SYMBOLIC COMPUTATION OF EXACT SOLUTIONS TO KDV EQUATION

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ABSTRACT. In this paper we show some exact solutions for the KdV equation. These solutions are obtained by He’s exp-function method, which is a powerful tool for solving nonlinear differential equations.

1 Introduction Nonlinear evolution and wave equations are partial differential equations (PDEs) involving first or second-order derivatives with respect to time. Such equations have been intensively studied for the past decades [7, 18], and several new methods to solve nonlinear PDEs, either numerically or analytically, are now available. When the dependent variable \( u \) in the PDE corresponds to a physical quantity (such as the surface height of a water wave, the magnitude of an electromagnetic wave, etc.), it is important to study the propagation or aggregation properties of \( u \). This motivates the study of methods to analytically solve evolution or wave equations via symbolic methods. The goal is to find exact traveling wave solutions. If these solutions do not change their form during propagation, they are called solitary waves. Solitary waves that preserve their shape upon collision are called solitons [19]. Solitary-waves and solitons arise due to a critical balance between dispersion and nonlinearity. Due to the complexity of the mathematics involved in finding exact solutions for these PDEs, the use of algorithmic techniques that can be implemented in the symbolic language of computer algebra systems becomes a necessity. Several computer algebra packages now exist to aid in the study of nonlinear PDEs [10, 9, 11]. For example, Painlevé analysis offers an algorithm for testing whether or not a PDE is a good candidate to be completely integrable. In addition, the PPainlevé method allows one to construct solitary wave solutions in explicit form. A more powerful technique is Hirota’s bilinear

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method [12] which allows one to find $N$-soliton solutions of large classes of completely integrable PDEs [15]. The story of the first observation of solitary waves is worth telling. In 1834, while riding horseback beside the narrow Union canal near Edinburgh in Scotland, J. Scott Russell noticed that a bow wave, rolling away from a large barge, traveled as a huge heap of water for quite a long distance before finally dispersing into smaller ripples. In order to study this intriguing phenomenon, Russell did extensive experiments in a large water tank. Further investigations of solitary waves were done by Airy, Stokes, Boussinesq and Rayleigh in an attempt to understand the mechanism behind this remarkable phenomenon [1]. The latter two scientists derived approximate models to describe solitary waves. In order to obtain his result, Boussinesq derived a one-dimensional nonlinear wave equation which now bears his name. The issue was finally resolved (in 1895) by two Dutchmen, Korteweg and de Vries, when they derived a nonlinear evolution equation governing long, one-dimensional surface gravity waves (with small amplitude) propagating in shallow water:

\[
\frac{\partial \eta}{\partial \tau} = \frac{3}{2} \sqrt{\frac{g}{h}} \frac{\partial}{\partial \xi} \left( \frac{1}{2} \eta^2 + \frac{2}{3} \sigma \eta + \frac{1}{3} \sigma \frac{\partial^2 \eta}{\partial \xi^2} \right)
\]

\[
\sigma = \frac{1}{3} h^3 - \frac{Th}{\rho g},
\]

where $\eta$ is the surface elevation of the wave above the equilibrium level $h$, $\alpha$ is a small arbitrary constant related to the uniform motion of the liquid, $g$ is the gravitational constant, $T$ is the surface tension, and $\rho$ is the density. The independent variables $\tau$ and $\xi$ are scaled versions of the time and space coordinates. Equation (1), which is called the Korteweg-de Vries (KdV) equation, can be brought into a non-dimensional form via the change of variables

\[
t = \rho \tau, \quad x = q \xi, \quad u = A \eta + B \alpha.
\]

where

\[
p = \frac{1}{A}, \quad q = -\frac{4\sqrt{h}}{A^2 \sqrt{g}}, \quad B = -\frac{2}{3}, \quad A = 2 \sqrt{\frac{h^2 (g/\rho) - 3T}{3g^2 \rho}}.
\]

After some algebra, one obtains

\[
u_t + 6u u_x + u_{xxx} = 0.
\]
where subscripts denote partial derivatives, e.g., \( u_{xxx} = \frac{\partial^3 u}{\partial x^3} \). Despite this early derivation of the KdV equation, it was not until 1960 that any new applications of the equation were discovered [1]. Amazingly, the KdV equation started to show up in a number of other physical contexts such as the study of stratified internal waves, ion-acoustic waves in plasma physics, lattice dynamics, and so on (further details can be found in Jeffrey and Kakutani [13], Miura [16], Ablowitz and Segur [2], Lamb [14], Calogero and Degasperis [4], Dodd et al. [6], and Novikov et al. [17]). Since the late 1960’s, the study of the properties of solitons, and the search for solitonic equations and methods to solve them, has been an active and exciting area of research.

Recently, He and Wu [8] proposed a straightforward and concise method, called Exp-function method, to obtain generalized solitonal solutions and periodic solutions of NLEEs.

### 2 Exact solutions to KdV equation. The Exp-function method

Using the transformation

\[
(5) \quad u = v(\xi), \quad \xi = \mu x + \lambda t,
\]

where \( \lambda, \mu \) are constants, equation (4) becomes

\[
(6) \quad \lambda v'(\xi) + 6\mu v(\xi)v'(\xi) + \mu^3 v''(\xi) = 0.
\]

In view of the Exp-function method, we assume that the solution of equation (6) can be expressed in the form

\[
(7) \quad v(\xi) = \frac{\sum_{n=c}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_m \exp(m\xi)} = \frac{a_{-c} \exp(-c\xi) + \cdots + a_d \exp(d\xi)}{b_{-p} \exp(-p\xi) + \cdots + b_q \exp(q\xi)},
\]

where \( c, d, p \) and \( q \) are positive integers which are unknown to be determined later, \( a_n \) and \( b_m \) are unknown constants.

In order to determine values of \( c \) and \( p \), we balance the linear term of highest order in equation (6) with the highest order nonlinear term, and the linear term of lowest order in equation (6) with the lowest order nonlinear term, respectively.

By simple calculation, we have

\[
(8) \quad v''(\xi) = \frac{k_1 \exp(7p + c)\xi + \cdots}{k_2 \exp(8p\xi) + \cdots}
\]
where the $k_i$ are some constants. Balancing highest order of Exp-function in equations (8) and (9), we have $7p + c = 2(3p + c)$ so that $c = p$.

Similarly, to determine values of $d$ and $q$, we balance the linear term of lowest order in equation (6)

\begin{equation}
\frac{v''(\xi)}{\cdots + k_4' \exp[-3q\xi]} 
\end{equation}

and

\begin{equation}
\frac{v(\xi)v'(\xi)}{\cdots + k_4' \exp[-8q\xi]} \end{equation}

where the $k_i'$ are some constants. Balancing lowest order of Exp-function in equations (10) and (11), we obtain $7q + d = 2(3q + d)$ so that $d = q$.

The considerations below say that any solution of the KdV equation (6) must have the form

\begin{equation}
\frac{a_1 \exp(c\xi) + \cdots + a_c \exp(d\xi)}{b_0 + b_1 \exp(q\xi)}
\end{equation}

We will consider two cases. In these cases we set $b_{-p} = 1$, that is, the trial solution has the form

\begin{equation}
\frac{a_1 \exp(c\xi) + \cdots + a_c \exp(d\xi)}{\exp(-p\xi) + \cdots + b_0 \exp(q\xi)}
\end{equation}

**2.1 Case 1:** $p = c = 1$ and $d = q = 1$ The trial solution equation (12) becomes

\begin{equation}
\frac{a_1 \exp(c\xi) + a_0 + a_{-1} \exp(-\xi)}{\exp(c\xi) + b_0 + b_{-1} \exp(-\xi)}
\end{equation}

Substituting (13) into (6) and equating to zero the coefficients of all powers of $\exp(\xi)$ yields a set of algebraic equations. Solving it with the aid of a computer, we obtain the following solutions.
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I. \[ a_1 = a_1, \quad b_0 = b_0, \quad a_{-1} = \frac{1}{4} a_1 b_0^2, \quad a_0 = a_1 b_0 + \mu^2 b_0, \quad b_{-1} = \frac{b_0^2}{4}, \]

\[ \lambda = -\mu^2 - 6\mu a_1, \quad \mu = \mu. \]

For \( b_0 \neq \pm 2 \) the soliton solutions corresponding to these values are

\[ u_1(x,t) = a_1 + \frac{4 \mu^2 b_0 e^{\mu(x+(\mu^2+6a_1)t)}}{(2e^{\mu x} + b_0 e^{(\mu^2+6a_1)t})^2}. \]

For \( b_0 = \pm 2 \) and real \( \mu \),

\[ u_2(x,t) = a_1 + \frac{\mu^2}{2} \text{sech}^2 \left( \frac{1}{2} (\mu (x - (\mu^2 + 6a_1)t)) \right), \]

\[ u_3(x,t) = a_1 - \frac{\mu^2}{2} \text{csch}^2 \left( \frac{1}{2} (\mu (x - (\mu^2 + 6a_1)t)) \right). \]

We obtain periodic solutions in the following cases:

\[ b_0 = 2 \text{ and } \mu = \sqrt{-1} m: \]

\[ u_4(x,t) = a_1 - 2m^2 \cos^2 \left( \frac{1}{2} (m (x + (m^2 - 6a_1) t)) \right). \]

\[ b_0 = -2 \text{ and } \mu = \sqrt{-1} m: \]

\[ u_5(x,t) = a_1 - 2m^2 \sin^2 \left( \frac{1}{2} (m (x + (m^2 - 6a_1) t)) \right). \]

2.2 Case 2: \( p = c = 2 \text{ and } d = q = 2 \) The trial solution equation (12) becomes

\[ v(\xi) = \frac{a_2 \exp(2\xi) + a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi) + a_{-2} \exp(-2\xi)}{\exp(2\xi) + b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi) + b_{-2} \exp(-2\xi)}. \]

Substituting (19) into (6) and equating to zero the coefficients of all powers of \( \exp(\xi) \) yields a set of algebraic equations. Solving it with the aid of a computer, we obtain many solutions. For space reasons, we only give some of them.

II. \[ a_2 = a_2, \quad b_0 = b_0, \quad a_1 = 0, \quad b_1 = 0, \quad a_{-2} = a_2 b_0^2/4, \quad a_{-1} = 0, \quad a_0 = a_2 b_0 + 4\mu^2 b_0, \quad b_{-1} = 0, \quad b_{-2} = b_0^2/4, \quad \lambda = -6\mu a_2 - 4\mu^3, \]

\[ u_6(x,t) = a_2 + \frac{16b_0\mu^2 e^{-2\mu(x-2(2\mu^2+3a_2)t)}}{(2 + b_0 e^{-2\mu(x-2(2\mu^2+3a_2)t)})^2}. \]
In the case when $b_0 = \pm 2$ we obtain solitonic and periodic solutions. More exactly,

(21) \[ u_7(x, t) = a_2 + 2\mu^2 \text{sech}^2(\mu(x - 2(2\mu^2 + 3a_2)t)). \]

(22) \[ u_8(x, t) = a_2 - 2\mu^2 \text{csch}^2(\mu(x - 2(2\mu^2 + 3a_2)t)). \]

For $b_0 = \pm 2$ and $\mu = \sqrt{\frac{1}{m}}$

(23) \[ u_9(x, t) = a_2 - 2m^2 \sec^2 \left( m(x + 2(2m^2 - 3a_2)t) \right). \]

(24) \[ u_{10}(x, t) = a_2 - 2m^2 \csc^2 \left( m(x + 2(2m^2 - 3a_2)t) \right). \]

III. \[ a_1 = a_1, \ a_2 = a_2, \ b_1 = 0, \ a_{-2} = \frac{1}{16} a^{2a_1^4}, \ a_{-1} = \frac{1}{4} a_0^3, \]

\[ a_0 = -\frac{1}{2} \frac{a_1^2 + a_0^2}{\mu^2}, \ b_{-1} = 0, \ b_{-2} = \frac{1}{16} \frac{a_0^4}{\mu^4}, \lambda = -6\mu a_2 - \mu^3. \]

(25) \[ u_{11}(x, t) = \frac{(4\mu^4 a_{11} e^{2\mu x} + a_1^2 e^{2\mu(\mu^2 + 6a_2)t}) a_2 + 4\mu^2 a_1 (a_2 + \mu^2) e^{(x + \mu(x + 6a_2)t)}}{(2\mu^2 e^{2\mu x} + a_1 e^{\mu(\mu^2 + 6a_2)t})^2}. \]

IV. \[ a_1 = a_1, \ a_2 = a_2, \ b_0 = b_0, \ b_1 = b_1, \lambda = -\mu^3 - 6\mu a_2, \]

\[ a_{-2} = \frac{a_2}{16\mu^8}(3a_1^4 + 3a_2^4 b_1^4 + 4\mu^4 a_2^2 b_0 b_1^2 - 8\mu^4 a_1 a_2 b_0 b_1 \]
\[ + 12\mu^2 a_1 a_2 b_1^2 - 12\mu^2 a_1 a_2 b_1^3 + 18a_1 a_2 b_1^2 - 12a_1 a_2 b_1^3 \]
\[ - 12a_1 a_2 b_1 - 4\mu^2 a_1 b_1 + 4\mu^2 a_2 b_1^2 + 4\mu^4 a_2 b_0), \]

\[ a_{-1} = -\frac{1}{4\mu^5}(12\mu^2 a_1 a_2 b_1 + 4\mu^4 a_1 b_1 - 15\mu^2 a_1 a_2 b_1^2 \]
\[ - 8\mu^4 a_1 a_2 b_1^2 - 3\mu^2 a_1^3 + 6\mu^2 a_2 b_1^3 + 4\mu^4 a_2 b_1^3 \]
\[ + 2a_1 b_1 - 6a_1 a_2 b_1^2 + 4\mu^4 a_2 b_0 b_1 + 6a_1 a_2 b_1^2 - 4\mu^4 a_1 a_2 b_0 \]
\[ - 2a_1 a_2 + 4\mu^6 a_2 b_0 b_1 - 4\mu^6 a_1 b_0), \]

\[ a_0 = \frac{1}{\mu^3}(2a_1 a_2 b_1 + \mu^2 a_1 b_1 + \mu^2 a_2 b_0 - a_0 b_1^2 - \mu^2 a_2 b_1^2 - a_1^2), \]

\[ b_{-2} = \frac{1}{16\mu^8}(3a_1^4 + 3a_2^4 b_1^4 + 4\mu^4 a_2^2 b_0 b_1 - 8\mu^4 a_1 a_2 b_0 b_1 \]
+ 12\mu^2a_1^2a_2b_1^2 - 12\mu^2a_1a_2^2b_1 + 18a_1^2a_2^2b_1^2 - 12a_1a_2b_1^3
- 12a_1a_2b_1 - 4\mu^2a_1^3b_1 + 4\mu^2a_2^3b_1^2 + 4\mu^4a_1^2b_0),
\]

\[b_{-1} = -\frac{1}{4\mu^6}(2a_2b_1^3 + 3\mu^2a_2^2b_1^2 - 6a_1a_2^2b_1^2 - 6\mu^2a_1a_2b_1^2
+ 4\mu^4a_2b_0b_1 + 6a_1^2a_2 + 3\mu^2a_1^2b_1 - 4\mu^4a_1b_0 - 2a_1^3).\]

(26) \[u_{12}(x,t) \]
\[= \frac{a_2 \exp(2\xi) + a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi) + a_{-2} \exp(-2\xi)}{\exp(2\xi) + b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi) + b_{-2} \exp(-2\xi)},\]
where \(\xi = \mu(x - (\mu^2 - 6a_2)t).\)

V. \(a_2 = a_{-2} = 0, a_1 = 2q\sqrt{-1}, a_0 = -4q, a_{-1} = -2q\sqrt{-1}, b_{-2} = 1,\)
\(b_{-1} = b_1 = 0, \mu = q\sqrt{-1}, \lambda = q\sqrt{-1}.\)

(27) \[u_{13}(x,t) = -\frac{2q(1 + \sin(\sqrt{q}(x + qt)))}{1 + \cos(2\sqrt{q}(x + qt))} \quad (q > 0).\]

VI. \(a_2 = a_{-2} = 0, a_1 = 2q\sqrt{-1}, a_0 = -4q, a_{-1} = -2q\sqrt{-1}, b_{-2} = 1,\)
\(b_{-1} = b_1 = 0, \mu = -q\sqrt{-1}, \lambda = -q\sqrt{-1}.\)

(28) \[u_{14}(x,t) = -\frac{2q(1 - \sin(\sqrt{q}(x + qt)))}{1 + \cos(2\sqrt{q}(x + qt))} \quad (q > 0).\]

VII. Other interesting periodic solutions are

(29) \[u_{15}(x,t) = -\frac{m^2}{2r}\left(4 + r \csc^2\left(\frac{m}{2r}(rx + m^2(r + 12)t)\right)\right),\]

(30) \[u_{16}(x,t) = \frac{m^2}{2r}\left(4 - r \sec^2\left(\frac{m}{2r}(rx + m^2(r - 12)t)\right)\right),\]

(31) \[u_{17}(x,t) \]
\[= \frac{m^2(r^2 - 24 + 8\cos(2m(x - 2m^2t)) + 8(16 - r^2)\sin(m(x - 2m^2t)))}{2(r + 4\cos(m(x - 2m^2t)))^2}.\]
Finally, other interesting solitonic solutions are

\begin{align*}
(32) \quad u_{18}(x,t) & = \frac{\mu^2}{2r} \left( 4 - r \cosh^2 \left( \frac{\mu}{2r} (rx - \mu^2(r + 12)t) \right) \right), \\
(33) \quad u_{19}(x,t) & = -\frac{\mu^2}{2r} \left( 4 - r \text{sech}^2 \left( \frac{\mu}{2r} (rx - \mu^2(r - 12)t) \right) \right), \\
(34) \quad u_{20}(x,t) & = -\frac{r^2 + 24 + 8 \sqrt{16 + r^2} \cosh (x + 2t) + 8 \cosh (2(x + 2t))}{2 \left( r - 4 \sinh (x + 2t) \right)^2}.
\end{align*}

Figures 1 and 2 illustrate graphically some solutions: the soliton solution $u_2(x,t)$ and the periodic solution $u_{14}(x,t)$.

FIGURE 1: Graphic of $u_2(x,t)$ for $a_1 = 0$, $\mu = 2$, $-1 \leq t \leq 1$ and $-3 \leq x \leq 3$. 
FIGURE 2: Graphic of $u_{14}(x,t)$ for $q = 2$, $-3 \leq t \leq 3$ and $-3 \leq x \leq 3$.

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