CANADIAN APPLIED
MATHEMATICS QUARTERLY
Volume 16, Number 1, Spring 2008

ASYMPTOTIC BEHAVIOR AND OSCILLATION OF SOLUTIONS OF THIRD-ORDER NONLINEAR NEUTRAL DELAY DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT. In this paper, we consider the third-order nonlinear neutral delay dynamic equations

$$\Big\{B(t)\big[A(t)\big(y(t)+p(t)y(\tau(t))\big)^{\Delta}\big]^{\Delta}\Big\}^{\Delta} + \int_a^b F(t,\xi,y(g(t,\xi)))\,\Delta\xi = 0,$$

on an arbitrary time scale $\mathbb{T}$ which is unbounded above. We establish some sufficient conditions which ensure that every solution of the above equations oscillates or converges to zero. To the best of our knowledge nothing is known regarding the qualitative behavior of these equations on time scales up to now, so this paper initiates the study. Not only are our results in this paper essentially new, but they also extend, improve and unify some known results of asymptotic behavior and oscillation of solutions for corresponding third-order nondelay dynamic equations on a time scale $\mathbb{T}$, third-order delay differential equations on $\mathbb{R}$ and third-order delay difference equations on $\mathbb{Z}$. Several examples are given to illustrate the applicability of our main results.

1 Introduction The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D. Thesis [**7**] in order to unify continuous and discrete analysis. Not only can this theory of the so-called "dynamic equations" unify the theories of differential equations and difference equations, but also it is able to

This work was supported by the Scientific Research Foundation of Education Department of Hunan Province of P. R. China (No. 06C242).
AMS subject classification: 34K11, 34K40, CLC number: O175.12.
Keywords: Asymptotic behavior, oscillation, third-order nonlinear neutral delay dynamic equation, time scale, Riccati transformation technique.

extend these classical cases to cases "in between," e.g., to the so-called q-difference equations. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications (see [**2**]).

Since Stefan Hilger formed the definition of derivatives and integrals on time scales, several authors have expounded on various aspects of this new theory, see the paper by Agarwal *et al.* [**1**] and the references cited therein. The books on the subject of time scale, i.e., measure chain, by Bohner and Peterson [**2, 3**] summarize and organize much of time scale calculus.

During recent years, there have been a number of results of the asymptotic behavior and oscillation of solutions for dynamic equations on time scales, and we refer the reader to the papers [**4, 6, 8, 9, 10**] and the references cited therein. However, most of the results obtained has centered around second-order dynamic equations on time scales, and there is very little known about the qualitative behavior of third-order dynamic equations on time scales.

The aim of this paper is to study the asymptotic behavior and oscillation of solutions of the third-order nonlinear neutral delay dynamic equations

$$\Big\{B(t)\big[A(t)\big(y(t)+p(t)y(\tau(t))\big)^{\Delta}\big]^{\Delta}\Big\}^{\Delta} + \int_a^b F(t,\xi,y(g(t,\xi)))\,\Delta\xi = 0, \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$, where $\sup\mathbb{T}=\infty, a,b\in\mathbb{T}$ and $a<b$.

We will make use of the following conditions in this paper:

(H1.1) $t_0\in\mathbb{T}$, $\mathbb{I}$ denotes the time scale interval $[t_0,\infty)$, that is, $\mathbb{I}=\{t : t\ge t_0, t\in\mathbb{T}\cap\mathbb{R}\}$, $A,B\in C_{rd}(\mathbb{I},\mathbb{R})$, $A(t),B(t)>0$, and

$$\int_{t_0}^{\infty}\frac{1}{A(t)}\,\Delta t=\int_{t_0}^{\infty}\frac{1}{B(t)}\,\Delta t=\infty;$$

(H1.2) $p\in C_{rd}(\mathbb{I},\mathbb{R})$ and $0\le p(t)\le 1$;

(H1.3) $\tau\in C_{rd}(\mathbb{I},\mathbb{T})$, $\tau(t)\le t$ and $\lim_{t\to\infty}\tau(t)=\infty$;

(H1.4) $[a,b]\subset\mathbb{T}$ is a time scale interval, $g(t,\xi)\in C_{rd}(\mathbb{I}\times[a,b],\mathbb{T})$, $g(t,a)\le g(t,\xi)\le t$ for $(t,\xi)\in\mathbb{I}\times[a,b]$, and $\lim_{t\to\infty}g(t,a)=\infty$;

(H1.5) $F \in C(\mathbb{I} \times [a,b] \times \mathbb{R}, \mathbb{R})$, and there exists a function $q(t,\xi) \in C_{rd}(\mathbb{I} \times [a,b], \mathbb{R})$, which is nonnegative and not identically zero on any half-line $[t_q, \infty) \times [a,b]$, $t_q \geq t_0$, such that

$$F(t,\xi,y)\operatorname{sgn} y \geq q(t,\xi)\, y \operatorname{sgn} y, \text{ for } (t,\xi) \in \mathbb{I} \times [a,b],\ y \in \mathbb{R} \text{ and } y \neq 0.$$

Recently, Erbe *et al.* [**6**] were concerned with a special case of (1.1), the third-order nonlinear nondelay dynamic equation

$$\Big(c(t)\big(a(t)y^{\Delta}(t)\big)^{\Delta}\Big)^{\Delta} + q(t)f(y(t)) = 0, \quad t \geq t_0, \tag{1.2}$$

where the functions $c(t)$, $a(t)$ and $q(t)$ are positive, real-valued, rd-continuous functions defined on the time scale interval $[t_0, \infty)$ and the following conditions hold. $f : \mathbb{R} \to \mathbb{R}$ is continuous with $uf(u) > 0$, $u \neq 0$ and satisfies the following condition: For each $k > 0$ there exists $M = M_k > 0$ such that

$$\frac{f(u)}{u} \geq M, \qquad |u| \geq k. \tag{1.3}$$

Erbe *et al.* [**6**] established some sufficient conditions which guarantee that every solution $x(t)$ of (1.2) is oscillatory or $\lim_{t\to\infty} x(t)$ exists (finite).

Also in 2005, Candan and Dahiya [**5**] studied another special case of (1.1), the third-order neutral functional differential equation on $\mathbb{R}$ with distributed deviating arguments

$$\Big\{a(t)\big[b(t)\big(y(t) + p(t)y(t-\tau)\big)'\big]'\Big\}' + \int_a^b q(t,\xi)y(g(t,\xi))\, d\xi = 0, \qquad t \geq t_0, \tag{1.4}$$

and got several sufficient conditions which ensure every solution of (1.4) is oscillatory or tends to zero as $t \to \infty$.

To the best of our knowledge nothing is known regarding the qualitative behavior of (1.1) on time scales up to now. By employing the Riccati transformation technique and the integral averaging technique, we obtain several sufficient conditions which guarantee that every solution of (1.1) either oscillates or tends to zero. Not only are our results in this paper essentially new, but they also extend, improve and unify some known results of asymptotic behavior and oscillation of solutions for corresponding third-order nondelay dynamic equations on a time scale $\mathbb{T}$, third-order delay differential equations on $\mathbb{R}$ and third-order delay difference equations on $\mathbb{Z}$, for example, some results of Erbe *et al.* [**6**] and Candan and Dahiya [**5**]. Some examples are shown to illustrate the applicability of our main results.

Recall that a solution of (1.1) is a nontrivial real function $y(t)$ such that

$$y(t) + p(t)y(\tau(t)) \in C^1_{rd}[t_y, \infty),$$
$$A(t)(y(t) + p(t)y(\tau(t)))^{\Delta} \in C^1_{rd}[t_y, \infty)$$

and

$$B(t)[A(t)(y(t) + p(t)y(\tau(t)))^{\Delta}]^{\Delta} \in C^1_{rd}[t_y, \infty)$$

for $t_y \geq t_0$ and satisfying (1.1) for $t \geq t_y$. Our attention is restricted to those solutions of (1.1) which exist on some half-line $[t_y, \infty)$ and satisfy $\sup\{|y(t)| : t > t_1\} > 0$ for any $t_1 \geq t_y$. A solution $y(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

2 Some lemmas

Lemma 2.1. *Suppose conditions* (H1.1)–(H1.5) *hold and* $y(t)$ *is an eventually positive solution of* (1.1). *Let*

$$x(t) = y(t) + p(t)y(\tau(t)), \quad t \in [t_0, \infty). \tag{2.1}$$

Then there exists a $t_* \in [t_0, \infty)$ *such that either*

(i) $x(t) > 0$, $x^{\Delta}(t) > 0$, $[A(t)x^{\Delta}(t)]^{\Delta} > 0$, $t \in [t_*, \infty)$,

or

(ii) $x(t) > 0$, $x^{\Delta}(t) < 0$, $[A(t)x^{\Delta}(t)]^{\Delta} > 0$, $t \in [t_*, \infty)$.

Proof. Let $y(t)$ be an eventually positive solution of (1.1), then there is a $t_1 \in [t_0, \infty)$ such that $y(t) > 0$ for $t \in [t_1, \infty)$. From (H1.2)–(H1.4) and (2.1), there exists a $t_2 \in [t_1, \infty)$ such that

$$\tau(t) \geq t_1, \quad g(t, \xi) \geq t_1, \quad t \in [t_2, \infty), \quad \xi \in [a, b], \tag{2.2}$$

and

$$y(\tau(t)) > 0, \quad y(g(t,\xi)) > 0,$$
$$x(t) \geq y(t) > 0, \quad t \in [t_2, \infty), \quad \xi \in [a, b]. \tag{2.3}$$

It follows from (1.1), (H1.5), (2.1) and (2.3) that for $t \in [t_2, \infty)$,

$$\begin{aligned} \left\{B(t)\left[A(t)x^{\Delta}(t)\right]^{\Delta}\right\}^{\Delta} &= -\int_a^b F(t,\xi,y(g(t,\xi)))\,\Delta\xi \\ &\leq -\int_a^b q(t,\xi)y(g(t,\xi))\Delta\xi \leq 0. \end{aligned} \tag{2.4}$$

Hence $B(t)[A(t)x^{\Delta}(t)]^{\Delta}$ is decreasing on $[t_2, \infty)$. We claim that

$$B(t)[A(t)x^{\Delta}(t)]^{\Delta} > 0, \quad t \in [t_2, \infty). \tag{2.5}$$

Assume the contrary, then there is a $t_3 \in [t_2, \infty)$ such that

$$B(t_3)(Ax^{\Delta})^{\Delta}(t_3) \leq 0.$$

Since

$$B(t)[A(t)x^{\Delta}(t)]^{\Delta} \leq B(t_3)(Ax^{\Delta})^{\Delta}(t_3), \quad t \in [t_3, \infty),$$

and $\int_a^b F(t,\xi,y(g(t,\xi)))\,\Delta\xi$ is not identically zero on any half-line $[t_q, \infty)$, it is clear that there exists a $t_4 \geq t_3$ such that

$$B(t_4)(Ax^{\Delta})^{\Delta}(t_4) := c_1 < B(t_3)(Ax^{\Delta})^{\Delta}(t_3) \leq 0.$$

Therefore, for $t \in [t_4, \infty)$ we have

$$B(t)[A(t)x^{\Delta}(t)]^{\Delta} \leq B(t_4)(Ax^{\Delta})^{\Delta}(t_4) = c_1 < 0$$

and

$$[A(t)x^{\Delta}(t)]^{\Delta} \leq \frac{c_1}{B(t)} < 0.$$

Integrating from t_4 to t, we obtain

$$A(t)x^{\Delta}(t) \leq A(t_4)x^{\Delta}(t_4) + c_1 \int_{t_4}^{t} \frac{1}{B(s)}\,\Delta s$$

for $t \in [t_4, \infty)$. Letting $t \to \infty$, then $A(t)x^\Delta(t) \to -\infty$ by (H1.1). Since $A(t)x^\Delta(t)$ is decreasing on $[t_4, \infty)$, there is a $t_5 \geq t_4$ such that

$$A(t)x^\Delta(t) \leq A(t_5)x^\Delta(t_5) < 0, \quad t \in [t_5, \infty).$$

Dividing by $A(t)$ and integrating from t_5 to t, we obtain

$$x(t) - x(t_5) \leq A(t_5)x^\Delta(t_5) \int_{t_5}^{t} \frac{1}{A(s)} \Delta s, \quad t \in [t_5, \infty),$$

which implies that $x(t) \to -\infty$ as $t \to \infty$ by (H1.1), a contradiction with (2.3). Hence (2.5) holds. Since $B(t) > 0$, we obtain

$$[A(t)x^\Delta(t)]^\Delta > 0, \quad t \in [t_2, \infty).$$

This implies that $A(t)x^\Delta(t)$ is strictly increasing on $[t_2, \infty)$. It follows from this that either $A(t)x^\Delta(t) < 0$ on $[t_2, \infty)$ or $A(t)x^\Delta(t)$ is eventually positive and the proof is complete. □

Lemma 2.2. *Suppose conditions* (H1.1)–(H1.5) *hold,* $y(t)$ *is an eventually positive solution of* (1.1) *and* $x(t)$ *is defined by* (2.1) *that satisfies Case* (i) *in Lemma* 2.1. *Then there exists a* $t_2 \in [t_0, \infty)$ *such that*

$$x^\Delta(t) \geq \frac{B(t)}{A(t)} \left(\int_{t_2}^{t} \frac{1}{B(\xi)} \Delta\xi \right) \left[A(t)x^\Delta(t)\right]^\Delta, \quad t \in [t_2, \infty). \tag{2.6}$$

Proof. By the assumption of Lemma 2.2 and the proof of Lemma 2.1, there is a $t_2 \in [t_0, \infty)$ such that

$$x(t) > 0, \quad x^\Delta(t) > 0, \quad [A(t)x^\Delta(t)]^\Delta > 0, \quad t \in [t_2, \infty),$$

and (2.4) hold. From (2.4) we see that $B(t)\left[A(t)x^\Delta(t)\right]^\Delta$ is decreasing on $[t_2, \infty)$. Thus, we have

$$\begin{aligned}
A(t)x^\Delta(t) &= A(t_2)x^\Delta(t_2) + \int_{t_2}^{t} [A(\xi)x^\Delta(\xi)]^\Delta \, \Delta\xi \\
&= A(t_2)x^\Delta(t_2) + \int_{t_2}^{t} \frac{B(\xi)\left[A(\xi)x^\Delta(\xi)\right]^\Delta}{B(\xi)} \, \Delta\xi \\
&\geq A(t_2)x^\Delta(t_2) + B(t)[A(t)x^\Delta(t)]^\Delta \int_{t_2}^{t} \frac{1}{B(\xi)} \, \Delta\xi \\
&\geq B(t)[A(t)x^\Delta(t)]^\Delta \int_{t_2}^{t} \frac{1}{B(\xi)} \, \Delta\xi
\end{aligned}$$

for $t \in [t_2, \infty)$, and this leads to (2.6). The proof is complete. □

Lemma 2.3 (Bohner and Peterson [**2,** p. 34, Theorem 1.90])**.** *Assume $g : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := g(\mathbb{T})$ is a time scale. By $\tilde{\Delta}$ we denote the (delta) derivative on $\widetilde{\mathbb{T}}$. Let $f : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $g^{\Delta}(t)$ and $f^{\tilde{\Delta}}(g(t))$ exist for $t \in \mathbb{T}^{\kappa}$, then $(f \circ g)^{\Delta} = (f^{\tilde{\Delta}} \circ g)g^{\Delta}$.*

Lemma 2.4 (Bohner and Peterson [**2,** p. 29, Theorem 1.76 (ii)])**.** *Assume $a, b \in \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$ is rd-continous. If $[a, b]$ consists of finitely many isolated points and $a < b$, then*

$$\int_a^b f(t)\,\Delta t = \sum_{t\in[a,b)} f(t)(\sigma(t) - t), \tag{2.7}$$

where $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ is the forward jump operator on $\mathbb{T}$.

3 Main results In this section, we establish some sufficient conditions which guarantee that every solution $y(t)$ of (1.1) oscillates on $[t_0, \infty)$ or converges to zero as $t \to \infty$, and extend and improve the results of Erbe *et al.* [**6**] and Candan and Dahiya [**5**]. Moreover, we obtain some results for (1.2), which supplement and perfect the results of Erbe *et al.* [**6**].

Theorem 3.1. *Suppose that conditions* (H1.1)–(H1.5) *hold. If there is a constant $\lambda > 0$ and a sufficiently large constant C such that*

$$\limsup_{t\to\infty} \int_C^t \int_a^b q(s,\xi)[1 - (1 + \lambda)p(g(s,\xi))]\,\Delta\xi\,\Delta s = \infty, \tag{3.1}$$

then every solution of (1.1) is oscillatory or tends to zero as $t \to \infty$.

Proof. Suppose that $y(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, assume that $y(t)$ is an eventually positive solution of (1.1). We now prove that $y(t)$ tends to zero as $t \to \infty$. Define $x(t)$ as in (2.1). By Lemma 2.1 either Case (i) or Case (ii) in Lemma 2.1 holds.

First of all, assume $x(t)$ satisfies Case (i) in Lemma 2.1. We proceed as in the proof of Lemma 2.1 to obtain that there exists a $t_2 \in [t_*, \infty)$ such that (2.3) and (2.4) hold for $t \in [t_2, \infty)$. From (H1.3) and (H1.4), there exists a $t_3 \in [t_2, \infty)$ such that

$$t_2 \le \tau(g(t, \xi)) \le g(t, \xi),\ t \in [t_3, \infty), \quad \xi \in [a, b]. \tag{3.2}$$

Then from (H1.2), (2.1), (2.3) and the condition $x^\Delta(t) > 0$ on $[t_*, \infty)$, we see that

$$
\begin{aligned}
(3.3) \qquad y(g(t,\xi)) &= x(g(t,\xi)) - p(g(t,\xi))y(\tau(g(t,\xi))) \\
&\geq x(g(t,\xi)) - p(g(t,\xi))x(\tau(g(t,\xi))) \\
&\geq x(g(t,\xi)) - p(g(t,\xi))x(g(t,\xi)) \\
&= x(g(t,\xi))[1 - p(g(t,\xi))], \quad t \in [t_3, \infty),\ \xi \in [a,b].
\end{aligned}
$$

For $t \in [t_3, \infty)$, $\xi \in [a,b]$, since $t_2 \leq g(t,\xi)$, we have $x(g(t,\xi)) \geq x(t_2) > 0$ and $y(g(t,\xi)) \geq x(t_2)[1 - p(g(t,\xi))]$. Thus, from (2.4) we get

$$
\begin{aligned}
&\left\{B(t)\left[A(t)x^\Delta(t)\right]^\Delta\right\}^\Delta \\
&\qquad \leq -x(t_2)\int_a^b q(t,\xi)[1 - p(g(t,\xi))]\,\Delta\xi, \quad t \in [t_3, \infty).
\end{aligned}
$$

Integrating the both sides of the last inequality from t_3 to t, we obtain

$$
\begin{aligned}
&B(t)\left[A(t)x^\Delta(t)\right]^\Delta - B(t_3)(Ax^\Delta)^\Delta(t_3) \\
&\qquad \leq -x(t_2)\int_{t_3}^t \int_a^b q(s,\xi)[1 - p(g(s,\xi))]\,\Delta\xi\,\Delta s, \quad t \in [t_3, \infty).
\end{aligned}
$$

Since $B(t)\left[A(t)x^\Delta(t)\right]^\Delta > 0$ on $[t_3, \infty)$, we have

$$
\begin{aligned}
&\int_{t_3}^t \int_a^b q(s,\xi)[1 - p(g(s,\xi))]\Delta\xi\Delta s \\
&\qquad < \frac{1}{x(t_2)} B(t_3)(Ax^\Delta)^\Delta(t_3), \quad t \in [t_3, \infty).
\end{aligned}
$$

Let $t \to \infty$, we see that

$$
\begin{aligned}
(3.4) \qquad &\int_{t_3}^\infty \int_a^b q(s,\xi)[1 - p(g(s,\xi))]\,\Delta\xi\,\Delta s \\
&\qquad \leq \frac{1}{x(t_2)} B(t_3)(Ax^\Delta)^\Delta(t_3) < \infty,
\end{aligned}
$$

On the other hand, it follows from (3.1) that

$$
\int_{t_3}^\infty \int_a^b q(s,\xi)[1 - p(g(s,\xi))]\,\Delta\xi\,\Delta s = \infty,
$$

which leads to a contradiction to (3.4). Therefore Case (i) in Lemma 2.1 is not possible.

Lastly, assume that $x(t)$ satisfies Case (ii) in Lemma 2.1. Proceeding as in the proof of Lemma 2.1, there is a $t_2 \in [t_*, \infty)$ such that

$$x(t) > 0,\ x^{\Delta}(t) < 0,\ [A(t)x^{\Delta}(t)]^{\Delta} > 0, \quad t \in [t_2, \infty),$$

(2.3) and (2.4) hold. Thus we get $\lim_{t\to\infty} x(t) := c_2 \geq 0$ and $x(t) \geq c_2$ for $t \in [t_2, \infty)$. We claim that $c_2 = 0$. Assume not, i.e., $c_2 > 0$, then we now show that this leads to a contradiction. By the properties of limit, for $\lambda > 0$ there is a $t_3 \in [t_2, \infty)$ such that $c_2 \leq x(t) < (1+\lambda)c_2$ for $t \in [t_3, \infty)$. From (H1.3) and (H1.4), there exists a $t_4 \in [t_3, \infty)$ such that $t_3 \leq \tau(g(t,\xi)) \leq g(t,\xi)$ for $t \in [t_4, \infty)$ and $\xi \in [a, b]$. Hence we have

$$\begin{aligned} &c_2 \leq x(\tau(g(t,\xi))) < (1+\lambda)c_2, \\ &c_2 \leq x(g(t,\xi)) < (1+\lambda)c_2, \quad t \in [t_4, \infty),\ \xi \in [a, b]. \end{aligned}$$

Then from (H1.2), (2.1) and (2.3), we see that

$$\begin{aligned} y(g(t,\xi)) &= x(g(t,\xi)) - p(g(t,\xi))y(\tau(g(t,\xi))) \\ &\geq x(g(t,\xi)) - p(g(t,\xi))x(\tau(g(t,\xi))) \\ &\geq c_2[1-(1+\lambda)p(g(t,\xi))], \quad t \in [t_4, \infty),\ \xi \in [a, b]. \end{aligned} \tag{3.5}$$

It follows from (2.4) and (3.5) that

$$\begin{aligned} &\left\{B(t)\left[A(t)x^{\Delta}(t)\right]^{\Delta}\right\}^{\Delta} \\ &\qquad \leq -\int_a^b q(t,\xi)y(g(t,\xi))\,\Delta\xi \\ &\qquad \leq -c_2\int_a^b q(t,\xi)[1-(1+\lambda)p(g(t,\xi))]\,\Delta\xi, \quad t \in [t_4, \infty). \end{aligned} \tag{3.6}$$

Integrating from t_4 to t, we get

$$\begin{aligned} B(t)\left[A(t)x^{\Delta}(t)\right]^{\Delta} &\leq B(t_4)(Ax^{\Delta})^{\Delta}(t_4) \\ &\quad - c_2\int_{t_4}^{t}\int_a^b q(s,\xi)[1-(1+\lambda)p(g(s,\xi))]\Delta\xi\Delta s, \quad t \in [t_4.\infty). \end{aligned}$$

Since $B(t)\left[A(t)x^{\Delta}(t)\right]^{\Delta} > 0$ on $[t_4, \infty)$, we conclude that

$$\int_{t_4}^{t}\int_{a}^{b} q(s,\xi)[1-(1+\lambda)p(g(s,\xi))]\,\Delta\xi\,\Delta s$$
$$< \frac{1}{c_2}B(t_4)(Ax^{\Delta})^{\Delta}(t_4), \quad t\in[t_4.\infty).$$

Therefore we obtain

$$\limsup_{t\to\infty}\int_{t_4}^{t}\int_{a}^{b} q(s,\xi)[1-(1+\lambda)p(g(s,\xi))]\,\Delta\xi\,\Delta s$$
$$\le \frac{1}{c_2}B(t_4)(Ax^{\Delta})^{\Delta}(t_4) < \infty,$$

which contradicts (3.1). Hence $c_2 = 0$, i.e., $\lim_{t\to\infty} x(t) = 0$. Since $0 < y(t) \le x(t)$ for all sufficiently large t, we have $\lim_{t\to\infty} y(t) = 0$. The proof is complete. □

Remark 3.1. By using the similar methods of proof of Theorem 3.1 in this paper, We find that the condition

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_1}^{t}(t-s^m)\left[Mr(s)q(s)-\frac{(r^{\Delta}(s))^2 a(s)}{4r(s)\delta(s,t_1)}\right]\Delta s=\infty$$

in Theorem 2 of Erbe *et al.* [**6**] is not indispensable. Namely, if we cancel the condition, then the conclusions of Theorem 2 of Erbe *et al.* [**6**] still hold. Thus, Theorem 3.1 extends Theorem 2 of [**6**] for (1.2) to (1.1).

Next, we present some new asymptotic behavior and oscillation results for solutions of (1.1), by using an integral averaging condition of Philos-type. We introduce a class of functions $\mathfrak{R}$ which will be extensively used in the sequel. Let

$$\mathbb{D}_0 = \{(t,s)\in\mathbb{T}\times\mathbb{T} : t > s \ge t_0\}$$

and

$$\mathbb{D} = \{(t,s)\in\mathbb{T}\times\mathbb{T} : t \ge s \ge t_0\}.$$

The function $H \in C_{rd}(\mathbb{D},\mathbb{R})$ is said to belong to the class $\mathfrak{R}$ if

(i) $H(t,t) \ge 0$ for $t \ge t_0$ and $H(t,s) > 0$ on $\mathbb{D}_0$;

(ii) H has a rd-continuous delta partial derivative $H^{\Delta_s}(t,s)$ on $\mathbb{D}_0$ with respect to the second variable.

Theorem 3.2. *Suppose that* (H1.1)–(H1.5) *and the following conditions hold:*

(H3.1) $\widetilde{\mathbb{T}} := g(\mathbb{T}, a) \subset \mathbb{T}$ *is a time scale,* $g^{\Delta}(t,a) > 0$ *is rd-continuous on* $\mathbb{T}^{\kappa}$*, and* $g(\sigma(t), a) = \sigma(g(t,a))$ *for all* $t \in \mathbb{T}$*;*

(H3.2) *There exists a function* $H(t,s) \in \mathfrak{R}$ *and a positive function* $r(t)$ *such that* $r^{\Delta}(t)$ *is rd-continuous on* $[t_0, \infty)$ *and for all sufficiently large* t_4 *and a certain* t_5*,*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_5)} \int_{t_5}^{t} \Big\{ H(t,s)r(s)Q(s) - \frac{[H^{\Delta_s}(t,s)r(\sigma(s)) + H(t,s)r^{\Delta}(s)]^2 A(g(s,a))}{4H(t,s)r(s)g^{\Delta}(s,a)V(s,t_4)} \Big\} \Delta s = \infty, \tag{3.7}$$

where t_5 *satisfies that* $t_5 > t_4$ *and* $g(t,a) > t_4$ *for* $t \in [t_5, \infty)$*, and*

$$\begin{aligned} Q(s) &= \int_a^b q(s,\xi)[1 - p(g(s,\xi))]\Delta\xi, \\ V(s,t_4) &= \int_{t_4}^{g(s,a)} \frac{1}{B(\xi)} \Delta\xi; \end{aligned} \tag{3.8}$$

(H3.3) *There is a constant* $\lambda > 0$ *and a sufficiently large constant* C *such that*

$$\limsup_{t\to\infty} \int_C^t \left[\frac{1}{A(u)} \int_u^{\infty} \Big(\frac{1}{B(z)} \int_z^{\infty} \Psi(s)\Delta s \Big) \Delta z \right] \Delta u = \infty, \tag{3.9}$$

where

$$\Psi(s) = \int_a^b q(s,\xi)[1 - (1+\lambda)p(g(s,\xi))]\,\Delta\xi. \tag{3.10}$$

Then every solution of (1.1) *is oscillatory or tends to zero as* $t \to \infty$*.*

Proof. Suppose that $y(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, assume that $y(t)$ is an eventually positive solution of (1.1). We now prove that $y(t)$ tends to zero as $t \to \infty$. Define $x(t)$ as in (2.1). By Lemma 2.1 either Case (i) or Case (ii) in Lemma 2.1 holds.

First, assume $x(t)$ satisfies Case (i) in Lemma 2.1. Proceeding as in the proof of Lemma 2.1 and Theorem 3.1, there exists a $t_3 \in [t_*, \infty)$ such that (2.3), (2.4), (3.2) and (3.3) hold for $t \in [t_3, \infty)$. For $t \in [t_3, \infty)$, $\xi \in [a, b]$, since $t_2 \leq g(t, a) \leq g(t, \xi)$ and $x^\Delta(t) > 0$, we have $x(g(t, \xi)) \geq x(g(t, a)) > 0$. Therefore, from (3.3) we obtain

$$y(g(t,\xi)) \geq x(g(t,a))[1 - p(g(t,\xi))], \quad t \in [t_3, \infty),\ \xi \in [a, b].$$

From (2.4) we get

$$\begin{aligned} &\left\{B(t)\left[A(t)x^\Delta(t)\right]^\Delta\right\}^\Delta \\ &\qquad \leq -\int_a^b q(t,\xi))y(g(t,a))\Delta\xi \\ &\qquad \leq -x(g(t,a))\int_a^b q(t,\xi))[1-p(g(t,\xi))]\Delta\xi \\ &\qquad := -x(g(t,a))Q(t), \quad t \in [t_3,\infty), \end{aligned} \tag{3.11}$$

where $Q(t)$ is defined by (3.8). Define the "Riccati" type function w by

$$w(t) = B(t)[A(t)x^\Delta(t)]^\Delta \frac{r(t)}{x(g(t,a))}, \quad t \in [t_3, \infty). \tag{3.12}$$

By the product rule $(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta$ and the quotient rule $\left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}$, where $f^\sigma(t) := f(\sigma(t))$, we obtain

$$\begin{aligned} w^\Delta(t) = &\left\{B(t)\left[A(t)x^\Delta(t)\right]^\Delta\right\}^\Delta \frac{r(t)}{x(g(t,a))} + B(\sigma(t))(Ax^\Delta)^\Delta(\sigma(t)) \\ &\times \frac{r^\Delta(t)x(g(t,a)) - r(t)[x(g(t,a))]^\Delta}{x(g(t,a))x(g(\sigma(t),a))}, \quad t \in [t_3,\infty). \end{aligned}$$

It follows from (3.11) and (3.12) that

$$w^\Delta(t) \leq -r(t)Q(t) + r^\Delta(t)\frac{w(\sigma(t))}{r(\sigma(t))}$$

$$- B(\sigma(t))(Ax^{\Delta})^{\Delta}(\sigma(t))\frac{r(t)[x(g(t,a))]^{\Delta}}{x(g(t,a))x(g(\sigma(t),a))}, \quad t \in [t_3, \infty).$$

From (H3.1) and Lemma 2.3 we get $[x(g(t,a))]^{\Delta} = x^{\Delta}(g(t,a))g^{\Delta}(t,a)$ for $t \in [t_3, \infty)$. From (2.6), there is a $t_4 \in [t_3, \infty)$ such that

$$x^{\Delta}(t) \geq \frac{B(t)}{A(t)}\Big(\int_{t_4}^{t} \frac{1}{B(\xi)}\Delta\xi\Big)[A(t)x^{\Delta}(t)]^{\Delta}, \quad t \in [t_4, \infty),$$

It follows from (H1.4) that there is a $t_5 \in (t_4, \infty)$ such that $g(t,a) > t_4$ for $t \in [t_5, \infty)$. Hence

$$\begin{aligned} x^{\Delta}(g(t,a)) &\geq \frac{B(g(t,a))}{A(g(t,a))}\left(\int_{t_4}^{g(t,a)} \frac{1}{B(\xi)}\Delta\xi\right)(Ax^{\Delta})^{\Delta}(g(t,a)) \\ &:= \frac{B(g(t,a))}{A(g(t,a))}V(t,t_4)(Ax^{\Delta})^{\Delta}(g(t,a)), \quad t \in [t_5, \infty), \end{aligned}$$

where V is defined by (3.8). Thus, for $t \in [t_5, \infty)$ we conclude

$$\begin{aligned} w^{\Delta}(t) \leq{}& -r(t)Q(t) + r^{\Delta}(t)\frac{w(\sigma(t))}{r(\sigma(t))} - B(\sigma(t))(Ax^{\Delta})^{\Delta}(\sigma(t)) \\ &\times \frac{r(t)g^{\Delta}(t,a)}{x(g(t,a))x(g(\sigma(t),a))}\,\frac{B(g(t,a))}{A(g(t,a))}V(t,t_4)(Ax^{\Delta})^{\Delta}(g(t,a)). \end{aligned}$$

Since $g(\sigma(t),a) \leq \sigma(t)$ and $x^{\Delta}(t) > 0$, we find $x(g(\sigma(t),a)) \leq x(\sigma(t))$ for $t \in [t_5, \infty)$. Since $g(t,a) \leq t \leq \sigma(t)$ and $B(t)\left[A(t)x^{\Delta}(t)\right]^{\Delta}$ is nonincreasing on $[t_5, \infty)$, we have

$$B(g(t,a))(Ax^{\Delta})^{\Delta}(g(t,a)) \geq B(\sigma(t))(Ax^{\Delta})^{\Delta}(\sigma(t)), \quad t \in [t_5, \infty).$$

Therefore we obtain

$$\begin{aligned} w^{\Delta}(t) \leq{}& -r(t)Q(t) + r^{\Delta}(t)\frac{w(\sigma(t))}{r(\sigma(t))} \\ &- \frac{\left[B(\sigma(t))(Ax^{\Delta})^{\Delta}(\sigma(t))\right]^2 r(t)g^{\Delta}(t,a)}{x(g(t,a))x(g(\sigma(t),a))}\frac{1}{A(g(t,a))}V(t,t_4) \\ ={}& -r(t)Q(t) + r^{\Delta}(t)\frac{w(\sigma(t))}{r(\sigma(t))} - \left[\frac{w(\sigma(t))x(g(\sigma(t),a))}{r(\sigma(t))}\right]^2 \end{aligned}$$

$$\times \frac{r(t)g^{\Delta}(t,a)}{x(g(t,a))x(g(\sigma(t),a))} \frac{1}{A(g(t,a))} V(t,t_4), \quad t \in [t_5,\infty).$$

Since $t \le \sigma(t), g^{\Delta}(t,a) > 0$ and $x^{\Delta}(t) > 0$, for $t \in [t_5,\infty)$ we find that

$$g(t,a) \le g(\sigma(t),a), \qquad x(g(t,a)) \le x(g(\sigma(t),a)),$$

and

$$\begin{aligned} w^{\Delta}(t) \le -r(t)Q(t) + r^{\Delta}(t)\frac{w(\sigma(t))}{r(\sigma(t))} \\ - \left[\frac{w(\sigma(t))}{r(\sigma(t))}\right]^2 \frac{r(t)g^{\Delta}(t,a)V(t,t_4)}{A(g(t,a))}. \end{aligned}$$

Hence, we get

$$\begin{aligned} &\int_{t_5}^{t} H(t,s)w^{\Delta}(s)\Delta s \\ &\quad \le -\int_{t_5}^{t} H(t,s)r(s)Q(s)\Delta s + \int_{t_5}^{t} H(t,s)r^{\Delta}(s)\frac{w(\sigma(s))}{r(\sigma(s))}\Delta s \\ &\quad - \int_{t_5}^{t} H(t,s)\left[\frac{w(\sigma(s))}{r(\sigma(s))}\right]^2 \frac{r(s)g^{\Delta}(s,a)V(s,t_4)}{A(g(s,a))}\Delta s, \quad t \in [t_5,\infty). \end{aligned}$$

Then from the integration by parts formula

$$\int_a^b f(s)g^{\Delta}(s)\Delta s = (f(s)g(s))|_a^b - \int_a^b f^{\Delta}(s)g(\sigma(s))\Delta s$$

we have

$$\begin{aligned} &\int_{t_5}^{t} H(t,s)w^{\Delta}(s)\Delta s \\ &\quad = \Big(H(t,s)w(s)\Big)\Big|_{s=t_5}^{s=t} - \int_{t_5}^{t} H^{\Delta_s}(t,s)w(\sigma(s))\Delta s \\ &\quad \le -\int_{t_5}^{t} H(t,s)r(s)Q(s)\Delta s + \int_{t_5}^{t} H(t,s)r^{\Delta}(s)\frac{w(\sigma(s))}{r(\sigma(s))}\Delta s \\ &\quad\quad - \int_{t_5}^{t} H(t,s)\left[\frac{w(\sigma(s))}{r(\sigma(s))}\right]^2 \frac{r(s)g^{\Delta}(s,a)V(s,t_4)}{A(g(s,a))}\Delta s, \quad t \in [t_5,\infty). \end{aligned}$$

Since $H(t,t) \geq 0$ and $w(t) > 0$ for $t \in [t_5, \infty)$, we obtain

$$\begin{aligned}
&- H(t,t_5)w(t_5) \leq H(t,t)w(t) - H(t,t_5)w(t_5) \\
&\qquad \leq - \int_{t_5}^{t} H(t,s)r(s)Q(s)\Delta s + \int_{t_5}^{t} \Big[H^{\Delta_s}(t,s) \\
&\qquad\quad + H(t,s)\frac{r^{\Delta}(s)}{r(\sigma(s))}\Big] w(\sigma(s))\Delta s \\
&\qquad\quad - \int_{t_5}^{t} H(t,s)\left[\frac{w(\sigma(s))}{r(\sigma(s))}\right]^2 \frac{r(s)g^{\Delta}(s,a)V(s,t_4)}{A(g(s,a))}\Delta s \\
&\qquad = - \int_{t_5}^{t} H(t,s)r(s)Q(s)\Delta s \\
&\qquad\quad + \int_{t_5}^{t} \frac{[H^{\Delta_s}(t,s)r(\sigma(s)) + H(t,s)r^{\Delta}(s)]^2 A(g(s,a))}{4H(t,s)r(s)g^{\Delta}(s,a)V(s,t_4)}\Delta s \\
&\qquad\quad - \int_{t_5}^{t} \Bigg\{ \frac{[H^{\Delta_s}(t,s)r(\sigma(s)) + H(t,s)r^{\Delta}(s)]\sqrt{A(g(s,a))}}{2\sqrt{H(t,s)r(s)g^{\Delta}(s,a)V(s,t_4)}} \\
&\qquad\quad - \sqrt{\frac{H(t,s)r(s)g^{\Delta}(s,a)V(s,t_4)}{A(g(s,a))}}\frac{w(\sigma(s))}{r(\sigma(s))}\Bigg\}^2 \Delta s \\
&\qquad \leq - \int_{t_5}^{t} H(t,s)r(s)Q(s)\Delta s \\
&\qquad\quad + \int_{t_5}^{t} \frac{[H^{\Delta_s}(t,s)r(\sigma(s)) + H(t,s)r^{\Delta}(s)]^2 A(g(s,a))}{4H(t,s)r(s)g^{\Delta}(s,a)V(s,t_4)}\Delta s, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [t_5, \infty).
\end{aligned}$$

Thus, for $t > t_5$ we get

$$\begin{aligned}
&\int_{t_5}^{t} \Bigg\{ H(t,s)r(s)Q(s) \\
&\qquad - \frac{[H^{\Delta_s}(t,s)r(\sigma(s)) + H(t,s)r^{\Delta}(s)]^2 A(g(s,a))}{4H(t,s)r(s)g^{\Delta}(s,a)V(s,t_4)}\Bigg\}\Delta s \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq H(t,t_5)w(t_5)
\end{aligned}$$

and

$$\frac{1}{H(t,t_5)}\int_{t_5}^{t}\Big\{H(t,s)r(s)Q(s) - \frac{[H^{\Delta_s}(t,s)r(\sigma(s))+H(t,s)r^{\Delta}(s)]^2A(g(s,a))}{4H(t,s)r(s)g^{\Delta}(s,a)V(s,t_4)}\Big\}\Delta s \le w(t_5).$$

Then we have

$$\limsup_{t\to\infty}\frac{1}{H(t,t_5)}\int_{t_5}^{t}\Big\{H(t,s)r(s)Q(s) - \frac{[H^{\Delta_s}(t,s)r(\sigma(s))+H(t,s)r^{\Delta}(s)]^2A(g(s,a))}{4H(t,s)r(s)g^{\Delta}(s,a)V(s,t_4)}\Big\}\Delta s \le w(t_5) < \infty,$$

which contradicts (3.7).

Finally, assume that $x(t)$ satisfies Case (ii) in Lemma 2.1. Since $x(t) > 0$ is decreasing on $[t_*,\infty)$, we obtain that $\lim_{t\to\infty} x(t) := c_2 \ge 0$ exists. We claim that $c_2 = 0$. Otherwise, i.e., $c_2 > 0$, we will find that this leads to a contradiction. Proceeding as in the proof of Theorem 3.1, there is a $t_4 \in [t_*,\infty)$ such that (3.6) holds. It follows from (3.6) that

$$\big\{B(t)\left[A(t)x^{\Delta}(t)\right]^{\Delta}\big\}^{\Delta} \le -c_2\Psi(t), \quad t\in[t_4,\infty),$$

where Ψ is defined by (3.10). Integrating the both sides of the last inequality from t to ∞, we get

$$\big(B(s)\left[A(s)x^{\Delta}(s)\right]^{\Delta}\big)\Big|_t^{\infty} \le -c_2\int_t^{\infty}\Psi(s)\Delta s, \quad t\in[t_4,\infty).$$

Since $B(s)\left[A(s)x^{\Delta}(s)\right]^{\Delta} > 0$ is decreasing on $[t_4,\infty)$, we see that

$$\lim_{s\to\infty} B(s)\left[A(s)x^{\Delta}(s)\right]^{\Delta} \ge 0$$

exists. Thus we have

$$-B(t)\left[A(t)x^{\Delta}(t)\right]^{\Delta} + c_2\int_t^{\infty}\Psi(s)\Delta s \le 0, \quad t\in[t_4,\infty).$$

Integrating the both sides of the last inequality from t to ∞ after dividing by $B(t)$ will lead to

$$\Big(-A(z)x^{\Delta}(z)\Big)\Big|_t^{\infty} + c_2\int_t^{\infty}\Big(\frac{1}{B(z)}\int_z^{\infty}\Psi(s)\Delta s\Big)\Delta z \le 0, \quad t\in[t_4,\infty).$$

Since $A(z)x^{\Delta}(z) < 0$ is increasing on $[t_4, \infty)$, we see that

$$\lim_{z\to\infty} A(z)x^{\Delta}(z) \le 0$$

exists. Hence

$$A(t)x^{\Delta}(t) + c_2 \int_t^{\infty} \Big(\frac{1}{B(z)} \int_z^{\infty} \Psi(s)\Delta s\Big)\Delta z \le 0, \quad t \in [t_4, \infty).$$

Dividing the both sides of the last inequality by $A(t)$ and integrating again from t_4 to t gives

$$x(u)\Big|_{t_4}^{t} + c_2 \int_{t_4}^{t} \left[\frac{1}{A(u)} \int_u^{\infty} \Big(\frac{1}{B(z)} \int_z^{\infty} \Psi(s)\Delta s\Big)\Delta z\right] \Delta u \le 0,\ t \in [t_4, \infty),$$

$$\begin{aligned} c_2 \int_{t_4}^{t} \left[\frac{1}{A(u)} \int_u^{\infty} \Big(\frac{1}{B(z)} \int_z^{\infty} \Psi(s)\Delta s\Big)\Delta z\right] \Delta u &\le x(t_4) - x(t) \\ &< x(t_4), \\ &t \in [t_4, \infty), \end{aligned}$$

and

$$\int_{t_4}^{t} \left[\frac{1}{A(u)} \int_u^{\infty} \left(\frac{1}{B(z)} \int_z^{\infty} \Psi(s)\Delta s\right)\Delta z\right] \Delta u < \frac{1}{c_2} x(t_4), \quad t \in [t_4, \infty).$$

Therefore we have

$$\limsup_{t\to\infty} \int_{t_4}^{t} \left[\frac{1}{A(u)} \int_u^{\infty} \Big(\frac{1}{B(z)} \int_z^{\infty} \Psi(s)\Delta s\Big)\Delta z\right] \Delta u \le \frac{1}{c_2} x(t_4) < \infty,$$

which is a contradiction to (3.9). Thus $c_2 = 0$, i.e., $\lim_{t\to\infty} x(t) = 0$. Since $0 < y(t) \le x(t)$ for all sufficiently large t, we see $\lim_{t\to\infty} y(t) = 0$. This completes the proof. □

Remark 3.2. Theorem 3.2 extends Theorem 2.1 of Candan and Dahiya [**5**] for (1.4) to (1.1).

Remark 3.3. By applying the similar methods of proof of Theorem 3.2, We obtain the following Theorem 3.3 for (1.2), which supplement and perfect the results of Erbe *et al.* [**6**].

Theorem 3.3. *Assume that condition* (1.3),

$$\int_{t_0}^{\infty} \frac{1}{c(t)}\Delta t = \int_{t_0}^{\infty} \frac{1}{b(t)}\Delta t = \infty$$

and

$$\int_{t_0}^{\infty} \left[\frac{1}{a(u)} \int_{u}^{\infty} \left(\frac{1}{c(z)} \int_{z}^{\infty} q(s)\Delta s\right)\Delta z\right] \Delta u = \infty$$

hold. Moreover, suppose that there exists a function $H(t,s) \in \mathfrak{R}$ *and a positive function* $r(t)$ *such that* $r^{\Delta}(t)$ *is rd-continuous on* $[t_0, \infty)$ *and for sufficiently large* t_1 *and a certain* $t_2 > t_1$,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_2)} \int_{t_2}^{t} \Big\{ H(t,s)Mr(s)q(s) - \frac{[H^{\Delta_s}(t,s)r(\sigma(s)) + H(t,s)r^{\Delta}(s)]^2 a(s)}{4H(t,s)r(s)\delta(s,t_1)} \Big\} \Delta s = \infty,$$

where $\delta(s,t_1) := \int_{t_1}^{t} \frac{1}{c(\xi)}\Delta\xi$. *Then every solution of* (1.2) *is oscillatory or tends to zero as* $t \to \infty$.

Remark 3.4. We observe that we don't require the restriction $H^{\Delta_s}(t,s) \le 0$ on $\in \mathbb{D}_0$ in Theorem 3.2. However, if the condition $H^{\Delta_s}(t,s) \le 0$ on $\in \mathbb{D}_0$ holds and in Theorem 3.2 we replace assumption (H3.2) with the following assumption:

(H3.4) there exists a function $H(t,s) \in \mathfrak{R}$ that satisfies $H^{\Delta_s}(t,s) \le 0$ on $\in \mathbb{D}_0$ and a positive function $r(t)$ such that $r^{\Delta}(t)$ is rd-continuous on $[t_0, \infty)$ and for all sufficiently large t_4 and a certain t_5,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_5)} \int_{t_5}^{t} H(t,s) \left[r(s)Q(s) - \frac{(r^{\Delta}(s))^2 A(g(s,a))}{4r(s)g^{\Delta}(s,a)V(s,t_4)} \right] \Delta s = \infty,$$

where Q and V are defined by (3.8) and t_5 satisfies that $t_5 > t_4$ and $g(t,a) > t_4$ for $t \in [t_5, \infty)$,

then the conclusions of Theorem 3.2 hold.

Remark 3.5. From Theorem 3.2 and Remark 3.4, we can obtain different sufficient conditions for asymptotic behavior and oscillation of solutions of (1.1) with different choices of $H(t,s)$ and $r(t)$. For example, let $H(t,s) = 1$ and $r(t) = t$, then $H^{\Delta_s}(t,s) = 0$, $r^{\Delta}(t) = 1$, and Theorem 3.2 yields the following results.

Corollary 3.1. *Suppose that* (H1.1)–(H1.5)*,* (H3.1) *and* (H3.3) *hold. Furthermore, assume for all sufficiently large* t_4 *and a certain* t_5*,*

$$\limsup_{t\to\infty}\int_{t_5}^{t}\left[sQ(s)-\frac{A(g(s,a))}{4sg^{\Delta}(s,a)V(s,t_4)}\right]\Delta s=\infty,$$

where Q *and* V *are defined by* (3.8) *and* t_5 *satisfies that* $t_5 > t_4$ *and* $g(t,a) > t_4$ *for* $t \in [t_5,\infty)$*. Then every solution of* (1.1) *is oscillatory or tends to zero as* $t\to\infty$*.*

Let $H(t,s)=1$ and $r(t)=1$, then $H^{\Delta_s}(t,s)=0$, $r^{\Delta}(t)=0$, and Theorem 3.2 implies the following results, which is different from Theorem 3.1.

Corollary 3.2. *Suppose that* (H1.1)–(H1.5) *and* (H3.3) *hold. Furthermore, suppose for all sufficiently large* t_4*,*

$$\int_{t_4}^{\infty}\int_{a}^{b}q(s,\xi)[1-p(g(s,\xi))]\,\Delta\xi\Delta s=\infty.$$

Then every solution of (1.1) *is oscillatory or tends to zero as* $t\to\infty$*.*

Let $H(t,s)=(t-s)^m,\ m\geq 1$, then

$$H^{\Delta_s}(t,s)=\begin{cases}-m(t-s)^{m-1}, & s=\sigma(s),\\ \dfrac{(t-\sigma(s))^m-(t-s)^m}{\sigma(s)-s}, & s<\sigma(s),\end{cases}$$

$H^{\Delta_s}(t,s)\leq 0$ for $s\leq t$, and from Remark 3.4 we have the following results, which involve an integral averaging condition of Kamenev-type.

Corollary 3.3. *Suppose that* (H1.1)–(H1.5)*,* (H3.1) *and* (H3.3) *hold. Moreover, assume that there exists a positive function* $r(t)$ *such that* $r^{\Delta}(t)$ *is rd-continuous on* $[t_0,\infty)$ *and for all sufficiently large* t_4 *and a certain* t_5*,*

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_5}^{t}(t-s)^m\Bigg[r(s)Q(s)-\frac{(r^{\Delta}(s))^2A(g(s,a))}{4r(s)g^{\Delta}(s,a)V(s,t_4)}\Bigg]\Delta s=\infty, \tag{3.13}$$

where $m \geq 1, Q$ and V are defined by (3.8) and t_5 satisfies that $t_5 > t_4$ and $g(t,a) > t_4$ for $t \in [t_5, \infty)$. Then every solution of (1.1) *is oscillatory or tends to zero as $t \to \infty$.*

Let $r(t) = 1$, then $r^{\Delta}(t) = 0$, and Corollary 3.3 indicates the following results.

Corollary 3.4. *Suppose* (H1.1)–(H1.5) *and* (H3.3) *hold. Further, assume for all sufficiently large t_4,*

$$\lim_{t\to\infty} \frac{1}{t^m} \int_{t_4}^{t} (t-s)^m \int_a^b q(s,\xi)[1 - p(g(s,\xi))]\,\Delta\xi\,\Delta s = \infty,$$

where $m \geq 1$. Then every solution of (1.1) *is oscillatory or tends to zero as $t \to \infty$.*

4 Some examples

Example 4.1. Consider the following third-order neutral delay dynamic equation

$$\left[y(t) + \frac{3}{4}y(t-\pi)\right]^{\Delta\Delta\Delta} + \int_{1/(3\pi)}^{2/(3\pi)} \left(2 - \frac{3}{2}e^{-\pi}\right)\frac{1}{\xi^2}e^{\frac{1}{\xi}}y\left(t - \frac{1}{\xi}\right)\Delta\xi = 0, \quad t \in \mathbb{T}. \tag{4.1}$$

Taking $\mathbb{T} = \mathbb{R}, t_0 > 0$ and $q(t,\xi) = (2 - \frac{3}{2}e^{-\pi})\frac{1}{\xi^2}e^{\frac{1}{\xi}}$, it is easy to see that (H1.1)–(H1.5) are satisfied. To apply Theorem 3.1, it remains to satisfy condition (3.1). Choosing $\lambda = \frac{1}{6} > 0$, then for a sufficiently large constant C, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \int_C^t \int_a^b q(s,\xi)[1 - (1+\lambda)p(g(s,\xi))]\,\Delta\xi\,\Delta s \\
&\quad = \limsup_{t\to\infty} \int_C^t \int_{1/(3\pi)}^{2/(3\pi)} \left(2 - \frac{3}{2}e^{-\pi}\right)\frac{1}{\xi^2}e^{\frac{1}{\xi}}\left[1 - (1+\frac{1}{6})\frac{3}{4}\right] d\xi\,ds \\
&\quad = \limsup_{t\to\infty} \int_C^t \frac{1}{8}\left(2 - \frac{3}{2}e^{-\pi}\right)(-e^{\frac{3\pi}{2}} + e^{3\pi})\,ds \\
&\quad = \limsup_{t\to\infty} \frac{1}{8}\left(2 - \frac{3}{2}e^{-\pi}\right)(-e^{\frac{3\pi}{2}} + e^{3\pi})(t - C) = \infty,
\end{aligned}$$

which shows (3.1) in Theorem 3.1 is satisfied. Thus, by Theorem 3.1 every solution of (4.1) is oscillatory or tends to zero as $t \to \infty$. An example of such a solution is $y(t) = e^t \cos t$, which is oscillatory.

Example 4.2. Consider the following third-order neutral delay dynamic equation

$$\left[y(t) + \frac{1}{e^2 - e} y(t-1)\right]^{\Delta\Delta\Delta} + \int_1^4 \frac{(e^{-1}-1)^2}{14e^2} \xi^2 y(t-2)\,\Delta\xi = 0, \quad t \in \mathbb{T}. \tag{4.2}$$

Choosing $\mathbb{T} = \mathbb{Z}$ and $q(t,\xi) = \frac{(e^{-1}-1)^2}{14e^2}\xi^2$, we find that (H1.1)–(H1.5) are satisfied. We will apply Theorem 3.1. From (2.7), we see that (4.2) becomes the third-order neutral delay difference equation

$$\Delta\left\{\Delta\left[\Delta\Big(y(t) + \frac{1}{e^2-e} y(t-1)\Big)\right]\right\} + \frac{(e^{-1}-1)^2}{e^2} y(t-2) = 0, \quad t \in \mathbb{Z},$$

where $\Delta y(t) = y(t+1) - y(t)$. Let $\lambda = \frac{1}{2}(e^2 - e - 1) > 0$, then from (2.7), for a sufficiently constant C we obtain

$$\begin{aligned}
&\limsup_{t\to\infty} \int_C^t \int_a^b q(s,\xi)[1 - (1+\lambda)p(g(s,\xi))]\,\Delta\xi\,\Delta s \\
&= \limsup_{t\to\infty} \int_C^t \int_1^4 \frac{(e^{-1}-1)^2}{14e^2}\xi^2 \left[1 - \left(1 + \frac{1}{2}(e^2-e-1)\right)\frac{1}{e^2-e}\right] \Delta\xi\,\Delta s \\
&= \limsup_{t\to\infty} \int_C^t \left\{\sum_{\xi=1}^{3} \frac{(e^{-1}-1)^2}{14e^2}\xi^2 \left[1 - \left(1 + \frac{1}{2}(e^2-e-1)\right)\frac{1}{e^2-e}\right]\right\} \Delta s \\
&= \limsup_{t\to\infty} \int_C^t \frac{(e^{-1}-1)^2(e^2-e-1)}{2e^2(e^2-e)}\,\Delta s = \infty,
\end{aligned}$$

which implies (3.1) in Theorem 3.1 is satisfied. Consequently, by Theorem 3.1 every solution of (4.2) is oscillatory or tends to zero as $t \to \infty$. Observe that $y(t) = e^{-t}$ is a solution of (4.2) , which satisfies $\lim_{t\to\infty} y(t) = 0$.

Example 4.3. Consider the following third-order nonlinear neutral delay dynamic equation

$$\left\{\sqrt{t}\left[\frac{1}{t^2}\left(y(t)+\frac{3}{4}\left(1-\frac{1}{\sqrt{2t}}\right)y(\tau(t))\right)^{\Delta}\right]^{\Delta}\right\}^{\Delta} + \int_0^1 \frac{\sqrt{t+\xi}}{t^2}y\left(\frac{t+\xi}{2}\right)e^{y^2(\frac{t+\xi}{2})}\,\Delta\xi = 0, \quad t \geq 3. \tag{4.3}$$

where $\tau(t)$ satisfies (H1.3). Taking $\mathbb{T} = \mathbb{R}$ and $q(t,\xi) = \sqrt{t+\xi}/t^2$, it is easy to check that (H1.1)–(H1.5) and (H3.1) are satisfied. To apply Corollary 3.3 of Theorem 3.2, it remains to satisfy condition (H3.3) and (3.13). Let $\lambda = 1/3 > 0$, then from (3.10) we have

$$\begin{aligned}\Psi(s) &= \int_a^b q(s,\xi)[1-(1+\lambda)p(g(s,\xi))]\,\Delta\xi \\ &= \int_0^1 \frac{\sqrt{s+\xi}}{s^2}\left[1-\left(1+\frac{1}{3}\right)\frac{3}{4}\left(1-\frac{1}{\sqrt{s+\xi}}\right)\right]d\xi \\ &= \frac{1}{s^2}, \quad s \in [3,\infty).\end{aligned}$$

Hence, for a sufficiently large constant C we see that

$$\begin{aligned}&\limsup_{t\to\infty}\int_C^t\left[\frac{1}{A(u)}\int_u^\infty\Big(\frac{1}{B(z)}\int_z^\infty\Psi(s)\Delta s\Big)\Delta z\right]\Delta u \\ &= \limsup_{t\to\infty}\int_C^t\left[u^2\int_u^\infty\Big(\frac{1}{\sqrt{z}}\int_z^\infty\frac{1}{s^2}ds\Big)dz\right]du \\ &= \limsup_{t\to\infty}\int_C^t\left[u^2\int_u^\infty\frac{1}{z\sqrt{z}}dz\right]du \\ &= \limsup_{t\to\infty}\int_C^t\frac{u^2}{2\sqrt{u}}du \\ &= \limsup_{t\to\infty}\frac{1}{5}(t^{\frac{5}{2}}-C^{\frac{5}{2}}) = \infty,\end{aligned}$$

which indicates (H3.3) is satisfied. Furthermore, from (3.8), we get

$$\begin{aligned}Q(s) &= \int_a^b q(s,\xi)[1-p(g(s,\xi))]\,\Delta\xi \\ &= \int_0^1 \frac{\sqrt{s+\xi}}{s^2}\left[1-\frac{3}{4}\left(1-\frac{1}{\sqrt{s+\xi}}\right)\right]d\xi \geq \frac{1}{s^2}, \quad s \in [3,\infty).\end{aligned}$$

For any sufficiently large t_4, there exists $t_5 = 2t_4 + 1 > t_4$ such that $g(t,a) = \frac{t}{2} > t_4$ for $t \in [t_5, \infty)$. Taking $m = 1, r(t) = t^2$, then we have

$$\frac{1}{t^m}\int_{t_5}^{t}(t-s)^m r(s)Q(s)\Delta s \geq \frac{1}{t}\int_{t_5}^{t}(t-s)s^2\frac{1}{s^2}\,ds$$
$$= \frac{1}{2t}(t-t_5)^2$$

for $t \in [t_5, \infty)$, which yields

$$\lim_{t\to\infty}\frac{1}{t^m}\int_{t_5}^{t}(t-s)^m r(s)Q(s)\Delta s = \infty. \tag{4.4}$$

On the other hand, we have

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t^m}\int_{t_5}^{t}(t-s)^m\frac{(r^\Delta(s))^2A(g(s,a))}{4r(s)g^\Delta(s,a)V(s,t_4)}\,\Delta s\\ &= \lim_{t\to\infty}\frac{1}{t}\int_{t_5}^{t}(t-s)\frac{4s^2(\frac{s}{2})^{-2}}{4s^2\,\frac{1}{2}\int_{t_4}^{s/2}\frac{1}{\sqrt{\xi}}d\xi}ds\\ &= \lim_{t\to\infty}4\sqrt{2}\Bigg[\int_{t_5}^{t}\frac{1}{s^2(\sqrt{s}-\sqrt{2t_4})}\,ds\\ &\quad -\frac{1}{t}\int_{t_5}^{t}\frac{1}{s(\sqrt{s}-\sqrt{2t_4})}ds\Bigg] < \infty.\end{aligned} \tag{4.5}$$

It follows from (4.4) and (4.5) that

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_5}^{t}(t-s)^m\left[r(s)Q(s)-\frac{(r^\Delta(s))^2A(g(s,a))}{4r(s)g^\Delta(s,a)V(s,t_4)}\right]\Delta s = \infty,$$

which indicates (3.13) in Corollary 3.3 is satisfied. Thus, all conditions of Corollary 3.3 are satisfied. By Corollary 3.3 every solution of (4.3) is oscillatory or tends to zero as $t \to \infty$.

Remark 4.1. For (4.3), We note that

$$\int_a^b q(s,\xi)\Delta\xi = \int_0^1\frac{\sqrt{s+\xi}}{s^2}d\xi \leq \frac{\sqrt{s+1}}{s^2} \quad \text{for } s \in [3,\infty).$$

Consequently, for any sufficiently large constant C, we get

$$\int_C^\infty \int_a^b q(s,\xi)\Delta\xi\Delta s \le \int_C^\infty \frac{\sqrt{s+1}}{s^2} ds < \infty.$$

For any $\lambda > 0$, we obtain

$$\int_C^\infty \int_a^b q(s,\xi)[1-(1+\lambda)p(g(s,\xi))]\Delta\xi\Delta s \le \int_C^\infty \int_a^b q(s,\xi)\Delta\xi\Delta s < \infty,$$

which shows (3.1) in Theorem 3.1 can not be satisfied. Thus, Theorem 3.1 can not be applied to (4.3).

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