\pi, \pi) \) for \( -B < t \leq 0 \) and any \( i \in V \), then the headings of agents can reach consensus.

Theorems 4–6 present the consensus criteria for the discrete-time multiagent systems with nonlinear local rules and time-varying delays. It should be especially pointed out that the above results include several well-known results as special cases [4].

### 3.3 Leader and asymmetric matrix

In [8], we have further investigated the consensus of a multiagent system with an active leader and asymmetric adjacency matrix. In particular, the state of the active leader is changing and unmeasured.

The state vectors of all agents of the multiagent system are given by

\[
\dot{x}_i = u_i \in \mathbb{R}^m, \quad i = 1, \ldots, n,
\]

where \( u_i \) are the control inputs. Assume also that the leader of the multiagent system is active. That is, its state variables keep changing. And its underlying dynamics is described by

\[
\begin{align*}
\dot{x}_0 &= v_0, \\
\dot{v}_0 &= a(t) = a_0(t) + \delta(t), \\
y &= x_0,
\end{align*}
\]

where \( y(t) = x_0(t) \) is the measured output and \( a(t) \) is the (acceleration) input. For each agent \( i \), the local control scheme consists of two different parts.
(A) A neighbor-based feedback law

\[ u_i = -k \left[ \sum_{j \in N_i} a_{ij} (x_i - x_j) + b_i (x_i - x_0) \right] + v, \]

where \( k > 0 \) and \( i = 1, \cdots, n \).

(B) A dynamic neighbor-based system to estimate \( v_0 \)

\[ \dot{v}_i = a_0 - \gamma k \left[ \sum_{j \in N_i} a_{ij} (x_i - x_j) + b_i (x_i - x_0) \right], \]

where \( 0 < \gamma < 1 \) and \( i = 1, \cdots, n \).

Suppose that the directed graph \( G \) is strongly connected, then one has the following theorem.

**Theorem 7** ([8]). For any given \( 0 < \gamma < 1 \), select a constant

\[ k > \frac{\max \xi_i}{2\gamma (1 - \gamma^2) \lambda}, \]

where \( \lambda \) is the minimal eigenvalue of \( M^s \). If the interconnection graph \( G \) keeps strongly connected or \( L \) is irreducible, then there exists some constant \( C \) satisfying

\[ \lim_{t \to \infty} \| \omega(t) \| \leq C \delta. \]

Moreover, if \( a(t) \) is known, that is, \( a(t) = a_0(t) \) or \( \delta = 0 \), then one has

\[ \lim_{t \to \infty} \| \omega(t) \| = 0. \]

If the interconnection graph \( G \) is not strongly connected, then one gets the following theorem.

**Theorem 8** ([8]). If \( b_{S_i+1} + \cdots + b_{S_{i+1}} > 0 \) for \( i = 1, \cdots, q \), then one can select some large enough \( k \) satisfying that each agent of the multiagent system can follow the leader or the tracking errors can be estimated.
4 Conclusions This paper has briefly reviewed the recent main advances in consensus of multiagent systems. To reveal the inherent dynamic mechanics of consensus in multiagent systems, some mathematical models were introduced, including Boids model, Vicsek model, and Couzin-Levin model. Furthermore, this paper reviews the recent main advances in the consensus of multiagent systems based on these mathematical models, such as the linear local updating rules, nonlinear local updating rules, and leader and asymmetric matrix cases. The research of multiagent systems brings us together with the goal of understanding collective behaviors in natural and man-made networked systems, such as the navigation mechanism of ants mill and sensor networks.

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