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DYNAMICS OF A QUASILINEAR REACTION DIFFUSION EQUATION WITH SINGULAR REACTION FUNCTIONS

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ABSTRACT. Degenerate reaction-diffusion equations of porous-medium type with singular reaction functions in both the differential equation and the boundary condition are investigated. The aim of this paper is to show the existence of a unique global time-dependent solution, the existence and uniqueness of a positive steady-state solution, and the convergence of the time-dependent solution to the positive steady-state solution. The convergence result of the time dependent solution exhibits some rather interesting distinctive behavior when compared with density-independent diffusion.

1 Introduction Degenerate reaction-diffusion equations have been treated by many researchers in recent years, and most of the discussions are for the global existence and the finite-time blow-up property of the solution. In this paper we investigate the global existence and the asymptotic behavior of the time-dependent solution in relation to positive steady-state solutions for some degenerate reaction-diffusion equations where the reaction functions may be singular. The basic problem under consideration is an extension of the logistic reaction diffusion equation that is given in the form

$$\begin{aligned} & u_t - d(x)\,\Delta u^m + \mathbf{c}(x)\cdot\nabla u^m = au^p - bu^q && (t>0,\ x\in\Omega) \\ (1.1)\qquad & \frac{\partial u}{\partial \nu} + \beta u = \sigma(x)\,u^{-\gamma} && (t>0,\ x\in\partial\Omega) \\ & u(0,x) = u_0(x) && (x\in\Omega) \end{aligned}$$

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where Ω is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, $\partial u/\partial\nu$ is the outward normal derivative of u on $\partial\Omega$, and Δ and ∇ are the Laplacian and gradient operators in Ω. The physical parameters m, a, b, β and γ are positive constants, p and q are any constants (not necessarily positive) satisfying $p < q$, $d(x)$ and $u_0(x)$ are positive functions on $\bar{\Omega}$, $\sigma(x)$ is nonnegative and $\mathbf{c}(x) \equiv (c_1(x), \ldots, c_n(x))$ is an arbitrary C^α-function in Ω. In the above problem the term au^p represents an internal source, bu^q is an internal sink, and $\sigma u^{-\gamma}$ acts as a boundary source. This type of source-sink functions are singular at $u = 0$ if $p < 0$ or $q < 0$, and so is the boundary source when $\gamma > 0$. The consideration of the convection term $\mathbf{c} \cdot \nabla u^m$ in (1.1) adds no complication in the analysis, and it includes the case where the diffusion term is given in the form $\nabla \cdot (d(x)\nabla u^m)$. In this situation, it suffices to replace the convection coefficient $\mathbf{c}(x)$ by $(\mathbf{c}(x) - \nabla d(x))$.

To investigate the asymptotic behavior of the solution of (1.1) we need to study the corresponding steady-state problem

$$\begin{aligned} -d\,(x)\,\Delta u^m + \mathbf{c}\,(x) \cdot \nabla u^m &= au^p - bu^q \quad (x \in \Omega) \\ \frac{\partial u}{\partial \nu} + \beta u &= \sigma\,(x)\,u^{-\gamma} \quad (x \in \partial\Omega)\,. \end{aligned} \tag{1.2}$$

Without much additional complication we also treat the following more general problem

$$\begin{aligned} u_t - d\,(x)\,\Delta u^m + \mathbf{c}\,(x) \cdot \nabla u^m & \\ = au^p - \sum_{i=1}^{N} b_i u^{q_i} & \quad (t > 0,\ x \in \Omega) \\ \frac{\partial u}{\partial \nu} + \beta u = \sigma\,(x)\,u^{-\gamma} & \quad (t > 0,\ x \in \partial\Omega) \\ u\,(0, x) = u_0\,(x) & \quad (x \in \Omega) \end{aligned} \tag{1.3}$$

where b_i and q_i, $i = 1, \ldots, N$, are any constants that satisfy the condition

$$b_i > 0,\ p < q_1 \le q_2 \le \cdots \le q_N. \quad (i = 1, \ldots, N)\,. \tag{1.4}$$

The constants p and q_i can be positive or negative, integers or non-integers so long as they satisfy (1.4).

It is well-known that in the Verhulst logistic population growth problem (1.1) where $m = p = 1$, $q > 1$ and $\sigma\,(x) = 0$, the corresponding steady-state problem (1.2) has only the trivial solution $u = 0$ if $a \le \lambda_1$ and it has a unique positive solution $u_s^*(x)$ if $a > \lambda_1$, where $\lambda_1 > 0$ is the smallest eigenvalue of the eigenvalue problem

$$(1.5)\qquad \begin{aligned} -d(x)\,\Delta\phi + \mathbf{c}(x)\cdot\nabla\phi &= \lambda\phi \quad && (x \in \Omega) \\ \frac{\partial\phi}{\partial\nu} + m\beta\phi &= 0 && (x \in \partial\Omega) \end{aligned}$$

corresponding to $m = 1$. Moreover, for any nontrivial nonnegative initial function $u_0(x)$ the corresponding time-dependent solution $u(t,x)$ of (1.1) converges to 0 as $t \to \infty$ if $a \le \lambda_1$, and it converges to $u^*(x)$ if $a > \lambda_1$ (cf. [**19,** p. 201], or [**21**]). It is interesting to know what is the effect of the value m (in the diffusion term) on the asymptotic behavior of the time-dependent solution, especially if the reaction functions are singular. On the other hand, if $p < 0$ or $q < 0$ then the reaction function in (1.1) is singular at $u = 0$. In this situation it is important to know whether there is a relationship among the exponents m, p and q so that a unique global solution to (1.1) exists and converges to a steady-state solution of (1.2) as $t \to \infty$. In case the steady-state problem (1.2) has multiple solutions, it is desired to know whether the time-dependent solution still converges to a steady-state and to which one if it does. The purpose of this paper is to investigate: (1) the existence (and uniqueness) of positive solutions of (1.2) for any value of $m > 0$, including the singular case $p < 0$ or $q < 0$; (2) the global existence and attraction property of the time-dependent solution of (1.1); and (3) the convergence of the time-dependent solution to positive steady-state solutions. In particular, we show that if $p < m \le q$ then Problem (1.2) has a unique positive solution $u_s^*(x)$, and for any $u_0(x) > 0$ on $\overline{\Omega}$ the solution $u(t,x)$ of (1.1) converges to $u_s^*(x)$ as $t \to \infty$. The convergence property holds true for every constant $a > 0$, including the case $a < \lambda_1$ and $\sigma(x) = 0$, and it gives a sharp contrast to the case $m = p = 1 < q$. We also show that the above conclusions hold true for the more general problem (1.3).

Parabolic and elliptic boundary problems with singular reaction functions arises from various fields of applied sciences, and many of the discussions in the earlier literature are devoted to semilinear reaction diffusion equations where the reaction function is given in the form $c_0 u^{-\alpha}$ for some constant $\alpha > 0$, where c_0 is either positive or negative. In modeling the dynamics of van der Waals force driven thin films of viscous fluids, a mathematical model for the thickness u of the thin film is governed by (1.2) with $m = 1$ and with the reaction function $(a - bu^{-\alpha})$, a special case of (1.2) with $p = 0$ and $q = -\alpha$ (cf. [**8, 13**]). Similar equation of (1.2) with the reaction function $au^p + bu^q$, where $p > 0$, $q < 0$

was treated in [**12, 23, 26**]. As a limiting case of models in chemical kinetics (Langmuir-Hinshelwood model) the authors of [**5, 6, 14**] investigated the finite-time quenching and blow up of u_t and Δu for Problem (1.1) with the reaction function $(-bu^{-\alpha})$. On the other hand, in a diffusion process of a gas that is in contact with a liquid in which the gas is absorbed by the liquid at the gas-liquid interface, the gas density is governed by (1.1) with the singular boundary condition

$$\frac{\partial u}{\partial \nu} + u = \sigma u^{-\gamma} \quad (t > 0,\ x \in \partial\Omega).$$

The differential equation in (1.1) with $m = 1$, $\mathbf{c}(x) = \mathbf{0}$, $p = q = 0$ and the above boundary condition was treated in [**17**] for $a = b = 0$ in the semi-infinite interval $\Omega = (0, \infty)$, and in [**18**] in a bounded domain with a given source function $f(x)$.

There are also a large amount of work which are devoted to degenerate reaction diffusion problems of the form (1.1) but are mostly for the case where the reaction function is either a nonsingular source au^p $(p > 0,\ b = 0)$ or a nonsingular sink $(-bu^q)$ $(q > 0,\ a = 0)$. The works in [**7, 10, 15, 16, 24, 25**] considered only an internal source while those in [**3, 9, 11**] are for an internal sink. The domain in the above works are either the whole space $\mathbb{R}^n$ or a bounded domain with either homogeneous Dirichlet boundary condition or Neumann-Robin type boundary condition with nonsingular boundary function. The main objectives of the above works are the global existence and blow-up property of the solution. The works in [**1, 4, 9, 22, 27**] are also concerned with the decay property of the solution. In this paper, our main concerns are the existence of positive steady-state solutions and the asymptotic behavior of the time-dependent solution $u(t, x)$ in relation to these steady-state solutions, especially the convergence of $u(t, x)$ to a unique positive steady-state solution. These results are stated in Section 2, and their proofs are given in Section 3. A concluding remark is given in Section 4.

2 The main results To ensure the existence of a classical solution to (1.1) and (1.2) we make the following:
Hypotheses (H)

(i) The constants m, a, b, β and γ are positive, and p and q are any constants satisfying $p < \min\{m, q\}$.

(ii) $d(x)$ and $u_0(x)$ are positive, $\sigma(x)$ is nonnegative, and these functions together with $\mathbf{c}(x) \equiv (c_1(x), \ldots, c_n(x))$ are all C^α-functions in their respective domains, where $\alpha \in (0, 1)$.

(iii) Ω is of class $C^{1+\alpha}$.

Under the above hypotheses we have the following existence and uniqueness result for the steady-state problem (1.2).

Theorem 1. *Let Hypotheses* (H) *hold, where p and q can be positive, zero, or negative. Then Problem* (1.2) *has a positive minimal solution $\underline{u}_s(x)$ and a positive maximal solution $\bar{u}_s(x)$ such that $0 < \underline{u}_s(x) \leq \bar{u}_s(x)$ on $\overline{\Omega}$. If $m \leq q$ and either $\mathbf{c}(x) \equiv \mathbf{0}$ or $m \leq 1+\gamma$ then $\underline{u}_s(x) = \bar{u}_s(x)$ $(\equiv u_s^*(x))$ and $u_s^*(x)$ is the unique positive solution of* (1.2).

For the time-dependent problem (1.1) we have the following global existence and asymptotic behavior of the solution.

Theorem 2. *Let Hypotheses* (H) *hold, and let $\underline{u}_s(x)$, $\bar{u}_s(x)$ be the respective positive minimal and maximal solutions of* (1.2). *Then*

(i) *for any $u_0(x) > 0$ on $\overline{\Omega}$, a unique global solution $u(t,x)$ to* (1.1) *exists and possesses the property*

$$\underline{u}_s(x) \leq u(t,x) \leq \bar{u}_s(x) \quad \text{as } t \to \infty, \tag{2.1}$$

(ii) *$u(t,x)$ converges to $\underline{u}_s(x)$ as $t \to \infty$ if $u_0(x) \leq \underline{u}_s(x)$, and it converges to $\bar{u}_s(x)$ if $u_0(x) \geq \bar{u}_s(x)$, and*

(iii) *$u(t,x)$ converges to a unique positive steady-state solution $u_s^*(x)$ as $t \to \infty$ if $m \leq q$ and either $\mathbf{c}(x) \equiv \mathbf{0}$ or $m \leq 1+\gamma$.*

The results in Theorems 1 and 2 hold true also for the more general problem (1.3). Specifically we have

Theorem 3. *Let Hypotheses* (H) *hold except that b and q are replaced, respectively, by $(b_1, \ldots, b_N)$ and $(q_1, \ldots, q_N)$ which satisfy condition* (1.5). *Then*

(i) *the steady-state problem of* (1.3) *has a positive minimal solution $\underline{u}_s(x)$ and a positive maximal solution $\bar{u}_s(x)$ such that $0 < \underline{u}_s(x) \leq \bar{u}_s(x)$ on $\bar{\Omega}$,*

(ii) *for any $u_0(x) > 0$ on $\overline{\Omega}$, a unique global solution $u(t,x)$ to* (1.3) *exists and satisfies* (2.1),

(iii) *$u(t,x)$ converges to $\underline{u}_s(x)$ as $t \to \infty$ if $u_0(x) \leq \underline{u}_s(x)$ and it converges to $\bar{u}_s(x)$ if $u_0(x) \geq \bar{u}_s(x)$, and*

(iv) *$u(t,x)$ converges to a unique positive steady-state solution $u_s^*(x)$ as $t \to \infty$ if $m \leq q$ and either $\mathbf{c}(x) \equiv \mathbf{0}$ or $m \leq 1+\gamma$.*

In the above theorems it is assumed that γ and $\sigma(x)$ are both positive. If $\gamma = 0$ or $\sigma(x) \equiv 0$, then the boundary condition in (1.1) (or (1.3)) becomes

$$\frac{\partial u}{\partial \nu} + \beta u = \sigma(x) \text{ or } \frac{\partial u}{\partial \nu} + \beta u = 0 \quad (t > 0,\, x \in \partial\Omega). \tag{2.2}$$

In this situation, we have the following

Corollary. *Let the conditions in Theorem* 3 *hold and let either* $\gamma = 0$ *or* $\sigma(x) \equiv 0$. *Then all the conclusions in* (i), (ii), (iii), *and* (iv) *of Theorem* 3 *remain true. In particular, these results hold true for Problem* (1.1).

Remark. It is seen from Theorem 2 that if $p < m$ (and $p < q$) then for any $a > 0$ and any positive initial function $u_0(x)$ the solution $u(t,x)$ of (1.1) enters the sector $\langle \underline{u}_s, \bar{u}_s \rangle$ as $t \to \infty$, and if, in addition, $m \le q$ then $u(t,x)$ converges to a unique positive steady-state solution. This implies that for any $a > 0$ the time-dependent solution moves away from 0 so that the trivial solution $u_s = 0$ is unstable. This is in contrast to the case $m = p = 1$ and $q = 1$ where $u_s = 0$ is a global attractor.

3 Proofs of the theorems

The proofs of the theorems in the previous section are based on the method of upper and lower solutions developed in [**20, 21**] for a more general class of reaction functions $f(x,u)$, $g(x,u)$. For the present problems these reaction functions are given by

$$f(x,u) = au^p - bu^q, \quad g(x,u) = m(\sigma(x)\, u^{m-\gamma-1} - \beta u^m) \tag{3.1}$$

for Problem (1.1) and (1.2), and

$$f(x,u) = au^p - \sum_{i=1}^{N} b_i u^{q_i}, \quad g(x,u) = m(\sigma(x)\, u^{m-\gamma-1} - \beta u^m) \tag{3.2}$$

for Problem (1.3). Since problem (1.1) is a special case of (1.3) and since the global existence and the asymptotic behavior of the time-dependent solution are determined by the corresponding steady-state problem, we only give the definition of upper and lower solutions for the steady-state problem of (1.3).

Definition. A pair of functions $\tilde{u}_s$, $\hat{u}_s$ in $C^2(\Omega) \cap C(\overline{\Omega})$ are called ordered upper and lower solutions of the steady-state problem of (1.3) if $\tilde{u}_s \geq \hat{u}_s$ and if

$$\begin{aligned} -d(x)\,\Delta\tilde{u}^m + \mathbf{c}(x)\cdot\nabla\tilde{u}^m &\geq a\tilde{u}^p - \sum_{i=1}^{N} b_i\tilde{u}^{q_i} \qquad (x \in \Omega) \\ \frac{\partial\tilde{u}_s}{\partial\nu} + \beta\tilde{u}_s &\geq \sigma(x)\,\tilde{u}^{-\gamma} \qquad (x \in \partial\Omega) \end{aligned} \tag{3.3}$$

and $\hat{u}_s$ satisfies (3.3) with inequalities reversed.

In the above definition, the constants γ, b_i, $i = 2, \ldots, N$, and the function $\sigma(x)$ can be positive or zero. In particular, if $b_1 = b$, $q_1 = q$ and $b_i = 0$ for all $i = 2, \ldots, N$, then it reduces to the definition of upper and lower solutions for problem (1.2). On the other hand, if $\gamma = 0$ or $\sigma(x) = 0$ then the boundary condition in (1.3) is reduced to that in (2.2) so that this definition is applicable to the linear boundary condition (2.2). For a given pair of ordered upper and lower solutions $\tilde{u}_s$, $\hat{u}_s$, we set

$$S \equiv \langle \hat{u}_s, \tilde{u}_s \rangle \equiv \left\{ u \in C(\bar{\Omega})\,; \hat{u}_s \leq u \leq \tilde{u}_s \right\}.$$

By considering the reaction functions in (3.1) we state the following existence results from Theorem 3.1 of [**20**] for the present problem (1.2).

Theorem A. *Let $\tilde{u}_s$, $\hat{u}_s$ be a pair of ordered upper and lower solutions of* (1.2)*, and let Hypotheses* (H) *hold. Then Problem* (1.2) *has a positive minimal solution $\underline{u}_s$ and a positive maximal solution $\bar{u}_s$ such that $0 < \underline{u}_s \leq \bar{u}_s$ on $\overline{\Omega}$. If $\underline{u}_s = \bar{u}_s$ ($\equiv u_s^*$), then u_s^* is the unique positive solution of* (1.2) *in S.*

For the time-dependent problem (1.1) we have the following conclusions from Theorems 2.1, 5.1 and 5.2 of [**21**].

Theorem B. *Let the conditions in Theorem* A *hold, and let $\underline{u}_s(x)$, $\bar{u}_s(x)$ be the respective positive minimal and maximal solutions of* (1.2)*. Then*

(i) *for any $u_0 \in S$, a unique global solution $u(t,x)$ to* (1.1) *exists and possesses the property*

$$\underline{u}_s(x) \leq u(t,x) \leq \bar{u}_s(x) \quad \text{as } t \to \infty, \tag{3.4}$$

(ii) *$u(t,x)$ converges to $\underline{u}_s(x)$ as $t \to \infty$ if $\hat{u}_s(x) \le u_0(x) \le \underline{u}_s(x)$, and it converges to $\bar{u}_s(x)$ if $\bar{u}_s(x) \le u_0(x) \le \tilde{u}_s(x)$, and*

(iii) *$u(t,x) \to u_s^*(x)$ as $t \to \infty$ if $\bar{u}_s(x) = \underline{u}_s(x)$ ($\equiv u_s^*(x)$).*

By using the reaction functions in (3.2) instead of (3.1) the results in Theorems A and B hold true for Problem (1.3) provided that the constants b, q in (H) are replaced, respectively, by $(b_1, \ldots, b_N)$ and $(q_1, \ldots, q_N)$ that satisfy condition (1.4).

Proof of Theorem 1. By Theorem A the existence of positive minimal and maximal solutions is ensured if there exist a pair of ordered positive upper and lower solutions. It is easy to see from the definition that for any constant M satisfying

$$M \ge \max\left\{ \left(\frac{a}{b}\right)^{1/r}, \left(\frac{\bar{\sigma}}{\beta}\right)^{1/(1+\gamma)} \right\}, \tag{3.5}$$

where $r \equiv q - p > 0$ and $\bar{\sigma} = \max\left\{\sigma(x) : x \in \overline{\Omega}\right\}$, $\tilde{u}_s = M$ is an upper solution. We seek a positive lower solution in the form $\hat{u}_s = (\delta\phi_m)^{1/m}$ for a sufficiently small constant $\delta > 0$, where ϕ_m is the (normalized) positive eigenfunction of (1.5) corresponding to the smallest eigenvalue $\lambda_m > 0$. Indeed, since $\hat{u}_s^m = \delta\phi_m$ and ϕ_m is strictly positive on $\overline{\Omega}$, $\hat{u}_s$ is a positive lower solution of (1.2) if

$$\begin{aligned} &- d(x)\,\Delta(\delta\phi_m) + \mathbf{c}(x)\cdot\nabla(\delta\phi_m) \\ &\qquad\qquad \le (\delta\phi_m)^{p/m}\left[a - b(\delta\phi_m)^{r/m}\right] \quad (x \in \Omega) \\ &\frac{1}{m}(\delta\phi_m)^{1/m-1}\frac{\partial}{\partial\nu}(\delta\phi_m) + \beta(\delta\phi_m)^{1/m} \\ &\qquad\qquad \le \sigma(x)(\delta\phi_m)^{-\gamma/m} \qquad\qquad (x \in \partial\Omega). \end{aligned}$$

In view of (1.5), the above relation is equivalent to

$$\begin{aligned} \lambda_m(\delta\phi_m) &\le (\delta\phi_m)^{p/m}\left[a - b(\delta\phi_m)^{r/m}\right] \quad (x \in \Omega) \\ \delta\left(\frac{\partial\phi_m}{\partial\nu} + m\beta\phi_m\right) &\le m\sigma(x)(\delta\phi_m)^{1-(1+\gamma)/m} \qquad (x \in \partial\Omega). \end{aligned} \tag{3.6}$$

It is clear from $a > 0$ and $p < m$ that the first inequality holds by a sufficiently small $\delta > 0$, while the second inequality is trivially satisfied by any $\delta > 0$. This shows that $\tilde{u}_s = M$ and $\hat{u}_s = (\delta\phi_m)^{1/m}$ are ordered upper and lower solutions of (1.2). The existence of minimal and maximal solutions $\underline{u}_s$, $\bar{u}_s$ and the relation $0 < \underline{u}_s \le \bar{u}_s$ follows from Theorem A.

To show the uniqueness of the positive solution we observe from (3.1) and the assumption $p < m \le q$ that

$$\begin{aligned}\frac{\partial}{\partial u}\left[\frac{f(x,u)}{u^m}\right] &= \frac{\partial}{\partial u}\left[u^{p-m}(a - bu^r)\right]\\ &= a(p-m)u^{p-m-1} - b(q-m)u^{q-m-1} < 0,\\ \frac{\partial}{\partial u}\left[\frac{g(x,u)}{u^m}\right] &= m\frac{\partial}{\partial u}\left[\sigma(x)u^{-(1+\gamma)} - \beta\right]\\ &= -m(1+\gamma)\sigma(x)u^{-(2+\gamma)} < 0\end{aligned} \tag{3.7}$$

for all $u > 0$. This implies that both $f(x,u)/u^m$ and $g(x,u)/u$ are decreasing functions of $u > 0$. By Theorem 3.3 of [**20**], $\underline{u}_s = \bar{u}_s$ $(\equiv u_s^*)$ and u_s^* is the unique positive solution in the sector $\left\langle (\delta\phi_m)^{1/m}, M\right\rangle$ if $\mathbf{c}(x) \equiv \mathbf{0}$.

To show the result for the case $m \le 1 + \gamma$ and any $\mathbf{c}(x) \ne \mathbf{0}$ we use a result of [**2**] (Theorem 3.2), which is a version of the Krein-Rutman Theorem. Choose a constant $A > 0$ such that

$$A \ge \max_{\substack{\hat{u}^m(x) \le w \le \tilde{u}^m(x)\\ x\in\Omega}} \left\{aw^{\frac{p}{m}-1} - bw^{\frac{q}{m}-1}\right\}.$$

Let $K : C^{\alpha}(\overline{\Omega}) \mapsto C^{2+\alpha}(\overline{\Omega})$ be the solution operator such that $w = K[f]$ is the solution of the boundary-value problem

$$\begin{aligned}-d(x)\Delta w + \mathbf{c}(x)\cdot\nabla w + Aw = f &\quad \text{in } \Omega,\\ \frac{\partial w}{\partial\nu} + m\beta w = 0 &\quad \text{on } \partial\Omega,\end{aligned}$$

and let $R : C^{2+\alpha-\mu}(\partial\Omega) \mapsto C^{2+\alpha}\left(\overline{\Omega}\right)$ be the solution operator such that $v = R[g]$ is the solution of the boundary-value problem

$$\begin{aligned}-d(x)\Delta v + \mathbf{c}(x)\cdot\nabla v + Av = 0 &\quad \text{in } \Omega,\\ \frac{\partial v}{\partial\nu} + m\beta v = g &\quad \text{on } \partial\Omega,\end{aligned}$$

where $0 < \mu < 1$ is a positive constant. By the maximum principle, both K and R are strongly positive operators in the sense that if $f \gneqq 0$

(resp. $g \gneqq 0$) then $K[f]$ (resp. $R[g]$) lies in the positive cone of $C^\alpha\left(\bar{\Omega}\right)$. Furthermore, K is a compact operator in the ordered Banach space $C^\alpha\left(\bar{\Omega}\right)$. Hence the following conclusions from [**2,** Theorem 3.2(iv)] are true:

> If λ is a constant, $F(x)$ is a positive $C^\alpha\left(\overline{\Omega}\right)$ function, and $y \in C^\alpha\left(\overline{\Omega}\right)$ is a nonnegative function, and if the equation
>
> $$\lambda v = K[Fv] + y \tag{3.8}$$
>
> has a positive solution, then $\lambda \geq r(KF)$, where $r(KF)$ is the spectral radius of the operator KF. Conversely, if $\lambda > r(KF)$ then for any nonnegative (resp. nonpositive) nontrivial function $y \in C^\alpha\left(\overline{\Omega}\right)$, the above equation (3.8) has a unique positive (resp. negative) solution.

In particular, if $\lambda = 1$, $r(KF) < 1$ and $y \leq 0$, then the equation (3.8) has a unique solution v which is nonpositive.

To show that the two steady-state solutions $\bar{u}_s$ and $\underline{u}_s$ are equal, we observe that the functions $\bar{w}_s \equiv \bar{u}_s^m$ and $\underline{w}_s \equiv \underline{u}_s^m$ satisfy the equations

$$\begin{aligned} \bar{w}_s &= K[(f^*(\bar{w}_s) + A)\bar{w}_s] + R[g^*(x, \bar{w}_s)], \\ \underline{w}_s &= K[(f^*(\underline{w}_s) + A)\underline{w}_s] + R[g^*(x, \underline{w}_s)] \end{aligned} \tag{3.9}$$

where

$$f^*(w) = aw^{\frac{p}{m}-1} - bw^{\frac{q}{m}-1}, \quad g^*(x, w) = m\sigma(x)\, w^{(m-1-\gamma)/m}. \tag{3.10}$$

Hence $v = \underline{w}_s$ is a positive solution of (3.8) with $\lambda = 1$, $F = f^*(\underline{w}_s) + A$ and $y = R[g^*(x, \underline{w}_s)]$. Since $R[g^*(x, \underline{w}_s)] > 0$, it follows that $1 > r(KF)$. Consider the function $w = \bar{w}_s - \underline{w}_s$. By subtracting equations in (3.9), it is easy to verify that w is a solution of (3.8) with $\lambda = 1$, $F = f^*(\underline{w}_s) + A$ and

$$y = K[\bar{w}_s(f^*(\bar{w}_s) - f^*(\underline{w}_s))] + R[g^*(x, \bar{w}_s) - g^*(x, \underline{w}_s)].$$

Since $p < m \leq \{q, 1+\gamma\}$, both f^* and g^* are nonincreasing in positive w. Hence the inequalities $\bar{w}_s \geq \underline{w}_s \geq 0$ ensure that $y \leq 0$. This and $1 > r(KF)$ imply that $w \leq 0$, which leads to $\bar{w}_s \leq \underline{w}_s$. Hence the two solutions are equal. □

Proof of Theorem 2. (i) Given any $u_0(x) > 0$ on $\bar{\Omega}$ there exist positive constants δ, M such that $(\delta\phi_m)^{1/m} \le u_0(x) \le M$. This implies that $u_0 \in S$, where $S = \left\langle (\delta\phi_m)^{1/m}, M \right\rangle$. By Theorem B, a unique global solution $u(t,x)$ to (1.1) exists and possesses the property (2.1).

(ii) Again by Theorem B, the solution $u(t,x)$ converges to $\underline{u}_s(x)$ as $t \to \infty$ if $(\delta\phi_m)^{1/m} \le u_0 \le \underline{u}_s$, and it converges to $\bar{u}_s(x)$ if $\bar{u}_s \le u_0 \le M$. The arbitrariness of δ and M leads to the conclusion in (ii).

(iii) By Theorem 1, $\underline{u}_s = \bar{u}_s (\equiv u_s^*)$ when $p < m \le q$ and either $\mathbf{c}(x) \equiv \mathbf{0}$ or $m \le 1 + \gamma$. The convergence of $u(t,x)$ to $u_s^*(x)$ follows from (2.1). This proves the theorem. □

Proof of Theorem 3. To prove Theorem 3 we again consider Problem (1.3) as a special case of the problem treated in [**20**] where the reaction functions $f(x,u)$, $g(x,u)$ are given by (3.2). It is obvious from (3.2) that for any constant M satisfying

$$M \ge \max\left\{ \left(\frac{a}{b_1}\right)^{1/r_1}, \left(\frac{\bar{\sigma}}{\beta}\right)^{1/(1+\gamma)} \right\}, \tag{3.11}$$

where $r_i = q_i - p > 0$ for $i = 1, \ldots, N$, $\tilde{u}_s = M$ is an upper solution of the steady-state problem of (1.3). Moreover, $\hat{u}_s = (\delta\phi_m)^{1/m}$ is a lower solution if the first inequality in (3.6) is replaced by

$$\lambda_m (\delta\phi_m) \le (\delta\phi_m)^{p/m} \left[a - \sum_{i=1}^{N} b_i (\delta\phi_m)^{r_i/m} \right].$$

Since the above inequality is equivalent to

$$\lambda_m (\delta\phi_m)^{(m-p)/m} \le a - \sum_{i=1}^{N} b_i (\delta\phi_m)^{r_i/m}$$

we see from $m > p$ that it is satisfied by a sufficiently small $\delta > 0$. This shows that the pair $\tilde{u}_s = M$ and $\hat{u}_s = (\delta\phi_m)^{1/m}$ are ordered positive upper and lower solutions. The conclusions in (i), (ii) and (iii) of the theorem follow from Theorems A and B for the steady-state problem of (1.3).

To show the uniqueness of the positive steady-state solution u_s^* we observe from (3.2), (1.4) and $p < m \le q_1$ that

$$\frac{\partial}{\partial u}\left[\frac{f(x,u)}{u^m}\right] = a(p-m)u^{p-m-1} - \sum_{i=1}^{N} b_i (q_i - m) u^{q_i-m-1} < 0$$

for $u > 0$. This shows that $f(x,u)/u^m$ is decreasing in u for $u > 0$. Since $g(x,u)/u^m$ is also decreasing in $u > 0$ we conclude from Theorem 3.3 of [**20**] that $\underline{u}_s = \bar{u}_s (\equiv u_s^*)$. To show this result for the case $m \leq 1+\gamma$ we use the relation (3.10) where $f^*(w)$ is given by

$$f^*(w) = aw^{p/m-1} - \sum_{i=1}^{N} b_i w^{q_i/m-1}.$$

Since

$$f_w^*(w) = a\left(\frac{p}{m}-1\right) w^{p/m-2} - \sum_{i=1}^{N} b_i \left(\frac{q_i}{m}-1\right) w^{q_i/m-2}$$

and $p < m \leq q_1 \leq q_i$ for $i = 2, \ldots, N$, we see that $f_w(w) < 0$ for $w > 0$. It follows from the same argument as that in the proof of Theorem 1 that $\underline{u}_s = \bar{u}_s \equiv u_s^*$ and u_s^* is the unique positive steady-state solution of (1.3). The conclusion in (iv) of the theorem follows again from Theorem B. □

Proof of the corollary. It is clear from the proof of Theorem 3 that $\tilde{u}_s = M$ and $\hat{u}_s = (\delta\phi_m)^{1/m}$ remain to be ordered upper and lower solutions of the steady-state problem of (1.3) when either $\gamma = 0$ or $\sigma(x) \equiv 0$, where M satisfies (3.11) with $\bar{\sigma} = 0$ if $\sigma(x) \equiv 0$. Since $f(x,u)/u^m$ is decreasing in $u > 0$ and $g(x,u)/u^m = m(\sigma(x)/u - \beta)$ or $-m\beta$, depending on $\gamma = 0$ or $\sigma(x) = 0$, we see that $g(x,u)/u^m$ is nonincreasing in $u > 0$. It follows again from Theorem 3.3 of [**20**] that $\underline{u}_s = \bar{u}_s$ $(\equiv u_s^*)$ and u_s^* is the unique positive steady-state solution. A similar argument as in the proof of Theorem 1 shows that $\underline{u}_s = \bar{u}_s$ $(\equiv u_s^*)$ if $m \leq 1+\gamma$. The conclusion of the corollary follows from Theorems A and B. □

4 Concluding remarks The discussion in the previous sections demonstrates that the method of upper and lower solutions leads to not only existence results for the time-dependent problem (1.1) and its corresponding steady-state problem (1.2) but also the dynamic behavior of the quasilinear reaction-diffusion problem. Specifically, through the construction of a pair of ordered upper and lower solutions of the elliptic boundary problem (1.2) it is ensured to have a maximal solution $\bar{u}_s$ and a minimal solution $\underline{u}_s$. Moreover the sector $\langle \underline{u}_s, \bar{u}_s \rangle$ between $\bar{u}_s$ and $\underline{u}_s$ is a global attractor of the parabolic problem (1.1) (in relation to $\langle \hat{u}_s, \tilde{u}_s \rangle$). This means that starting from any initial function u_0 in $\langle \hat{u}_s, \tilde{u}_s \rangle$ the corresponding solution $u(t,x)$ of (1.1) enters the sector $\langle \underline{u}_s, \bar{u}_s \rangle$ as $t \to \infty$.

In particular, if the maximal solution coincides with the minimal solution, as in the case $m < \min\{q, 1+\gamma\}$, then their common value $u_s^*(x)$ is a global attractor of the quasilinear parabolic problem. In terms of Lyapunov stability theory this means that $u_s^*(x)$ is asymptotically stable with a stability region $\langle \hat{u}_s, \tilde{u}_s \rangle$. Since the quasilinear elliptic problem (1.2) can be reduced to a semilinear elliptic problem, upper and lower solutions for (1.2) can often be constructed by using the techniques or known results for semilinear equations. This approach can also be used to study the dynamics of other quasilinear parabolic equations with singular or nonsingular reaction functions.

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