ABSTRACT. We consider the Dirichlet problem for the Stokes problem in a plane polygonal domain (simply connected) with a reentrant corner at the origin. In addition to the velocity field \( \mathbf{u} \), the tensor field \( \sigma = \nabla \mathbf{u} \) is introduced as a supplementary unknown leading us to a mixed formulation. Note that contrarily to [4], that our mixed formulation of the continuous problem is not hybridized. To discretize the mixed formulation, given a fixed triangulation, each of the two lines of the tensor field \( \sigma \) is approximated on each triangle of the given triangulation by a Raviart-Thomas vector field of degree 0 and \( \mathbf{u} \) by a constant vector field. Under suitable refinement conditions on the regular family of triangulations under consideration \( \{ T_h \} \), we prove that despite to the reentrant corner, that the convergence of the sequence of approximate solutions \( (\sigma_h, p_h, \mathbf{u}_h) \) to \( (\sigma, p, \mathbf{u}) \) is still of order 1 in the \( L^2 \)-norm. Finally, the discretized mixed formulation is hybridized by introducing a Lagrange multiplier \( \lambda_h \) to relax the continuity of the normal trace of \( \mathbf{u}_h \) across interelement edges of the triangulation \( T_h \). A formula expressing \( \lambda_h \) in terms of \( \mathbf{u}_h \) and \( \sigma_h \) is established. Under some additional assumptions, it is proved that \( (\lambda_h)_{e_h} \to \frac{1}{h} \int_{e_h} \mathbf{u} \, ds \) as \( h \to 0^+ \). Without these additional assumptions, the behaviour of the \( \{ \mathbf{u}_h \} \) at the boundary is studied. The jumps of \( \mathbf{u}_h \) as one crosses from one triangle to an adjacent one are studied and also compared with those of the means of \( \mathbf{u} \) on each triangle of the triangulation \( T_h \). In the late section about the numerical implementation, we first derive explicit formulas for \( \sigma_{h|K} \) and \( \mathbf{u}_{h|K} \) on each triangle \( K \) of the triangulation \( T_h \) in terms of the Lagrange multiplier \( \lambda_{h|\partial K} \). We then reduce our problem to the resolution of a linear system with explicit coefficients in terms of the geometry of the triangulation with only the Lagrange
multiplier $\lambda_h$ on each edge and the discrete pressure $p_h$ on each triangle as unknowns.

0 Introduction Let $\Omega$ be a plane domain with a polygonal boundary $\Gamma$, the union of a finite number $N$ of linear segments $\Gamma_j$ numbered according to the positive orientation. We denote by $\omega_j$ the angle between $\Gamma_{j+1}$ and $\Gamma_j$. Without loss of generality, we may assume for simplicity that $\omega_j < \pi$ for every $j$ except for $j = N$ ($\Gamma_{N+1} := \Gamma_1$). To simplify further, we assume that $S_N$, the corner point between $\Gamma_1$ and $\Gamma_N$, has been translated to the origin. In addition, we assume that $\Gamma_1$ is included in the positive abscissa semi-axis $(Ox_1)$ while $\Gamma_N$ is situated in the lower half-plane. The angle counted counterclockwise between $\Gamma_1$ and $\Gamma_N$ which is bigger than $\pi$ is denoted $\omega_N$. To alleviate the notation, we set $\omega := \omega_N$.

We now consider the stationary Stokes problem for an incompressible viscous fluid confined in $\Omega$: given $\vec{f} = (f_1, f_2) \in (L^2(\Omega))^2$ a massic density of forces, find the functions $\vec{u} = (u_1, u_2) \in (H^1_0(\Omega))^2$ the velocity field of the fluid and $p \in L^2_0(\Omega)$ (the space of square integrable functions on $\Omega$ with mean zero) the pressure, satisfying:

\[
\begin{aligned}
- \nu \Delta \vec{u} + \text{grad} p &= \vec{f} \quad \text{in } \Omega, \\
\text{div } \vec{u} &= 0 \quad \text{in } \Omega, \\
\vec{u} &= \vec{0} \quad \text{on } \Gamma.
\end{aligned}
\]

We consider the mixed formulation of problem (0.1), where the gradient of the velocity field is introduced as a new unknown (in [4], a hybridized mixed formulation is considered): find $((\sigma, p), \vec{u}) \in X \times Y$ such that

\[
\begin{aligned}
\nu \int_{\Omega} \sigma : \tau \, dx + \int_{\Omega} \vec{u} \cdot \text{div} (\nu \tau - q \delta) \, dx &= 0, \quad \forall (\tau, q) \in X, \\
\int_{\Omega} \text{div} (\nu \sigma - p \delta) \cdot \vec{v} \, dx &= - \int_{\Omega} \vec{f} \cdot \vec{v} \, dx, \quad \forall \vec{v} \in Y,
\end{aligned}
\]

where

\[
X = \{(\tau, q) \in L^2(\Omega)^4 \times L^2_0(\Omega); \text{div} (\nu \tau - q \delta) \in L^2(\Omega)^2\},
\]

\[
Y = L^2(\Omega)^2.
\]

Let $\{T_h\}$ be a regular family of triangulations over $\Omega$ [2] and $T_h$ a fixed triangulation belonging to $\{T_h\}$. To this triangulation, we associate the
following finite dimensional “approximating” subspaces $X_h$ of $X$ and $Y_h$ of $Y$:

$$X_h = \{(\tau_h, q_h) \in X, \forall K \in T_h, q_h|_K \in P_0(K) \text{ and } ((\tau_h)|_K)_{(i,\cdot)} \in RT_0(K) \text{ for } i = 1, 2\}$$

$$Y_h = \{\vec{v}_h \in Y; \forall K \in T_h: (\vec{v}_h)|_K \in (P_0(K))^2\}.$$ 

$RT_0(K)$, denotes the three dimensional vector space of vector fields on $K$ which are of the form

$$\vec{v}(x_1, x_2) = \left(\begin{array}{c} a \\ b \\ c \\ \end{array} \right) + c \left(\begin{array}{c} x_1 \\ x_2 \end{array} \right),$$

where $a$, $b$ and $c$ are real constants [9, 11] ($RT_0(K)$ coincides with the space $R_1$ defined in [11, page III-2] and with the space $D_1$ defined in [9, p. 550]). $P_0(K)$ denotes the one dimensional space of constant functions on $K$. To these two finite dimensional subspaces $X_h$, and $Y_h$ we associate the discrete problem: find $((\sigma_h, p_h), \vec{u}_h) \in X_h \times Y_h$ solution of

$$\begin{aligned}
&\nu \int_{\Omega} \sigma_h : \tau_h \, dx \\
&+ \int_{\Omega} \vec{u}_h : \text{div}(\nu \tau_h - q_h \delta) \, dx = 0, \quad \forall (\tau_h, q_h) \in X_h, \\
&\int_{\Omega} \text{div}(\nu \sigma_h - p_h \delta) : \vec{v}_h \, dx + \int_{\Omega} \vec{v}_h \cdot \vec{v}_h \, dx = 0, \quad \forall \vec{v}_h \in Y_h.
\end{aligned}$$

(0.3)

Problem (0.3) possesses one and only one solution.

Due to the re-entrant corner of $\Omega$ at the origin, $\vec{u}$ is not in general in the Sobolev space $(H^2(\Omega))^2$. But $\vec{u}$ belongs to the weighted Sobolev space $(H^{2,\alpha}(\Omega))^2$ for every $\alpha > 1 - \eta_0(\omega)$ where

$$\eta_0(\omega) = \inf\{\xi \in \mathbb{R}_+^2; z = \xi + i\eta \text{ verifies } \sin^2 \omega z = z^2 \sin^2 \omega, \ z \neq 1\}$$

([1, Theorem II.1, p. 260].) and

$$H^{2,\alpha}(\Omega) := \{v \in H^1(\Omega); r^\alpha \partial^\beta v \in L^2(\Omega), \forall \beta \in \mathbb{N}^2 \text{ such that } |\beta| = 2\},$$

$$\|v\|_{H^{2,\alpha}(\Omega)} := \left(\|v\|^2_{H^1(\Omega)} + \sum_{|\beta| = 2} \|r^\alpha \partial^\beta v\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$
\((r(.))\text{ denotes the euclidean distance to the origin).}

Due to this fact, to recapture a convergence of order 1 in the \(L^2\)-norm of the approximate solutions \(((\sigma_h, p_h), \tilde{u}_h)\) to the exact solution \(((\sigma, p), \tilde{u})\), we must impose two refinement conditions on the regular family of triangulations \(\{T_h\}\). We suppose that for a certain \(\alpha \in (1 - \eta_0(\omega), 1)\), there exists a constant \(c > 0\) such that \(\forall T_h \in \{T_h\}\) and \(\forall K \in T_h\)

\[(b.1)\] \(h_K \leq ch^\frac{1}{1-\alpha}\) for every \(K \in T_h\) possessing a vertex at 0;

\[(b.2)\] \(h_K \leq c\inf_K r^\alpha\) for every \(K \in T_h\) which do not admit a vertex at 0.

Under these conditions on the regular family of triangulations \(\{T_h\}\) we show that

\[(0.4)\] \(\|\sigma - \sigma_h\|_{L^2(\Omega)^d} \leq Ch\left(|\tilde{u}|_{H^2(\Omega)^d} + |p|_{H^1(\Omega)}\right),\)

\[(0.5)\] \(\|p - p_h\|_{L^2(\Omega)} \leq ch\left(|\tilde{u}|_{H^2(\Omega)^d} + |p|_{H^1(\Omega)}\right)\)

and

\[(0.6)\] \(\|\tilde{u} - \tilde{u}_h\|_{L^2(\Omega)^d} \leq ch\left(|\tilde{u}|_{H^1(\Omega)^d} + |\tilde{u}|_{H^2(\Omega)^d} + |p|_{H^1(\Omega)}\right).\)

If, moreover, we make the stronger assumption that the right-hand side member \(\tilde{f} \in (H^1(\Omega))^2\), then we are able to prove that

\[(0.7)\] \(\|\tilde{u}_h - P_h\tilde{u}\|_{L^2(\Omega)^d} \leq ch^2\left(|\tilde{u}|_{H^2(\Omega)^d} + |p|_{H^1(\Omega)} + |\tilde{f}|_{H^1(\Omega)^2}\right)\).

We introduce in the second section of our paper, the hybrid formulation of the discrete problem (0.3), by introducing a Lagrange multiplier \(\tilde{\lambda}_h\) to relax the continuity of the normal trace of \(\sigma_h\) across interelement edges of the triangulation \(T_h\). From the hybrid formulation (57), we derive an integral formula connecting the Lagrange multiplier \(\tilde{\lambda}_h\) to \(\tilde{u}_h\) and \(\sigma_h\). Supposing, moreover, \(\tilde{f} \in (H^1(\Omega))^2\), \(\alpha \in (1 - \eta_0(\omega_N), \frac{1}{2})\) and that there is a strictly positive constant \(\tilde{c}\) independent of \(h\) such that

\(h_K \geq \tilde{c}h^\frac{1}{1-\alpha}\) for every triangle \(K\) of \(T_h\), and for every triangulation \(T_h \in \{T_h\}\), we show that for any side \(e_h\) of the triangulation \(T_h\):

\[
\left|\langle \tilde{\lambda}_h\rangle_{e_h} - \frac{1}{|e_h|} \int_{e_h} \tilde{u} ds\right| \leq ch^{\frac{1}{1-\alpha}}\left(|\tilde{u}|_{H^2(\Omega)^d} + |p|_{H^1(\Omega)} + |\tilde{f}|_{H^1(\Omega)^2}\right)
\]
which implies that \((\tilde{X}_h)_{e_h} - (1/|e_h|) \int_{e_h} \tilde{u} \, ds \to 0\) uniformly in \(e_h\) when \(h \to 0^+\) (\(|e_h|\) denotes the length of \(e_h\)).

Returning now to the general case i.e. without making additional assumptions on \(\tilde{f}, \alpha\) and \(\{T_h\}\), we study the behaviour of the approximate solutions \(\{\tilde{u}_h\}\) near the boundary and the jumps of \(\tilde{u}_h\) as one crosses from one triangle of the triangulation \(T_h\) to an adjacent one. In particular, it is proved that for every \(\beta \in (0, \eta_0(\omega))\), that there exists a constant \(c > 0\) independent of \(K\) and \(h\) such that

\[
|\tilde{u}_{h,K}| \leq h^\beta \|\tilde{u}\|_{C^\beta(\partial T_h)} + ch(|\tilde{u}|_{H^2(\Omega)} + |p|_{H^1(\Omega)}) \quad \forall K \subset BL(h),
\]

where \(BL(h)\) denotes the boundary layer of \(T_h\) formed by the triangles of \(T_h\) having one side or one vertex contained in the boundary \(\Gamma\) of \(\Omega\).

We study the jumps of the approximate solutions \(\tilde{u}_h\) between adjacent triangles and give also a result comparing these jumps with those of the means of the exact solution \(\tilde{u}\) on each triangle of \(T_h\).

### 1 Error estimates for the mixed formulation

#### 1.1 Position of the problem

Let \(\tilde{f} = (f_1, f_2) \in (L^2(\Omega))^2\) be given and let us consider \(\tilde{u} = (u_1, u_2) \in (H^1_0(\Omega))^2\) and \(p \in L^2_0(\Omega)\), solution of the Stokes problem:

\[
\begin{align*}
-\nu \Delta \tilde{u} + \bar{\nabla} p &= \tilde{f} \quad \text{in } \Omega, \\
\text{div} \tilde{u} &= 0 \quad \text{in } \Omega, \\
\tilde{u} &= 0 \quad \text{on } \Gamma,
\end{align*}
\]

where \(\nu > 0\) denotes the viscosity.

Due to the assumptions made on \(\Omega\), in the introduction, the solution \(\tilde{u} \in (H^{2, \alpha}(\Omega))^2\) for every \(\alpha > 1 - \eta_0(\omega)\) where

\[
\eta_0(\omega) = \inf\{\xi \in \mathbb{R}_+; \ z = \xi + i\eta \text{ verifies } \sin^2 \omega z = z^2 \sin^2 \omega, \ z \neq 1\}
\]

([1, Theorem II.1, p. 260]). By Hölder’s inequality, it follows that \(\tilde{u} \in W^{2,p}(\Omega)\) for every \(p\) such that \(1 < p < \frac{4}{\alpha + 1}\). In particular \(\tilde{u} \in C^0(\Omega)^2\).

Introducing the variable \(\sigma := \bar{\nabla} \tilde{u}\), we can rewrite the Stokes system

\[
\begin{align*}
-\text{div} (\nu \sigma - p \delta) &= \tilde{f} \quad \text{in } \Omega, \\
\text{div} \tilde{u} &= 0 \quad \text{in } \Omega, \\
\sigma &= \bar{\nabla} \tilde{u} \quad \text{in } \Omega, \\
\tilde{u} &= 0 \quad \text{on } \Gamma,
\end{align*}
\]
where \( \delta \) is the identity tensor field. Let us also recall that for any measurable tensor field \( \tau \) on \( \Omega \), that \( \text{div} \ (\tau) \) denotes the vector field on \( \Omega \) defined by \( (\text{div} \, \tau)_i = \sum_{j=1}^{2} \partial \tau_{ij} / \partial x_j \) \((i = 1,2)\).

Let \( X \) be the Hilbert space defined by
\[
X := \{ (\tau, q) \in L^2(\Omega)^4 \times L^2_0(\Omega) ; \text{div} \, (\nu \tau - q \delta) \in L^2(\Omega)^2 \}
\]
and endowed with the Hilbertian norm
\[
\| (\tau, q) \| = (\| \tau \|_{L^2(\Omega)}^2 + \| q \|_{L^2_0(\Omega)}^2 + \| \text{div} \, (\nu \tau - q \delta) \|_{L^2(\Omega)^2}^2)^{\frac{1}{2}}.
\]
Moreover, we also introduce the Hilbert space \( Y := L^2(\Omega)^2 \) endowed with its usual Hilbertian norm. For two tensor fields \( \sigma, \tau \) on \( \Omega \), the scalar product \( \sigma : \tau \) is defined by
\[
\sigma : \tau := \sum_{i,j=1}^{2} \sigma_{ij} \tau_{ij}.
\]

For every \( (\tau, q) \in X \), we have by Green’s formula
\[
\nu \int_{\Omega} \sigma : \tau \, dx = \int_{\Omega} \text{grad} \, (\bar{u}) : (\nu \tau - q \delta) \, dx = - \int_{\Omega} \text{div} \, (\nu \tau - q \delta) \cdot \bar{u} \, dx.
\]

Thus, we have obtained the equation
\[
(3) \quad \nu \int_{\Omega} \sigma : \tau \, dx + \int_{\Omega} \text{div} \, (\nu \tau - q \delta) \cdot \bar{u} \, dx = 0.
\]

On the other hand since \( \text{div} \, (\nu \sigma - p \delta) = - \bar{f} \), we have also \( \forall \, \bar{v} \in Y \) that
\[
(4) \quad \int_{\Omega} \text{div} \, (\nu \sigma - p \delta) \cdot \bar{v} \, dx = - \int_{\Omega} \bar{f} \cdot \bar{v} \, dx.
\]

Thus, if \((\bar{u}, p)\) is the solution of the Stokes system (1), then \((\sigma := \text{grad} \, \bar{u}, p, \bar{u})\) is solution of the system
\[
(5) \quad \begin{cases}
\nu \int_{\Omega} \sigma : \tau \, dx + \int_{\Omega} \bar{u} \cdot \text{div} \, (\nu \tau - q \delta) \, dx = 0, \forall \, (\tau, q) \in X, \\
\int_{\Omega} \text{div} \, (\nu \sigma - p \delta) \cdot \bar{v} \, dx = - \int_{\Omega} \bar{f} \cdot \bar{v} \, dx, \forall \, \bar{v} \in Y.
\end{cases}
\]

which is called the mixed formulation of (1).

We now prove that the solution of (5) is unique.
Theorem 1.1. The mixed formulation (5) possesses one and only one solution.

Proof. We have already proved before that problem (5) has at least one solution. It remains to show that this solution is unique. For that, let us show that if \(((\sigma, p), \overline{u}) \in X \times Y\) checks

\[
\begin{align*}
\nu \int_{\Omega} \sigma : \tau \, dx + \int_{\Omega} \overline{u} \cdot \text{div} \left( \nu \tau - q \delta \right) \, dx &= 0, \quad \forall (\tau, q) \in X, \\
\int_{\Omega} \text{div} \left( \nu \sigma - p \delta \right) \cdot \bar{v} \, dx &= 0, \quad \forall \bar{v} \in Y,
\end{align*}
\]

then \(((\sigma, p), \overline{u}) = 0.

By taking \((\tau, q) = (\sigma, p)\) in the first equation of (6) and \(\bar{v} = \overline{u}\) in the second equation of (6), we obtain \(\int_{\Omega} \sigma : \sigma \, dx = 0\), from which follows \(\sigma = 0\). Consequently, the first equation of (6) is reduced to :

\[
\int_{\Omega} \overline{u} \cdot \text{div} \left( \nu \tau - q \delta \right) \, dx = 0, \quad \forall (\tau, q) \in X.
\]

Let us take \(q = 0\) and solve in \(\Omega\) the Dirichlet problem \(\Delta \overline{\Psi} = \overline{u}\) with \(\overline{\Psi} \in H^{1}(\Omega)^{2}\). Let us set \(\tau = \nabla \overline{\Psi}, \tau \in L^2(\Omega)^4\) and \(\text{div} \tau = \overline{u} \in Y\). Therefore \((\tau, 0) \in X\). Applying (7) to \((\tau, 0)\), we obtain \(\int_{\Omega} \overline{u}^2 \, dx = 0\), from which we deduce \(\overline{u} = 0\).

From \((0, p) \in X\), it follows that \(p \in H^1(\Omega) \cap L_0^2(\Omega)\). By the second equation of (6) it follows that \(\int_{\Omega} \nabla \overline{p} \cdot \bar{v} \, dx = 0, \forall \bar{v} \in Y\). Choosing \(\bar{v} = \nabla \overline{p}\), it follows that \(p = \text{constant}\), but as \(p \in L_0^2(\Omega)\), it follows that \(p = 0\).

Now, we want to introduce the discrete formulation of the problem (5). In that purpose, let us consider a triangulation \(T_h\) of \(\overline{\Omega}\) and let us define the approximation spaces \(X_h\) and \(Y_h\) for the spaces \(X\) and \(Y\), respectively,

\[
X_h = \{(\tau_h, q_h) \in X \quad \forall K \in T_h, \quad q_h|_K \in P_0(K) \quad \text{and} \quad (\tau_h|_K)_{(i, \cdot)} \in RT_0(K) \text{ for } i = 1, 2\}
\]

\[
Y_h = \{\bar{u}_h \in Y; \quad \forall K \in T_h; \quad (\bar{u}_h)|_K \in (P_0(K))^2\}
\]

where \(P_0(K)\) denotes the vector space of constant functions on \(K\). We can now introduce the finite dimensional problem: find \(((\sigma_h, p_h), \overline{u}_h) \in \)
We want to prove that the finite dimensional problem (8) always has, one and only one solution \(((\sigma_h, p_h), \bar{u}_h)) \in X_h \times Y_h.

**Proposition 1.2.** Problem (8) possesses one and only one solution \(((\sigma_h, p_h), \bar{u}_h)) \in X_h \times Y_h.

**Proof.** To every element \(((\sigma_h, p_h), \bar{u}_h)) \in X_h \times Y_h$, we associate the element of its dual $X'_h \times Y'_h$ where

$$X_h \rightarrow \mathbb{R} : (\tau_h, q_h) \mapsto \nu \int_{\Omega} \sigma_h \cdot \tau_h \, dx + \int_{\Omega} \bar{u}_h \cdot \text{div} (\nu \tau_h - q_h \delta) \, dx,$$

$$Y_h \rightarrow \mathbb{R} : \bar{v}_h \mapsto \int_{\Omega} \bar{v}_h \cdot \text{div} (\nu \sigma_h - p_h \delta) \, dx.$$

Let us call this mapping $\Phi_h$; it is a linear mapping from $X_h \times Y_h$ into its dual. We have to prove that $\Phi_h$ is injective and surjective. But $\Phi_h$ is linear and the arrival space has the same finite dimension as the departure space. Thus by a well known theorem of linear algebra, it suffices to prove that $\Phi_h$ is injective. Thus let \(((\sigma_h, p_h), \bar{u}_h)) \in X_h \times Y_h$ be such that

$$\int_{\Omega} \text{div} (\nu \sigma_h - p_h \delta) \cdot \bar{v}_h \, dx = 0, \quad \forall \bar{v}_h \in Y_h,$$

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$$\nu \int_{\Omega} \sigma_h \cdot \tau_h \, dx + \int_{\Omega} \bar{u}_h \cdot \text{div} (\nu \tau_h - q_h \delta) \, dx = 0, \quad \forall (\tau_h, q_h) \in X_h.$$

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From (9), it follows that $\int_{\Omega} \bar{u}_h \cdot \text{div} (\nu \sigma_h - p_h \delta) \, dx = 0$, and then by taking, $\tau_h = \sigma_h$ and $q_h = p_h$ in (10), we get $\int_{\Omega} \sigma_h \cdot \sigma_h \, dx = 0$, implying $\sigma_h = 0$. Knowing that $\sigma_h = 0$, (10) reduces to

$$\int_{\Omega} \bar{u}_h \cdot \text{div} (\nu \tau_h - q_h \delta) \, dx = 0, \quad \forall (\tau_h, q_h) \in X_h.$$

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By Lemma 1.2 in [10], there exists

\[ \tilde{Q}^1_h, \tilde{Q}^2_h \in \{ q_h \in H(\text{div}, \Omega) : \forall K \in T_h : q_h|_K \in RT_0(K) \} \]

such that \( \text{div} \tilde{Q}^1_h = u^1_h \) and \( \text{div} \tilde{Q}^2_h = u^2_h \). With \( \tau_h = \begin{pmatrix} Q^1_{h,1} & Q^1_{h,2} \\ Q^2_{h,1} & Q^2_{h,2} \end{pmatrix} \) and \( q_h = 0 \) we obtain from (11) that \( \int_{\Omega} |u_h|^2 \, dx = 0 \), implying \( u_h = 0 \).

Now, it remains to show that \( p_h = 0 \). We have \( p_h|_K = \text{cte} \), and owing to the fact that \( \sigma_h = 0 \) it follows that \( p_h \delta \in H(\text{div}, \Omega) \) therefore \( p_h = \text{cte} \) in \( \Omega \). Since \( p_h \in L^2_0(\Omega) \), this constant can be only zero.

### 1.2 Error estimates

We now want to prove some approximation result of \( \sigma_h \) by \( \sigma_h \) in the \( L^2(\Omega) \)-norm. Let us first introduce the following subspace \( V^f_h \) of \( X_h \)

\[
V^f_h = \left\{ (\tau_h, q_h) \in X_h ; \int_{\Omega} \text{div} (\nu \tau_h - q_h \delta) \cdot \bar{v}_h \, dx \right. \\
\quad \quad \quad - \left. \int_{\Omega} \bar{f} \cdot \bar{v}_h \, dx , \forall \bar{v}_h \in Y_h \right\}.
\]

An alternative description of the space \( V^f_h \) is given by the following lemma.

**Lemma 1.3.**

\[
V^f_h = \left\{ (\tau_h, q_h) \in X_h ; \text{div} (\nu \tau_h)|_K = -\frac{1}{|K|} \int_{K} \bar{f} \, dx , \forall K \in T_h \right\}.
\]

**Proof.** The set \( \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, K \in T_h \) is a basis for \( Y_h \) where \( 1_K \) is the characteristic function of the triangle \( K \). Then \( (\tau_h, q_h) \in X_h \) is in \( V^f_h \) if and only if for every \( K \in T_h \)

\[
\int_{K} \text{div} (\nu \tau_h) = -\int_{K} \bar{f} \, dx.
\]

Owing to the fact that \( (\tau_h, q_h) \) is in \( X_h \), we obtain that \( \text{div} (\nu \tau_h) \) is constant on each triangle \( K \in T_h \). Then \( \int_{K} \text{div} (\nu \tau_h) = |K| \text{div} (\nu \tau_h) \), and replacing in the preceding equation, we obtain the result.

**Corollary 1.4.** \( \forall (\tau_h, q_h) \in V^f_h, \text{div} (\nu (\sigma_h - \tau_h)) = \text{div} ((p_h - q_h) \delta) \).
Proof. The second equation of the system (8) expresses exactly that 
\((\sigma_h, p_h) \in V_h^f\). Then, by Lemma 1.3, we have
\[
div (\nu \sigma_h - p_h \delta) |_K = div (\nu \tau_h - q_h \delta) |_K, \quad \forall \ K \in T_h.
\]
Since \(div [(\nu \sigma_h - p_h \delta) - (\nu \tau_h - q_h \delta)] \in L^2(\Omega)\), this implies that
\[
div [(\nu \sigma_h - p_h \delta) - (\nu \tau_h - q_h \delta)] = 0.
\]
\[\square\]

Corollary 1.5. \(\forall (\tau_h, q_h) \in V_h^f\),
\begin{align*}
(12) \quad &\int_\Omega \sigma : (\sigma_h - \tau_h) \, dx = 0, \\
(13) \quad &\int_\Omega \sigma_h : (\sigma_h - \tau_h) \, dx = 0, \\
(14) \quad &\int_\Omega (\sigma - \sigma_h) : (\sigma_h - \tau_h) \, dx = 0.
\end{align*}

Proof. 1) \((\sigma_h - \tau_h, p_h - q_h) \in X_h \subset X\), it follows from the first equation of system (5) that
\[
\nu \int_\Omega \sigma : (\sigma_h - \tau_h) \, dx + \int_\Omega \bar{u} \, div (\nu (\sigma_h - \tau_h) - (p_h - q_h) \delta) \, dx = 0.
\]
By application of Corollary 1.4 we obtain (12).

2) \((\sigma_h - \tau_h, p_h - q_h) \in X_h\), it follows from the first equation of system (8) that
\[
\nu \int_\Omega \sigma_h : (\sigma_h - \tau_h) \, dx + \int_\Omega \bar{u}_h \, div (\nu (\sigma_h - \tau_h) - (p_h - q_h) \delta) \, dx = 0.
\]
By application of Corollary 1.4, we obtain (13).

3) Subtracting (13) from (12) gives us (14). \[\square\]

Corollary 1.6.
\begin{equation}
\|\sigma - \sigma_h\|_{L^2(\Omega)^4} = \inf_{\tau_h \in W_h^f} \|\sigma - \tau_h\|_{(L^2(\Omega)^4)^*},
\end{equation}
where \(W_h^f = \{\tau_h \in L^2(\Omega)^4; \exists q_h \text{ such that } (\tau_h, q_h) \in V_h^f\} \).
Proof.

\[
\|\sigma - \sigma_h\|_{L^2(\Omega)}^2 = \int_{\Omega} (\sigma - \sigma_h) : (\sigma - \sigma_h) \, dx \\
= \int_{\Omega} (\sigma - \sigma_h) : (\sigma - \tau_h) \, dx \\
+ \int_{\Omega} (\sigma - \sigma_h) : (\tau_h - \sigma_h) \, dx \\
= \int_{\Omega} (\sigma - \sigma_h) : (\sigma - \tau_h) \, dx
\]

the second integral being zero by (14). It follows that

\[
\|\sigma - \sigma_h\|_{L^2(\Omega)}^2 \leq \|\sigma - \sigma_h\|_{L^2(\Omega)} \|\sigma - \tau_h\|_{L^2(\Omega)}.
\]

Simplifying the two members by \(\|\sigma - \sigma_h\|_{L^2(\Omega)}\), we obtain

\[
\|\sigma - \sigma_h\|_{L^2(\Omega)} \leq \|\sigma - \tau_h\|_{L^2(\Omega)}, \quad \forall \tau_h \in W_h^f.
\]

From this inequality, we have

\[
\|\sigma - \sigma_h\|_{L^2(\Omega)} \leq \inf_{\tau_h \in W_h^f} \|\sigma - \tau_h\|_{L^2(\Omega)}.
\]

But the second equation of the system (8) implies that \(\sigma_h \in W_h^f\). Thus (15) is proved.

It is known that \(\text{div} (\nu \sigma - p \delta) = -\bar{\mathbf{f}} \in L^2(\Omega)^2\) and that \((\nu \sigma - p \delta) \in W^{1,r}(\Omega)^4\) for \(r > 1\), \(r\) sufficiently near 1 since as seen previously \(\bar{\mathbf{u}} \in H^{2,\alpha}(\Omega)^2\) for some \(\alpha \in (0, \frac{1}{2})\). Thus, we can apply the Raviart-Thomas interpolation operator to \((\nu \sigma - p \delta)\), which we denote by \(\pi_h^1\). Then we define

\[
\pi_h(\sigma, p) = \left( \frac{1}{\nu} \left[ \pi_h^1(\nu \sigma - p \delta) + \rho_h p \delta \right], \rho_h p \right),
\]

where we have ([4, p. 87])

1°) \(\forall K \in T_h\),

\[
\frac{1}{\nu} \left[ \pi_h^1(\nu \sigma - p \delta) + \rho_h p \delta \right]_K = \frac{1}{\nu} \left[ \pi_h^1(\nu \sigma - p \delta) + (\rho_h p)_{|K} \delta \right]_K \in RT_h(K)^2.
\]
2°) \( \rho_h \) denotes the operator of orthogonal projection of \( L_0^2(\Omega) \) onto the subspace \( \{ q_h \in L_0^2(\Omega) : q_h|_K \in P_0(K), \forall K \in T_h \} \), i.e.,

\[
(\rho_h q)_K := \rho_K q := \frac{1}{|K|} \int_K q \, dx, \quad \forall K \in T_h.
\]

**Theorem 1.7.** Let \( \{T_h\} \) be a family of triangulations over \( \Omega \) satisfying the following hypotheses:

(i) the family of triangulations \( \{T_h\} \) is regular, i.e., there exists a constant \( c \) such that \( \max_{K \in T_h} \frac{1}{\rho_K} \leq c \) where \( h_K \) denotes the diameter of \( K \) and \( \rho_K \) its interior diameter;

(ii) \( h_K \leq c h^{\frac{1}{2}} \) for every triangle \( K \in T_h \) which has one of its vertices at the origin and for every triangulation \( T_h \) belonging to the family \( \{T_h\} \) (\( h := \max_{K \in T_h} h_K \));

(iii) \( h_K \leq c(\inf_{x \in K} r^\alpha(x))h \) for every triangle \( K \in T_h \) without any vertices at the origin and for every triangulation \( T_h \) belonging to the family \( \{T_h\} \) (\( r(x) \) denotes the distance from the point \( x \) to the origin).

Then there exists a constant \( C \) independent of \( h \) such that

\[
\| \sigma - \sigma_h \|_{L^2(\Omega)^4} \leq Ch \left( \| \tilde{\omega} \|_{H^{2,\alpha} (\Omega)^2} + \| p \|_{H^{1,\alpha} (\Omega)} \right)
\]

for \( \alpha > 1 - \gamma_0(\omega) \).

**Proof.** Since the normal component of \( \pi_h^1(\nu \sigma - p \delta) \) is continuous on the interface between two triangles, \( \pi_h^1(\nu \sigma - p \delta) \in H(\text{div}, \Omega)^2 \).

Let \( K \in T_h \). We have

\[
(\text{div}(\pi_h^1(\nu \sigma - p \delta)))|_K = -\frac{1}{|K|} \int_K \tilde{f} \, dx.
\]

Indeed,

\[
\int_K \text{div}(\pi_h^1(\nu \sigma - p \delta)) \, dx = \int_{\partial K} \pi_h^1(\nu \sigma - p \delta) \cdot \tilde{n} \, ds
\]

\[
= \sum_{e \subset \partial K} \int_e \pi_h^1(\nu \sigma - p \delta) \cdot \tilde{n} \, ds
\]

\[
= \sum_{e \subset \partial K} \int_e (\nu \sigma - p \delta) \cdot \tilde{n} \, ds
\]
(by the definition of $\pi_K$)
\[
\frac{1}{\nu} \int_{\partial K} (\nu \sigma - p \delta) \cdot \bar{n} \, ds
= \int_K \text{div} (\nu \sigma - p \delta) \, dx
= - \int_K \bar{f} \, dx.
\]

Therefore by Lemma 1.3, it follows that $\pi_h(\sigma, p) \in V_h^\bar{f}$. Then by application of Corollary 1.6, we obtain
\[
\|\sigma - \sigma_h\|_{L^2(\Omega)^4} \leq \left\| \frac{1}{\nu} \left[ \pi_h^1(\nu \sigma - p \delta) + \rho_h p \delta \right] \right\|_{L^2(\Omega)^4}
= \left\| \frac{1}{\nu} (\nu \sigma - p \delta) - \frac{1}{\nu} \pi_h^1(\nu \sigma - p \delta) \right\|_{L^2(\Omega)^4}
\leq \frac{1}{\nu} \| (\nu \sigma - p \delta) - \pi_h^1(\nu \sigma - p \delta) \|_{L^2(\Omega)^4}
+ \frac{1}{\nu} \| p - \rho_h p \|_{L^2(\Omega)}.
\]

**Case 1:** Let us suppose that the triangle $K \in T_h$ has one of its vertices at the origin. By Proposition 1.12 of [10] applied to each line of the tensor $(\nu \sigma - p \delta)$, we obtain
\[
\| (\nu \sigma - p \delta) - \pi_h^1(\nu \sigma - p \delta) \|_{L^2(K)^4}
\leq c^2 |B_K|^4 |B_K^{-1}|^{2+2\alpha} \int_K |x|^{2\alpha} |D(\nu \sigma - p \delta)|^2 \, dx,
\]
where the derivative of a tensor $a = (a_{ij})_{1 \leq i,j \leq n}$ is given by $(Da)_{ijk} = \partial a_{ij} / \partial x_k$.

By using the traditional properties $|B_K| \leq \sqrt{2} h_K$ and $|B_K^{-1}| \leq \sqrt{2} / \rho_K$, the previous bound of $\| (\nu \sigma - p \delta) - \pi_h^1(\nu \sigma - p \delta) \|_{L^2(K)^4}$ becomes
\[
\| (\nu \sigma - p \delta) - \pi_h^1(\nu \sigma - p \delta) \|_{L^2(K)^4}
\leq c^2 |B_K|^4 |B_K^{-1}|^{2+2\alpha} \int_K |x|^{2\alpha} |D(\nu \sigma - p \delta)|^2 \, dx,
\]
\[
\leq c^2 h_k^4 \rho_K^{(2+2\alpha)} \int_K |x|^{2\alpha} |D(\nu \sigma - p\delta)|^2 \, dx
\]
\[
\leq c^2 h_k^{2(1-\alpha)} \int_K |x|^{2\alpha} |D(\nu \sigma - p\delta)|^2 \, dx
\]
\[
\leq c^2 h_k^{2(1-\alpha)} \int_K |x|^{2\alpha} |D(\nu \sigma - p\delta)|^2 \, dx
\]
\[
\leq c^2 h_k^2 \int_K |x|^{2\alpha} |D(\nu \sigma - p\delta)|^2 \, dx.
\]

**Case 2:** Let us suppose that none of the vertices of the triangle \( K \in T_h \) is at the origin. Using Proposition 1.12 of [10] with \( \alpha = 0 \) and an arbitrary vertex of \( K \), we obtain

\[
\|(\nu \sigma - p\delta) - \pi_K^1(\nu \sigma - p\delta)\|_{L^2(K)}^2 \leq c^2 |B_K|^4 |B_K^{-1}|^2 \int_K |D(\nu \sigma - p\delta)|^2 \, dx
\]
\[
\leq \frac{c^2 h_k^4 \rho_K^{-2}}{(\inf_{x \in K} r^\alpha(x))^2} \int_K |x|^{2\alpha} |D(\nu \sigma - p\delta)|^2 \, dx
\]
\[
\leq \frac{c^2 h_k^2}{(\inf_{x \in K} r^\alpha(x))^2} \int_K |x|^{2\alpha} |D(\nu \sigma - p\delta)|^2 \, dx.
\]

Therefore by hypothesis (iii), we obtain:

\[
(22) \quad \|(\nu \sigma - p\delta) - \pi_K^1(\nu \sigma - p\delta)\|_{L^2(K)}^2 \leq c^2 h_k^2 \int_K |x|^{2\alpha} |D(\nu \sigma - p\delta)|^2 \, dx.
\]

From the inequalities (21) and (22), it follows that:

\[
(23) \quad \|(\nu \sigma - p\delta) - \pi_K^1(\nu \sigma - p\delta)\|_{L^2(\Omega)}^2 \leq c^2 h_k^2 \sum_{K \in T_h} \int_K |x|^{2\alpha} |D(\nu \sigma - p\delta)|^2 \, dx
\]
\[
\leq c^2 h_k^2 \int_\Omega |x|^{2\alpha} |D(\nu \sigma - p\delta)|^2 \, dx.
\]

Now, we want also to prove that \( \|p - \rho_h p\|_{L^2(\Omega)} \) is bounded by some constant times \( h \), more precisely, to show that

\[
(24) \quad \|p - \rho_h p\|_{L^2(\Omega)} \leq c h |p|_{H^{1,\alpha}(\Omega)}.
\]
Case 1: $K$ has one of its vertices at the origin.

\[ \|p - \rho_K p\|_{L^2(K)}^2 \leq |\det B_K| \int_K |p \circ F_K(\hat{x}) - \frac{1}{|K|} \int_K p \circ F_K(\hat{y}) \, d\hat{y}|^2 \, d\hat{x} \]

\[ \leq |\det B_K| \int_K |\hat{x}|^{2\alpha} |\nabla (p \circ F_K)|^2 \, d\hat{x} \]

(by Lemma 1.8 of [10])

\[ \leq c |B_K^{-1}|^{2\alpha} \int_K |x|^{2\alpha} |Dp(x) \circ B_K|^2 \, dx \]

\[ \leq c |B_K^{-1}|^{2\alpha} |B_K|^2 \int_K |x|^{2\alpha} |\nabla p(x)|^2 \, dx \]

\[ \leq ch_k^{2(1-\alpha)} \int_K |x|^{2\alpha} |\nabla p|^2(x) \, dx \]

\[ \leq ch_k^2 \int_K |x|^{2\alpha} |\nabla p|^2(x) \, dx. \]

Case 2: No vertex of the triangle $K \in T_h$ is at the origin.

\[ \|p - \rho_K p\|_{L^2(K)}^2 \leq c |\det B_K| \int_K |\nabla (p \circ F_K)|^2(\hat{x}) \, d\hat{x} \]

(by Lemma 1.8 of [10] applied with $\alpha = 0$)

\[ \leq ch_k^2 \int_K |\nabla p|^2(x) \, dx \]

\[ \leq c \frac{h_k^2}{(\inf_{x \in K} r^n(x))^2} \int_K |x|^{2\alpha} |\nabla p|^2 \, dx \]

\[ \leq ch_k^2 \int_K |x|^{2\alpha} |\nabla p|^2 \, dx. \]

Summing over $K$, we obtain

\[ \|p - \rho_h p\|_{L^2(\Omega)} = \left( \sum_{K \in T_h} \|p - \rho_K p\|_{L^2(K)}^2 \right)^{\frac{1}{2}} \leq ch |p|_{H^{1,\alpha}(\Omega)}. \]
which establishes bound (24).

From the inequalities (19), (23) and (24), it follows that

$$\|\sigma - \sigma_h\|_{L^2(\Omega)^4} \leq c h (\|\widetilde{u}\|_{H^{2,\infty}(\Omega)^2} + |p|_{H^{1,\infty}(\Omega)}).$$

Let us now bound the error on $p_h$ in the $L^2$-norm, i.e., $\|p - p_h\|_{0,\Omega}$.

**Theorem 1.8.** Let $((\sigma, p), \widetilde{u})$ be the solution of the continuous problem and let $((\sigma_h, p_h), \widetilde{u}_h)$ be the solution of the discrete problem. Under the hypotheses of Theorem 1.7, there exists a constant $c > 0$ independent of $h$ such that

$$\|p - p_h\|_{L^2(\Omega)} \leq c h \left(\|\widetilde{u}\|_{H^{2,\infty}(\Omega)^2} + |p|_{H^{1,\infty}(\Omega)}\right).$$

**Proof.** According to [4, p. 92] for every $(\tau, q)$ such that $\text{div}(\tau) = 0$ and $\int_\Omega q \, dx = 0$, we have

$$\|q\|_{L^2(\Omega)} \leq c \|\tau\|_{L^2(\Omega)^4}.$$

Since $\text{div}(\pi_h^1(\nu \sigma - p\delta) - (\nu \sigma_h - p_h\delta)) = 0$ and because $\pi_h(\sigma, p)$ and $(\sigma_h, p_h)$ belong to the space $V^V$, and $\int_\Omega (\rho_h p - p_h) \, dx = 0$, then (26) applies to our case, which implies that

$$\|\rho_h p - p_h\|_{L^2(\Omega)} \leq c \left(\|\frac{1}{\nu} [\pi_h^1(\nu \sigma - p\delta) + \rho_h \delta] - \sigma_{h}\|_{L^2(\Omega)^4}
\leq c \left(\frac{1}{\nu} \|\pi_h^1(\nu \sigma - p\delta) + \rho_h \delta - \sigma - \sigma_h\|_{L^2(\Omega)^4}
\leq c \|\pi_h^1(\nu \sigma - p\delta) - (\nu \sigma - p\delta)\|_{L^2(\Omega)^4}
\leq c h \left(\|\tilde{u}\|_{H^{2,\infty}(\Omega)^2} + |p|_{H^{1,\infty}(\Omega)}\right)\right).$$

By the triangular inequality, (24) and (27), we obtain (25).
It remains thus to bound the error on the velocity field $\tilde{u}_h$. Firstly, we establish the following.

**Proposition 1.9.** Let $\{T_h\}$ be a regular family of triangulations over $\Omega$. Then, there exists a constant $c > 0$ independent of $h$ such that

$$\|\tilde{u} - \tilde{u}_h\|_{L^2(\Omega)^2} \leq c \left( \inf_{\tilde{v}_h \in X_h} \|\tilde{u} - \tilde{v}_h\|_{L^2(\Omega)^2} + \|\sigma - \sigma_h\|_{L^2(\Omega)^2} \right).$$

**Proof.** Let $(\tau_h, q_h) \in X_h$. By the first equation of system (5), we obtain

$$\nu \int_\Omega \sigma : \tau_h \, dx + \int_\Omega \tilde{u} \text{div} \left( \nu \tau_h - q_h \delta \right) \, dx = 0,$$

and by the first equation of system (8), we obtain

$$\nu \int_\Omega \sigma_h : \tau_h \, dx + \int_\Omega \tilde{u}_h \text{div} \left( \nu \tau_h - q_h \delta \right) \, dx = 0.$$

Subtracting (30) from (29), we obtain

$$\nu \int_\Omega (\sigma - \sigma_h) : \tau_h \, dx + \int_\Omega (\tilde{u} - \tilde{u}_h) \text{div} \left( \nu \tau_h - q_h \delta \right) \, dx = 0.$$

Adding $\int_\Omega (\tilde{u}_h - \tilde{v}_h) \text{div} \left( \nu \tau_h - q_h \delta \right) \, dx$ to each member of the previous equality, then dividing the two members by

$$\| (\tau_h, q_h) \|_{X_h} := \left( \|\tau_h\|^2_{L^2(\Omega)^2} + \|q_h\|^2_{L^2(\Omega)^2} + \|\text{div} \left( \nu \tau_h - q_h \delta \right)\|^2_{L^2(\Omega)^2} \right)^{\frac{1}{2}},$$

we obtain

$$\frac{\nu \int_\Omega (\sigma - \sigma_h) : \tau_h \, dx}{\| (\tau_h, q_h) \|_{X_h}} + \frac{\int_\Omega (\tilde{u} - \tilde{v}_h) \text{div} \left( \nu \tau_h - q_h \delta \right) \, dx}{\| (\tau_h, q_h) \|_{X_h}} = \frac{\int_\Omega (\tilde{u}_h - \tilde{v}_h) \text{div} \left( \nu \tau_h - q_h \delta \right) \, dx}{\| (\tau_h, q_h) \|_{X_h}}.$$

But

$$\frac{|\nu \int_\Omega (\sigma - \sigma_h) : \tau_h \, dx|}{\| (\tau_h, q_h) \|_{X_h}} + \frac{|\int_\Omega (\tilde{u} - \tilde{v}_h) \text{div} \left( \nu \tau_h - q_h \delta \right) \, dx|}{\| (\tau_h, q_h) \|_{X_h}}$$
From (32), it now follows that

\[
\| \sigma - \sigma_h \|_{L^2(\Omega)} + \| \tau_h \|_{L^2(\Omega)} \leq \nu \| \sigma - \sigma_h \|_{L^2(\Omega)} + \| \tau_h \|_{L^2(\Omega)}.
\]

By application of Corollary 1.15 of [10] to each component of \((\tilde{u}_h - \tilde{v}_h)\), we obtain

\[
\sup_{(\tau_h, q_h) \in X_h} \frac{1}{\| (\tau_h, q_h) \|_{X_h}} \int_{\Omega} (\tilde{u}_h - \tilde{v}_h) \div (\nu \tau_h - q_h \delta) \, dx \geq \beta^* \| \tilde{u}_h - \tilde{v}_h \|_{L^2(\Omega)}.
\]

(33) and (34) imply

\[
\| \tilde{u}_h - \tilde{v}_h \|_{L^2(\Omega)} \leq \frac{1}{\beta} \left( \nu \| \sigma - \sigma_h \|_{L^2(\Omega)} + \| \tilde{u} - \tilde{v}_h \|_{L^2(\Omega)} \right).
\]

By the triangle inequality, it follows from the previous inequality

\[
\| \tilde{u} - \tilde{u}_h \|_{L^2(\Omega)} \leq \| \tilde{u} - \tilde{v}_h \|_{L^2(\Omega)} + \| \tilde{u}_h - \tilde{v}_h \|_{L^2(\Omega)} \\
\leq \nu \| \sigma - \sigma_h \|_{L^2(\Omega)} + \left( 1 + \frac{1}{\beta^*} \right) \| \tilde{u} - \tilde{v}_h \|_{L^2(\Omega)}.
\]

Taking the infimum over \(\tilde{v}_h\) running in \(Y_h\), we obtain the result. \(\Box\)

We are now in a position to bound the error on the velocity field \(\tilde{u}_h\).

**Theorem 1.10.** Let \(\{T_h\}\) be a family of triangulations over \(\Omega\) as in Theorem 1.7. Then, there exists a constant \(c > 0\) independent of \(h\) such that

\[
\| \tilde{u} - \tilde{u}_h \|_{L^2(\Omega)} \leq ch \left( |\tilde{u}|_{H^1(\Omega)} + |\tilde{u}|_{H^2(\Omega)} + |p|_{H^1(\Omega)} \right)
\]
Proof. We define $P_h \bar{u} \in Y_h$ as the vector function whose restriction to each triangle $K$ of $T_h$ is equal to the average of $\bar{u}$ on $K$. Then

$$\|\bar{u} - P_h \bar{u}\|_{L^2(\Omega)^2}^2 = \sum_{K \in T_h} \int_K |\bar{u}(x) - \frac{1}{|K|} \int_K \bar{u}(y) dy|^2 dx. \tag{37}$$

Denoting by

$$A = (a_1, a_2), \quad B = (a_1 + b_1, a_2 + b_2) \quad \text{and} \quad C = (a_1 + c_1, a_2 + c_2)$$

the coordinates of the three vertices of the triangle $K$ cited in the counterclockwise order, the affine mapping

$$F_K : \hat{K} \rightarrow K : \hat{y} = \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \end{pmatrix} \mapsto F_K(\hat{y}) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \end{pmatrix}$$

maps the reference triangle $\hat{K}$ with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$ bijectively onto the triangle $K$. In the following, the matrix $(b_1, c_1)$ will also be denoted $B_K$. By this affine change of variables, we obtain

$$\int_K \bar{u}(y) dy = \frac{1}{|K|} \int_{\hat{K}} \bar{u}(F_K(\hat{y})) d\hat{y}. \tag{38}$$

(37) and (38) imply

$$\|\bar{u} - P_h \bar{u}\|_{L^2(\Omega)^2}^2 = \sum_{K \in T_h} \int_K \left| \bar{u}(x) - \frac{1}{|K|} \int_{\hat{K}} \bar{u}(F_K(\hat{y})) d\hat{y} \right|^2 dx = \sum_{K \in T_h} |\det B_K| \int_{\hat{K}} \left| \bar{u}(F_K(\hat{x})) \right|^2 d\hat{x} - \frac{1}{|K|} \int_{\hat{K}} \bar{u}(F_K(\hat{y})) d\hat{y} \right|^2 d\hat{x}$$

$$= \sum_{K \in T_h} |\det B_K| \int_{\hat{K}} \left| \bar{v}(\hat{x}) - \frac{1}{|K|} \int_{\hat{K}} \bar{v}(\hat{y}) d\hat{y} \right|^2 d\hat{x}$$

(39)

$$\leq c \sum_{K \in T_h} |\det B_K| \int_{\hat{K}} |\nabla \bar{v}(\hat{x})|^2 d\hat{x},$$

(where $\bar{v} := \bar{u} \circ F_K$)
by Poincaré inequality for functions of mean zero (see [3, 5]). As

$$\nabla \vec{u} = \begin{pmatrix} \frac{\partial u_1}{\partial \vec{x}_1} & \frac{\partial u_1}{\partial \vec{x}_2} \\ \frac{\partial u_2}{\partial \vec{x}_1} & \frac{\partial u_2}{\partial \vec{x}_2} \end{pmatrix}$$

where

$$\frac{\partial u_i}{\partial \vec{x}_1}(\vec{x}) = \left( b_1 \frac{\partial u_i}{\partial x_1} + b_2 \frac{\partial u_i}{\partial x_2} \right) (F_K(\vec{x})), $$

$$\frac{\partial u_i}{\partial \vec{x}_2}(\vec{x}) = \left( c_1 \frac{\partial u_i}{\partial x_1} + c_2 \frac{\partial u_i}{\partial x_2} \right) (F_K(\vec{x})), \quad (i = 1, 2).$$

It follows that

\begin{equation}
(40) \quad \int_{\vec{K}} |\nabla \vec{u}(\vec{x})|^2 d\vec{x} \leq 2 \int_{\vec{K}} \left( b_1^2 + c_1^2 \right) \left( \left( \frac{\partial u_1}{\partial x_1} \right)^2 + \left( \frac{\partial u_2}{\partial x_1} \right)^2 \right) (F_K(\vec{x})) d\vec{x} 
+ 2 \left( b_2^2 + c_2^2 \right) \left( \left( \frac{\partial u_1}{\partial x_2} \right)^2 + \left( \frac{\partial u_2}{\partial x_2} \right)^2 \right) (F_K(\vec{x})) d\vec{x} 
\leq 2 (b_1^2 + c_1^2 + b_2^2 + c_2^2) \int_{\vec{K}} |\nabla \vec{u}|^2 (F_K(\vec{x})) d\vec{x} 
\leq 2 \frac{|B_K|^2}{|\det B_K|} \int_{\vec{K}} |\nabla \vec{u}|^2(x) dx 
\leq 4 \frac{h_{\vec{K}}^2}{|\det B_K|} \int_{\vec{K}} |\nabla \vec{u}|^2(x) dx.
\end{equation}

Inequalities (39) and (40) imply that

$$\| \vec{u} - P_h \vec{u} \|_{L^2(\Omega)^2} \leq c \sum_{K \in T_h} h_{\vec{K}}^2 \int_K |\nabla \vec{u}|^2 dx 
\leq c h^2 \int_{\Omega} |\nabla \vec{u}|^2 dx \quad \text{or} \quad h = \max_{K \in T_h} h_K.$$

Therefore,

\begin{equation}
(41) \quad \| \vec{u} - P_h \vec{u} \|_{L^2(\Omega)^2} \leq c h \| \vec{u} \|_{H^1(\Omega)^2}.
\end{equation}
Using inequalities (28), (1.7) and (41), we obtain the desired bound (36).

If we assume, moreover, that the right hand side $\bar{f} \in H^1(\Omega)^2$, we have a result of superconvergence for $\bar{u}_h$. The proof of this result requires the following lemma.

**Lemma 1.11.** $\forall \gamma \in H(\text{div}, \Omega)^2 \cap W^{1,p}(\Omega)^4$ for some $p > 1$ (this hypothesis insures the existence of $\pi^H_1 \gamma$), we have

$$
\int_{\Omega} \text{div} (\gamma) \cdot \bar{v}_h \, dx = \int_{\Omega} \text{div} (\pi^H_1 \gamma) \cdot \bar{v}_h \, dx, \quad \forall \bar{v}_h \in Y_h.
$$

**Proof.** We have $\forall \bar{v}_h \in Y_h$

$$
\int_{\Omega} \text{div} (\pi^H_1 \gamma) \cdot \bar{v}_h \, dx = \sum_{K \in T_h} \int_{K} \text{div} (\pi^H_1 \gamma) \cdot \bar{v}_h \, dx
$$

$$
= \sum_{K \in T_h} \int_{\partial K} (\pi^H_1 \gamma \bar{\gamma}) \cdot \bar{v}_h \, ds \quad \text{(by Green formula)}
$$

$$
= \sum_{K \in T_h} \int_{\partial K} (\gamma \bar{\gamma}) \cdot \bar{v}_h \, ds \quad \text{(by the definition of $\pi^H_1$)}
$$

$$
= \sum_{K \in T_h} \int_{K} \text{div} (\gamma) \cdot \bar{v}_h \, dx
$$

by applying Green formula to each term $\int_{\partial K} [\gamma] \bar{\gamma} \cdot \bar{v}_h \, ds$,

$$
= \int_{\Omega} \text{div} (\gamma) \cdot \bar{v}_h \, dx.
$$

We arrive now to our superconvergence result.

**Proposition 1.12.** If $\bar{f} \in H^1(\Omega)^2$, then

$$
\| \bar{u}_h - P_h \bar{u} \|_{L^2(\Omega)^2} \leq c h^2 (|\bar{u}|_{H^{1.6}(\Omega)^2} + |\bar{p}|_{H^{1.6}(\Omega)} + |\bar{f}|_{H^1(\Omega)^2}).
$$
Proof. We adapt the proof of Farhloul ([F] p. 106) to our case, i.e., in the case $\tilde{u}$ does not necessarily belong to $H^3(\Omega)^2$.

Let us solve the mixed formulation of Stokes’ problem with the right hand side $(P_h \tilde{u} - \tilde{u}_h)$: find $(\gamma, s) \in X$ and $\tilde{z} \in Y$ which gives a solution to

$$
\begin{cases}
\nu \int_{\Omega} \gamma : \tau \, dx + \int_{\Omega} \tilde{z} \cdot \text{div} (\nu\gamma - q\delta) \, dx = 0, & \forall (\tau, q) \in X, \\
\int_{\Omega} \text{div} (\nu\gamma - s\delta) \cdot \tilde{v} \, dx \\
+ \int_{\Omega} (P_h \tilde{u} - \tilde{u}_h) \cdot \tilde{v} \, dx = 0, & \forall \tilde{v} \in Y = L^2(\Omega)^2.
\end{cases}
$$

(44)

The velocity field $\tilde{z}$ solution of problem (44) verifies

$$
\|\tilde{z}\|_{H^1(\Omega)^2} \leq c \|P_h \tilde{u} - \tilde{u}_h\|_{L^2(\Omega)^2}.
$$

In equation (44)$_{(ii)}$, let us choose $\tilde{v} = P_h \tilde{u} - \tilde{u}_h$. Then

$$
\|P_h \tilde{u} - \tilde{u}_h\|_{L^2(\Omega)^2}^2 = - \int_{\Omega} \text{div} (\nu\gamma - s\delta) \cdot (P_h \tilde{u} - \tilde{u}_h) \, dx
$$

$$
= - \int_{\Omega} \text{div} (\pi_h^1(\nu\gamma - s\delta)) \cdot (P_h \tilde{u} - \tilde{u}_h) \, dx
$$

(by Lemma 1.11)

$$
= - \int_{\Omega} \text{div} (\pi_h^1(\nu\gamma - s\delta)) \cdot (\tilde{u} - \tilde{u}_h) \, dx
$$

$$
+ \int_{\Omega} \text{div} (\pi_h^1(\nu\gamma - s\delta)) \cdot (\tilde{u} - P_h \tilde{u}) \, dx, 
$$

where the second term is zero since

$$
(P_h \tilde{u})_{|K} = \frac{1}{|K|} \int_{K} \tilde{u} \, dx.
$$

Thus

$$
\|P_h \tilde{u} - \tilde{u}_h\|_{L^2(\Omega)^2}^2 = - \int_{\Omega} \text{div} (\pi_h^1(\nu\gamma - s\delta)) \cdot (\tilde{u} - \tilde{u}_h) \, dx.
$$

(46)
From the first equation of the continuous problem (5) and the first equation of the discrete problem (8), we obtain by subtraction

\[ \nu \int_{\Omega} (\sigma - \sigma_h) : \tau_h \, dx + \int_{\Omega} (\bar{u} - \bar{u}_h) \cdot \text{div} (\nu \tau_h - q_h \delta) \, dx = 0, \quad \forall (\tau_h, q_h) \in X_h. \]

Taking

\[ (\tau_h, q_h) = \pi_h(\gamma, s) = \left( \frac{1}{\nu} \left[ \pi_h^1(\nu \gamma - s \delta) + \rho_h(s) \delta \right], \rho_h(s) \right) \]

in the previous equation, we obtain

\[ (47) \quad \nu \int_{\Omega} (\sigma - \sigma_h) : \pi_h^1(\nu \gamma - s \delta) + \rho_h(s) \delta \, dx + \int_{\Omega} (\bar{u} - \bar{u}_h) \cdot \text{div} (\pi_h^1(\nu \gamma - s \delta)) \, dx = 0. \]

From (46) and (47), it follows that

\[ (48) \quad ||P_h \bar{u} - \bar{u}_h||_{L^2(\Omega)^2}^2 = \nu \int_{\Omega} (\sigma - \sigma_h) : \left( \frac{1}{\nu} \left[ \pi_h^1(\nu \gamma - s \delta) + \rho_h(s) \delta \right] - \gamma \right) \, dx + \nu \int_{\Omega} (\sigma_h - \sigma) : \gamma \, dx = \nu \int_{\Omega} (\sigma - \sigma_h) : \left( \frac{1}{\nu} \left[ \pi_h^1(\nu \gamma - s \delta) + \rho_h(s) \delta \right] - \gamma \right) \, dx + \int_{\Omega} \tilde{\varepsilon} \cdot \text{div} (\nu(\sigma - \sigma_h) - (p - p_h)\delta) \, dx, \]

by the first equation of (44).

On the other hand, from the second equation of problem (5) and the second equation of problem (8), we obtain by subtraction

\[ (49) \quad \int_{\Omega} \text{div} (\nu(\sigma - \sigma_h) - (p - p_h)\delta) \cdot \bar{v}_h \, dx = 0, \quad \forall \bar{v}_h \in Y_h. \]
By (48) and (49), we have

\[ (50) \quad \| P_h \tilde{u} - \tilde{u}_h \|_{L^2(\Omega)}^2 = \nu \int_{\Omega} (\sigma - \sigma_h) : \left( \frac{1}{\nu} \left[ \pi_h^1 (\nu \gamma - s \delta) + \rho_h (s) \delta \right] - \gamma \right) \, dx \]

\[ + \int_{\Omega} (\tilde{z} - P_h \tilde{z}) : \text{div} (\nu (\sigma - \sigma_h) - (p - p_h) \delta) \, dx \]

\[ \leq \nu \| \sigma - \sigma_h \|_{L^2(\Omega)}^4 \left\| \left[ \frac{1}{\nu} \left[ \pi_h^1 (\nu \gamma - s \delta) + \rho_h (s) \delta \right] - \gamma \right] \|_{L^2(\Omega)}^4 \right. \]

\[ + \| \tilde{z} - P_h \tilde{z} \|_{L^2(\Omega)^2} \| \text{div} (\nu (\sigma - \sigma_h) - (p - p_h) \delta) \|_{L^2(\Omega)}^2 \]

\[ \leq \nu \| \sigma - \sigma_h \|_{L^2(\Omega)}^4 \left\| \left[ \frac{1}{\nu} \left[ \pi_h^1 (\nu \gamma - s \delta) + \rho_h (s) \delta \right] - \gamma \right] \|_{L^2(\Omega)}^4 \right. \]

\[ + \| \tilde{z} - P_h \tilde{z} \|_{L^2(\Omega)^2} \| \tilde{f} - P_h \tilde{f} \|_{L^2(\Omega)^2} \].

But,

\[ \nu \| \sigma - \sigma_h \|_{L^2(\Omega)}^4 \left\| \frac{1}{\nu} \left[ \pi_h^1 (\nu \gamma - s \delta) + \rho_h (s) \delta \right] - \gamma \right\|_{L^2(\Omega)}^4 \]

\[ \leq \nu \| \sigma - \sigma_h \|_{L^2(\Omega)}^4 \left\| \frac{1}{\nu} \left[ \pi_h^1 (\nu \gamma - s \delta) - (\nu \gamma - s \delta) \right] \|_{L^2(\Omega)}^4 \right. \]

\[ + \| \rho_h (s) - s \|_{L^2(\Omega)} \].

Using the inequalities (1.7), (23) and (24) to bound the right hand side of the previous inequality, we obtain

\[ (51) \quad \nu \| \sigma - \sigma_h \|_{L^2(\Omega)}^4 \left\| \frac{1}{\nu} \left[ \pi_h^1 (\nu \gamma - s \delta) + \rho_h s \delta \right] - \gamma \right\|_{L^2(\Omega)}^4 \]

\[ \leq c h \left( |\tilde{u}|_{H^2(\Omega)^2} + |p|_{H^1(\Omega)} \right) \left( |\tilde{u}|_{H^2(\Omega)^2} + |p|_{H^1(\Omega)} \right) \left( |\tilde{z}|_{H^2(\Omega)^2} + |p|_{H^1(\Omega)} \right) \]

\[ \leq c h^2 \left( |\tilde{u}|_{H^2(\Omega)^2} + |p|_{H^1(\Omega)} \right) \left( |\tilde{z}|_{H^2(\Omega)^2} + |p|_{H^1(\Omega)} \right) \]

\[ \leq c h^2 \left( |\tilde{u}|_{H^2(\Omega)^2} + |p|_{H^1(\Omega)} \right) \| P_h \tilde{u} - \tilde{u}_h \|_{L^2(\Omega)^2} \]

as \((\tilde{z}, s)\) is the solution of the Stokes’ problem with right hand side \(P_h \tilde{u} - \tilde{u}_h\).
On the other hand, by (41), we get
\[ \| z - P_h z \|_{L^2(\Omega)} \leq c h \| z \|_{H^1(\Omega)} \leq c h \| P_h \tilde{u} - \tilde{u}_h \|_{L^2(\Omega)}, \] (by (45)),
\[ \| \tilde{f} - P_h \tilde{f} \|_{L^2(\Omega)} \leq c h \| \tilde{f} \|_{H^1(\Omega)}. \]

The inequalities above imply that
\[ \| z - P_h z \|_{L^2(\Omega)} \leq c h^2 \| z \|_{H^1(\Omega)} \],
\[ \| f - P_h f \|_{L^2(\Omega)} \leq c h^2 \| f \|_{H^1(\Omega)}. \]

Then, from (50), (51) and the previous inequality, it follows that
\[ \| P_h \tilde{u} - \tilde{u}_h \|_{L^2(\Omega)}^2 \]
\[ \leq c h^2 (| \tilde{u} |_{H^2(\Omega)}^2 + | \tilde{f} |_{H^1(\Omega)}^2 ) \| P_h \tilde{u} - \tilde{u}_h \|_{L^2(\Omega)}^2. \]

Dividing both sides of the inequality by \( \| P_h \tilde{u} - \tilde{u}_h \|_{L^2(\Omega)}^2 \), we obtain (43).

2 Hybrid formulation, behaviour at the boundary and jumps of the approximate solutions

Before introducing a hybrid formulation of the discrete problem (8), we are going to define some appropriate spaces for the hybridization

\[ \tilde{X}_h = \{ (\tau_h, q_h) \in L^2(\Omega) \times L^2_0(\Omega); \tau_h|_K \in RT_0(K)^2 \] and \( q_h|_K \in P_0(K), \forall K \in T_h \}, \]
\[ \tilde{Y}_h = \{ \tilde{v}_h \in \{ L^2(\Omega) \}^2; \forall K \in T_h : \tilde{v}_h|_K \in (P_0(K))^2 \} \equiv Y_h, \]
\[ \tilde{M}_h = \{ \tilde{\mu}_h : \partial T_h \rightarrow \mathbb{R}; \tilde{\mu}_h|_e \in (P_0(e))^2, \forall e \subset \partial T_h \] and \( \tilde{\mu}_h|_e = 0 \) if the edge \( e \subset \Gamma \}. \]

By \( \partial T_h \), we mean the union of all edges of the triangulation \( T_h \) and \( \partial \) means the “interior” of the edge \( e \). Now we are going to write a hybrid formulation of the discrete problem (8). This hybrid formulation is based
on the following integration by parts

\[ \int_{\Omega} \nabla \bar{u}_h : \tau_h \, dx = \int_{\Omega} \nabla \bar{u}_h : (\nu \tau_h - q_h \delta) \, dx \]

\[ = \sum_{K \in T_h} \int_{K} \nabla \bar{u}_h : (\nu \tau_h - q_h \delta) \, dx \]

\[ = - \sum_{K \in T_h} \int_{K} \text{div}(\nu \tau_h - q_h \delta) \cdot \bar{u}_h \]

\[ + \sum_{K \in T_h} \int_{\partial K} \bar{a}_h(\nu \tau_h - q_h \delta) \cdot \bar{n} \, ds \]

Thus,

\[ \int_{\Omega} \nabla \bar{u}_h : \tau_h \, dx = - \sum_{K \in T_h} \int_{K} \text{div}(\nu \tau_h - q_h \delta) \cdot \bar{u}_h \, dx \]

\[ + \sum_{K \in T_h} \int_{\partial K} \bar{a}_h(\nu \tau_h - q_h \delta) \cdot \bar{n} \, ds \]

Comparing (56) with the first equation of the discrete problem (8), it appears logical to introduce the following problem, namely, “hybrid formulation” of the discrete problem (8), i.e., find \((\bar{\sigma}_h, \bar{p}_h) \in \bar{X}_h\) and \((\bar{u}_h, \bar{\lambda}_h) \in \bar{Y}_h \times \bar{M}_h\) such that

\[ \begin{align*}
\nu \int_{\Omega} \bar{\sigma}_h : \tau_h \, dx + \sum_{K \in T_h} \int_{K} \text{div}(\nu \tau_h - q_h \delta) \cdot \bar{u}_h \, dx \\
- \sum_{K \in T_h} \int_{\partial K} \bar{\lambda}_h(\nu \tau_h - q_h \delta) \cdot \bar{n}_K \, ds = 0, \quad \forall (\tau_h, q_h) \in \bar{X}_h,
\end{align*} \]

\[ \begin{align*}
\sum_{K \in T_h} \int_{K} \text{div}(\nu \bar{\sigma}_h - \bar{p}_h \delta) \cdot \bar{u}_h \, dx \\
+ \int_{\Omega} \bar{f} \cdot \bar{v}_h \, dx = 0, \quad \forall \bar{v}_h \in \bar{Y}_h,
\end{align*} \]

\[ \begin{align*}
\sum_{K \in T_h} \int_{\partial K} \bar{\mu}_h(\nu \bar{\sigma}_h - \bar{p}_h \delta) \cdot \bar{n} \, ds = 0, \quad \forall \bar{\mu}_h \in \bar{M}_h.
\end{align*} \]

Due to the third equation of the system (57) the tensor \((\nu \bar{\sigma}_h - \bar{p}_h \delta)\) must check the constraint of continuity of its normal component at the
interfaces of the two adjacent triangles, this constraint having been introduced using Lagrange multipliers.

The left member of the second equation of system (57) then becomes

\[
\sum_{K \in \mathcal{T}_h} \int_K \text{div}(\nu \tilde{\tau}_h - \tilde{p}_h \delta) \cdot \tilde{v}_h \, dx + \int_{\Omega} \tilde{f} \cdot \tilde{v}_h \, dx
\]

\[
= \int_{\Omega} \text{div}(\nu \tilde{\tau}_h - \tilde{p}_h \delta) \cdot \tilde{v}_h \, dx + \int_{\Omega} \tilde{f} \cdot \tilde{v}_h \, dx
\]

This implies that the second equation of system (57) is identically the same as the second equation of system (8). Only the first equation of system (57) is different from the first equation of system (8). But, in the particular case when \((\tau_h, q_h) \in X_h \subset \tilde{X}_h, \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tilde{\lambda}_h \cdot (\nu \tau_h - q_h \delta) \, ds = 0\) and \(\text{div}(\nu \tau_h - q_h \delta) \in L^2(\Omega)\). Consequently both first equations of systems (57) and (8) also coincide in this particular case. But by Proposition 1.2, system (8) possesses one and only one solution. Thus we have already proved the following.

**Proposition 2.1.** Suppose \(((\tilde{\tau}_h, \tilde{p}_h), \tilde{u}_h, \tilde{\lambda}_h) \in \tilde{X}_h \times \tilde{Y}_h \times \tilde{M}_h\) is a solution of the hybrid formulation (57). Then necessarily \(\tilde{\tau}_h = \sigma_h, \tilde{p}_h = p_h\) and \(\tilde{u}_h = \bar{u}_h\) where \(((\sigma_h, p_h), \bar{u}_h)\) is the unique solution of the discrete mixed formulation (8).

That (57) possesses at most one solution is up to now not completely clear. By Proposition 2.1 unicity is clear for \(((\tilde{\tau}_h, \tilde{p}_h), \tilde{u}_h)\) but not yet for \(\tilde{\lambda}_h\). Let us define the following tensor fields.

\[
\tilde{P}_{K,e} = \frac{|e|}{2|K|} \begin{pmatrix}
    x_1 - E_1 & x_2 - E_2 \\
    |e| & 0
\end{pmatrix}
\]

and

\[
\tilde{Q}_{K,e} = \frac{|e|}{2|K|} \begin{pmatrix}
    0 & 0 \\
    x_1 - E_1 & x_2 - E_2
\end{pmatrix},
\]

where \(E\) denotes the vertex opposite to the side \(e\) of the triangle \(K\). In what follows, we denote by \(\tilde{\sigma}_1^h\) and \(\tilde{\sigma}_2^h\) each of the two lines of the tensor field \(\sigma_h\) and by \(\lambda_1^h\) and \(\lambda_2^h\) each of the two components of \(\tilde{\lambda}_h\).
Proposition 2.2. Let $e$ be an arbitrary edge of the triangulation $T_h$ over $\Omega$ belonging to the triangle $K$. Then

$$\lambda^1_{h|e} = u^1_{h,K} + \frac{1}{2|K|} \int_K (\bar{x} - \bar{E}) \cdot \vec{\sigma}^1_h \, dx$$

$$\lambda^2_{h|e} = u^2_{h,K} + \frac{1}{2|K|} \int_K (\bar{x} - \bar{E}) \cdot \vec{\sigma}^2_h \, dx$$

where $E$ denotes the vertex of $K$ opposite to the side $e$, $|K|$ the area of $K$ and $u^i_{h,K}$ ($i = 1, 2$) the two components of the restriction $\overline{u}_h, K$ of the vector field $\overline{u}_h$ to the triangle $K$.

Proof. Let us choose $\tau_h$ equal to the tensor field $\hat{P}_{K,e}$ in $K$ and $0$ out of $K$ and $q_h = 0$ in the first equation of (57)

$$\nu \int_K \vec{\sigma}_h : \hat{P}_{K,e} \, dx + \int_K \text{div} (\nu \hat{P}_{K,e}) \cdot \overline{u}_h \, dx - \int_e \vec{\lambda}_h (\nu \hat{P}_{K,e}) \cdot \vec{n}_K \, ds = 0.$$

Now, it results immediately from formula (58) that

$$\text{div} (\hat{P}_{K,e}) = \frac{|e|}{2|K|} \begin{pmatrix} \text{div} (\bar{x} - \bar{E}) & 0 \end{pmatrix} = \frac{|e|}{|K|} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\hat{P}_{K,e} \cdot \vec{n}_K = \frac{|e|}{2|K|} \begin{pmatrix} (\bar{x} - \bar{E}) \cdot \vec{n}_K \\ 0 \end{pmatrix}^T = \begin{cases} (1, 0)^T & \text{if } x \in e \\ (0, 0)^T & \text{if } x \in \partial K \setminus e. \end{cases}$$

Moreover, by using the explicit expression of $\hat{P}_{K,e}$ in the integral $\int_K \vec{\sigma}_h : \hat{P}_{K,e} \, dx$, we obtain from (61) the first equation of the formula (60).

Alternatively, taking in the first equation of (57), $\tau_h$ equal to the tensor field $\hat{Q}_{K,e}$ in $K$ and $0$ out of $K$ and $q_h = 0$, one can prove in the same way the second equation of (60).

We remark that the two formulas (60) may be rewritten in the form of the single vector formula

$$\vec{\lambda}_{h|e} = \overline{u}_{h,K} + \frac{1}{2|K|} \int_K \sigma_h \cdot (\bar{x} - \bar{E}) \, dx$$

We are now able to show that the hybrid formulation (57) has one and only one solution.
**Proposition 2.3.** The discrete hybrid formulation (57) admits one and only one solution.

**Proof.** From proposition 2.1 and 2.2, it results that if the system (57) possesses a solution, it is unique. To show the existence, we consider the linear mapping

\[ \Phi_h : \tilde{X}_h \times \tilde{Y}_h \times M_h \rightarrow (\tilde{X}_h \times \tilde{Y}_h \times M_h)' \]

\[ = \tilde{X}_h' \times \tilde{Y}_h' \times M_h' : ((\tilde{\sigma}_h, \tilde{\mu}_h), \tilde{u}_h, \tilde{v}_h) \]

\[ \mapsto \left( (\tau_h, q_h) \in \tilde{X}_h \rightarrow \nu \int_\Omega \tilde{\sigma}_h \tau_h \, dx \right. \]

\[ + \sum_{K \in T_h} \int_K \tilde{\text{div}} (\nu \tau_h - q_h \delta) \cdot \tilde{u}_h \, dx \]

\[ - \sum_{K \in T_h} \int_{\partial K} \tilde{\sigma}_h (\nu \tau_h - q_h \delta) \cdot \tilde{n}_K \, ds \in \mathbb{R}, \]

\[ \tilde{v}_h \in \tilde{Y}_h \mapsto \sum_{K \in T_h} \int_K \tilde{\text{div}} (\nu \tilde{\sigma}_h - \tilde{\mu}_h \delta) \cdot \tilde{v}_h \, dx \in \mathbb{R}, \]

\[ \tilde{\mu}_h \in M_h \mapsto \sum_{K \in T_h} \int_{\partial K} \tilde{\mu}_h (\nu \tilde{\sigma}_h - \tilde{\mu}_h \delta) \cdot \tilde{n} \, ds \in \mathbb{R} \right). \]

By a classical result of elementary linear algebra

\[ \dim \tilde{X}_h \times \tilde{Y}_h \times M_h = \dim \ker \Phi_h + \dim \text{Ran} \Phi_h. \]

But we know already that \( \Phi_h \) is injective and consequently

\[ \dim \text{Ran} \Phi_h = \dim \tilde{X}_h \times \tilde{Y}_h \times M_h = \dim \tilde{X}_h' \times \tilde{Y}_h' \times M_h'. \]

Thus \( \Phi_h \) is surjective. \( \square \)

**Proposition 2.4.** Let \( \{T_h\} \) be a family of triangulations over \( \tilde{\Omega} \) satisfying the hypotheses of Theorem 1.7. Let \( e \) be any edge of the triangulation \( T_h \) and \( K \) any triangle of \( T_h \) for which \( e \) is a side. Then, there exists a constant \( c > 0 \) independent of \( h \) and \( K \) such that

\[ \lambda^i_{h|e} = \frac{1}{|e|} \int_e u^i \, ds \]

\[\text{(62)}\]
\[ \leq \left| u_h^i - \frac{1}{|K|} \int_K u^i \, dx \right| + ch(\|\tilde{u}\|_{H^{2,\infty}(\Omega)}^2 + |p|_{H^{1,\infty}(\Omega)}) \]

where \( u^i \) and \( \lambda_h^i \) (\( i = 1, 2 \)) are the components of \( \tilde{u} \) and \( \lambda_h \), respectively.

**Proof.**

\[
\frac{1}{2|K|} \int_K (\bar{x} - \bar{E}) \cdot \nabla u^i \, dx = \frac{1}{2|K|} \int_K \left( \text{div} \left( u^i(\bar{x} - \bar{E}) \right) - 2u^i \right) \, dx \\
= \frac{1}{2|K|} \int_{\partial K} u^i(\bar{x} - \bar{E}) \cdot \bar{n}_K \, ds - \frac{1}{|K|} \int_K u^i \, dx, \\
= \frac{1}{2|K|} h_e \int_e u^i \, ds - \frac{1}{|K|} \int_K u^i \, dx,
\]

because \( (\bar{x} - \bar{E}) \cdot \bar{n}_K \) is equal to the length of the altitude \( h_e \) relative to the edge \( e \) if \( x \in e \) and is 0 if \( x \) is on the two other sides of the triangle \( K \) (let us recall that \( E \) denotes the vertex of the triangle \( K \) opposite to the side \( e \)). We have thus established that

\[
\frac{1}{2|K|} \int_K (\bar{x} - \bar{E}) \cdot \nabla u^i \, dx = \frac{h_e}{h_e|e|} \int_e u^i \, ds - \frac{1}{|K|} \int_K u^i \, dx \\
= \frac{1}{|e|} \int_e u^i \, ds - \frac{1}{|K|} \int_K u^i \, dx.
\]

Thus,

\[
\frac{1}{|e|} \int_e u^i \, ds = \frac{1}{|K|} \int_K u^i \, dx + \frac{1}{2|K|} \int_K (\bar{x} - \bar{E}) \cdot \nabla u^i \, dx.
\]

From (63) and Proposition 2.2, we get

\[
\left| \lambda_h^i \right|_{|e|} - \frac{1}{|e|} \int_e u^i \, ds \leq \left| u_h^i - \frac{1}{|K|} \int_K u^i \, dx \right| \\
+ \frac{1}{2|K|} \left| \int_K (\bar{x} - \bar{E}) \cdot (\bar{\sigma}_h - \nabla u^i) \, dx \right|.
\]
But

\begin{align*}
(65) \quad \frac{1}{2|K|} \left| \int_K (\overline{x} - \overline{E}) \cdot (\overline{\sigma}_h^i - \overline{\nabla} u^i) \, dx \right| & \\
& \leq \frac{1}{2|K|} \left( \int_K |\overline{x} - \overline{E}|^2 \, dx \right)^{\frac{1}{2}} \cdot \left( \int_K |\overline{\sigma}_h^i - \overline{\nabla} u^i|^2 \, dx \right)^{\frac{1}{2}} \\
& \leq \frac{2}{\pi \rho_K^2} (h_K^2 \pi h_K^2)^{\frac{1}{2}} ||\sigma_h^i - \overline{\nabla} u^i||_{L^2(K)}^2 \\
& \leq 2 \left( \max_{K \in \mathcal{T}_h : T_h \in \{T_h\} \rho_K} \frac{h_K}{\rho_K} \right)^2 ||\sigma_h^i - \overline{\nabla} u^i||_{L^2(\Omega)}^2 \\
& \leq c h(||\overline{u}||_{H^2,0(\Omega)}^2 + |p|_{H^{1,0}(\Omega)}) \quad \text{(by theorem 1.7)}.
\end{align*}

From (65) and (64), we get (62).

The weak point in (62) is that in general we can’t say anything on the behaviour of $|u_h|_{K} - \frac{1}{|K|} \int_K u \, dx$. From (36), (41), (1.7) and the triangle inequality it follows that

$$
\| \overline{u}_h - P_h \overline{u} \|_{L^2(\Omega)} \leq c h(||\overline{u}||_{H^2,0(\Omega)}^2 + |p|_{H^{1,0}(\Omega)})
$$

which implies that

\begin{align*}
(66) \quad \left| u_h^i, K - \frac{1}{|K|} \int_K u^i \, dx \right| & \leq \frac{1}{|K|^{\frac{1}{2}}} ||\overline{u}_h - P_h \overline{u}||_{L^2(\Omega)} \\
& \leq c \frac{h}{|K|^{\frac{1}{2}}} (||\overline{u}||_{H^2,0(\Omega)}^2 + |p|_{H^{1,0}(\Omega)}).
\end{align*}

But the ratio $h/|K|^{\frac{1}{2}}$ is always $\geq \sqrt{2}$. However, if we suppose that $f \in H^1(\Omega)$, $\alpha < 1/2$ and some additional hypothesis on the family of triangulations $\{T_h\}$, we have the following result

**Proposition 2.5.** In addition to the assumptions of Theorem 1.7, let us suppose, moreover, that $f \in H^1(\Omega)$, $\alpha < 1/2$ and that there exists a strictly positive constant $\tilde{\gamma}$ independent of $h$ such that $h_K \geq \tilde{\gamma} h^{1+\alpha}$ for every triangle $K$ of $T_h$ and every triangulation $T_h \in \{T_h\}$. Then, for every edge $e_h$ of $T_h$

$$
\bar{\lambda}_{h|e_h} - \frac{1}{|e_h|} \int_{e_h} \overline{u} \, ds \leq c \frac{h^{1+\alpha}}{1+\alpha} (||\overline{u}||_{H^2,0(\Omega)}^2 + |p|_{H^{1,0}(\Omega)} + |\overline{f}|_{H^{1}(\Omega)}^2)
$$
and \( \hat{X}_{h|e_h} - \frac{1}{|e_h|} \int_{e_h} \tilde{u} \, ds \to 0 \) uniformly in \( e_h \) when \( h \to 0^+ \) (\( |e_h| \) denotes the length of \( e_h \)).

**Proof.** This time, we have

\[
\left| \bar{u}_h,K - \frac{1}{|K|} \int_K \bar{u} \, dx \right| \leq \frac{1}{|K|^2} \| \bar{u}_h - P_h \bar{u} \|_{L^2(\Omega)^2}^2 \\
\leq c \frac{h^2}{|K|^2} \left( |\bar{u}|_{H^2(\Omega)^2} + |p|_{H^1(\Omega)} + |\bar{f}|_{H^1(\Omega)^2} \right)
\]

(by Proposition 1.12)

\[
\leq ch^{\frac{1-\alpha}{2}} \left( |\bar{u}|_{H^2(\Omega)^2} + |p|_{H^1(\Omega)} + |\bar{f}|_{H^1(\Omega)^2} \right)
\]

(by the hypothesis \( h_K \geq \gamma h_{\text{min}} \)).

Using Proposition 2.4 (equation 62), we obtain

\[
\left| \hat{X}_{h|e_h} - \frac{1}{|e_h|} \int_{e_h} \bar{u} \, ds \right| \leq ch^{\frac{1-\alpha}{2}} \left( |\bar{u}|_{H^2(\Omega)^2} + |p|_{H^1(\Omega)} + |\bar{f}|_{H^1(\Omega)^2} \right).
\]

Returning to the general case, i.e., without supposing the previously stated additional hypotheses on \( f, \alpha \) and \( \{T_h\} \), let us study the behaviour of the approximate solutions \( \bar{u}_h \) near the boundary. When the triangle \( K \in T_h \) has an edge contained in the boundary \( \Gamma \), we have the following.

**Corollary 2.6.** If \( K \) is a triangle of the triangulation \( T_h \) which has a side \( e \subset \Gamma \equiv \partial \Omega \), then

\[
(67) \quad \left| \bar{u}_h,K - \frac{1}{|K|} \int_K \bar{u} \, dx \right| \leq ch^{\frac{1-\alpha}{2}} \left( |\bar{u}|_{H^2(\Omega)^2} + |p|_{H^1(\Omega)} \right),
\]

for some constant \( c > 0 \) independent of \( h \) and \( K \).
Proof. Let us call $e$ the edge of $K$ contained in $\Gamma$. By the definition of $\bar{M}_h$ (54), we have that $\lambda_{h|e} = 0$. From Proposition 2.2, we then get

\begin{equation}
(68) \quad u_{h,K}^i = -\frac{1}{2|K|} \int_K (\ddot{x} - \ddot{E}) \cdot \ddot{\eta}_h^i \, dx.
\end{equation}

From (63) and $\ddot{u}_e = 0$, we get that

\begin{equation}
(69) \quad -\frac{1}{|K|} \int_K u^i \, dx = \frac{1}{2|K|} \int_K (\ddot{x} - \ddot{E}) \cdot \nabla u^i \, dx.
\end{equation}

From (68) and (69), we get

\begin{equation}
(70) \quad u_{h,K}^i = \frac{1}{|K|} \int_K u^i \, dx = \frac{1}{2|K|} \int_K (\ddot{x} - \ddot{E}) \cdot (\nabla u^i - \ddot{\eta}_h^i) \, dx.
\end{equation}

From (70) and (65), we get the result. \hfill \Box

Corollary 2.7. We keep the same setting as in Proposition 2.4. Then there exists a constant $c > 0$ independent of $h$ and $e$ such that

\begin{equation}
(71) \quad \left| \lambda_{h|e} - \frac{1}{|e|} \int_e \ddot{u} \, ds \right| \leq ch(|\ddot{u}|_{H^2(\Omega)} + |p|_{H^1(\Omega)})
\end{equation}

for all edges $e$ of a triangle $K \in T_h$ which has one of its side contained in the boundary of $\Omega$.

Proof. This is an immediate consequence of (62) and (67). \hfill \Box

Corollary 2.8. Let $K$ be a triangle of $T_h$ which has a vertex $A$ on $\Gamma$. Then we have also that

\begin{equation}
(72) \quad \left| \ddot{u}_{h,K} - \frac{1}{|K|} \int_K \ddot{u} \, dx \right| \leq ch(|\ddot{u}|_{H^2(\Omega)} + |p|_{H^1(\Omega)})
\end{equation}

for some constant $c > 0$ independent of $h$ and $K$.

Proof. By Proposition 2.2 and formula (63) we have for every triangle $K' \in T_h$ and every edge $e'$ of $K'$$u_{h,K'}^i = \frac{1}{|K'|} \int_{K'} u^i \, dx$
\[ = \lambda_{h,e'}^i - \frac{1}{|e'|} \int_{e'} u^i ds + \frac{1}{2|K'|} \int_{K'} (\vec{x} - \vec{E'}) \cdot (\vec{\nabla} u^i - \vec{\sigma}_h^i) dx. \]

where \( E' \) denotes the vertex opposite to the side \( e' \). By (65) there exists a constant \( c > 0 \) independent of \( K' \) such that

\[
\left| \frac{1}{2|K'|} \int_{K'} (\vec{x} - \vec{E'}) \cdot (\vec{\nabla} u^i - \vec{\sigma}_h^i) dx \right| \leq ch((|\vec{u}|_{H^2(\Omega)}^2 + |p|_{H^1(\Omega)})).
\]

Consequently if

\[
\left| u_{h,K'}^i - \frac{1}{|K'|} \int_{K'} u^i ds \right| \leq ch((|\vec{u}|_{H^2(\Omega)}^2 + |p|_{H^1(\Omega)})),
\]

then

\[
\left| \lambda_{h,e'}^i - \frac{1}{|e'|} \int_{e'} u^i ds \right| \leq ch((|\vec{u}|_{H^2(\Omega)}^2 + |p|_{H^1(\Omega)}))
\]

and reciprocally. Let us consider the subfamily \( F_{A,h} \) of all triangles of the triangulation \( T_h \) admitting \( A \) as a vertex. As \( \{T_h\} \) is a regular family of triangulations, the number of elements of the family \( F_{A,h} \) is bounded by a constant independent of \( h \). Let us choose first, one of the two triangles of the family \( F_{A,h} \) which has one of its sides contained in the boundary \( \Gamma \) of \( \Omega \). Let us call it \( K_1 \). By Corollary 2.6 choosing for \( e \) the side of \( K_1 \) contained in the boundary \( \Gamma \) of \( \Omega \), we get

\[
\left| \tilde{u}_{h,K_1} - \frac{1}{|K_1|} \int_{K_1} \tilde{u} dx \right| \leq c \ h((|\vec{u}|_{H^2(\Omega)}^2 + |p|_{H^1(\Omega)}))
\]

and thus by what we have said just above

\[
\left| \tilde{\lambda}_{h,e'}^i - \frac{1}{|e'|} \int_{e'} \tilde{u} ds \right| \leq c \ h((|\vec{u}|_{H^2(\Omega)}^2 + |p|_{H^1(\Omega)}))
\]

for every side \( e' \) of the triangle \( K_1 \). From this last inequality results

\[
\left| \tilde{u}_{h,K_2} - \frac{1}{|K_2|} \int_{K_2} \tilde{u} dx \right| \leq c \ h((|\vec{u}|_{H^2(\Omega)}^2 + |p|_{H^1(\Omega)}))
\]

for the adjacent triangle \( K_2 \in F_{A,h} \) to \( K_1 \). This inequality implies

\[
\left| \tilde{\lambda}_{h,e'}^i - \frac{1}{|e'|} \int_{e'} \tilde{u} ds \right| \leq ch((|\vec{u}|_{H^2(\Omega)}^2 + |p|_{H^1(\Omega)}))
\]
for every side \( e' \) of the triangle \( K_2 \). Considering now the triangle \( K_3 \in F_{A,h} \) adjacent to \( K_2 \) and different from \( K_1 \). We obtain

\[
\left| \bar{u}_{h,K_3} - \frac{1}{|K_3|} \int_{K_3} \bar{u} \, dx \right| \leq ch(|\bar{u}|_{H^{2,\alpha}({\Omega})}^2 + |p|_{H^{1,\alpha}({\Omega})}).
\]

Proceeding in this way iteratively, we obtain

\[
\left| \bar{u}_{h,K} - \frac{1}{|K|} \int_{K} \bar{u} \, dx \right| \leq ch(|\bar{u}|_{H^{2,\alpha}({\Omega})}^2 + |p|_{H^{1,\alpha}({\Omega})})
\]

for every triangle \( K \in F_{A,h} \).

In the following, we will say that a triangle \( K \) of the triangulation \( T_h \) is contained in the boundary layer, if it has a side or a vertex contained in the boundary \( \Gamma \) of \( \Omega \). For these triangles, we have the following result.

**Corollary 2.9.** For all \( \beta \in (0, \eta_0(\omega)) \) and all triangles \( K \) of \( T_h \) contained in the boundary layer, we have with a constant \( c > 0 \) independent of \( h \) and \( \bar{u} \) such that

\[
(73) \quad |\bar{u}_{h,K}| \leq h^\beta||\bar{u}||_{C^{0,\beta}({\overline{\Omega}})}^2 + ch(|\bar{u}|_{H^{2,\alpha}({\Omega})}^2 + |p|_{H^{1,\alpha}({\Omega})}).
\]

**Proof.** As \( \bar{u} \in H^{2,\alpha}({\Omega}) \) for \( \alpha > 1 - \eta_0(\omega) \), it follows by Hölder’s inequality that \( \bar{u} \in W^{2,p}({\Omega}) \) for all \( p \) such that

\[
1 < p < \frac{1}{1 - \frac{\eta_0(\omega)}{2}} < \frac{1}{1 - \frac{2}{p}} < 2
\]

the third inequality following from [1, p. 260]. As \( 2/p < 2 < 2/p + 1 \) by [2, p. 114], \( W^{2,p}({\Omega}) \hookrightarrow C^{0,\frac{2}{p} - \frac{2}{p} \omega}({\overline{\Omega}}) \). The inequalities \( 1 < p < \frac{1}{1 - \frac{\eta_0(\omega)}{2}} \)

are equivalent to \( 0 < 2 - \frac{2}{p} < \eta_0(\omega) \) and thus \( \bar{u} \in C^{0,\beta}({\overline{\Omega}}) \) for all \( \beta \in (0, \eta_0(\omega)) \). As \( \bar{u} = 0 \) on \( \Gamma \equiv \partial \Omega \), we deduce that \( ||\bar{u}(x)||_{C^{0,\beta}({\overline{\Omega}})} \leq h^\beta||\bar{u}||_{C^{0,\beta}({\overline{\Omega}})} \) for all \( x \) belonging to some triangle contained in the boundary layer. Thus for all triangle \( K \) contained in the boundary layer, we have also that

\[
\left| \frac{1}{|K|} \int_{K} \bar{u}(x) \, dx \right| \leq h^\beta||\bar{u}||_{C^{0,\beta}({\overline{\Omega}})}.
\]

The result then follows by (67) or (72) and the triangle inequality. □
The preceding corollary 2.9 shows that $\bar{u}_h$ tends to satisfy the Dirichlet boundary condition as $h$ tends to zero. Denoting by $BL(h)$ the boundary layer, we have $\|\bar{u}_h\|_{\infty, BL(h)} \leq ch^\beta$.

We now want to elucidate what is the behaviour of the discontinuity $\bar{u}_h(K_2) - \bar{u}_h(K_1)$ for two adjacent triangles $K_1$ and $K_2$ of $T_h$ as $h$ tends to zero.

**Proposition 2.10.** Let $K_1$ and $K_2$ be two adjacent triangles of $T_h$. Let $e$ be the common edge of $K_1$ and $K_2$, $E_1$ be the opposite vertex to $e$ in $K_1$ and $E_2$ be the opposite vertex to $e$ in $K_2$. Then

$$u_h'(K_1) - u_h'(K_2) = \frac{-1}{2|K_1|} \int_{K_1} (\bar{x} - \bar{E}_1) \cdot \bar{\sigma}_h^1 \, dx + \frac{1}{2|K_2|} \int_{K_2} (\bar{x} - \bar{E}_2) \cdot \bar{\sigma}_h^2 \, dx$$

where $i = 1, 2$.

**Proof.** Let $\tau_h$ be the tensor field defined by (see formula (58))

$$
\begin{cases}
\bar{P}_{K_1,e} & \text{in } K_1, \\
-\bar{P}_{K_2,e} & \text{in } K_2, \\
0 & \text{outside } K_1 \cup K_2.
\end{cases}
$$

The normal component of $\tau_h$ is continuous when crossing from one triangle to an adjacent one. Let us take $q_h = 0$. With this choice of $(\tau_h, q_h)$ the first equation of system (57) reduces to the first equation of system (8) and we get

$$\int_{K_1} \sigma_h : \bar{P}_{K_1,e} \, dx - \int_{K_2} \sigma_h : \bar{P}_{K_2,e} \, dx + u_h^1(K_1) |e| - u_h^1(K_2) |e| = 0.$$  

Replacing $\bar{P}_{K_1,e}$ and $\bar{P}_{K_2,e}$ by their explicit expression (58) in (75), we obtain (74) for $i = 1$. The proof is done in a similar way for $i = 2$ using the tensor field (59) in place of (58).

**Lemma 2.11.** Keeping the same setting as in Proposition 2.10, we have

$$\int_{K_1} u' \, dx - \int_{K_2} u' \, dx = \frac{-1}{2|K_1|} \int_{K_1} (\bar{x} - \bar{E}_1) \cdot \nabla u' \, dx$$
\[
+ \frac{1}{2|K_2|} \int_{K_2} (\bar{x} - \bar{E}_2) \cdot \nabla u^i \, dx \quad (i = 1, 2).
\]

**Proof.** The result follows immediately from formula (63). \qed

**Proposition 2.12.** Let \( \{T_h\} \) be a family of triangulations over \( \Omega \) satisfying the hypotheses of Theorem 1.7. There exists a constant \( c \) independent of \( h \) and \( \bar{u} \) such that for any pair of adjacent triangles \( K_1 \) and \( K_2 \) of the triangulation \( T_h \) such that

\[
(u_h^i(K_2) - u_h^i(K_1)) - \left( \frac{1}{|K_1|} \int_{K_1} u^i \, dx - \frac{1}{|K_2|} \int_{K_2} u^i \, dx \right) \leq ch(|\bar{u}|_{H^2(\Omega)} + |p|_{H^1(\Omega)})
\]

where \( i = 1, 2 \).

**Proof.** From (74) and (2.11), we get

\[
\begin{align*}
(u_h^i(K_1) - u_h^i(K_2)) & - \left( \frac{1}{|K_1|} \int_{K_1} u^i \, dx - \frac{1}{|K_2|} \int_{K_2} u^i \, dx \right) \\
&= \frac{1}{2|K_2|} \int_{K_2} (\bar{x} - \bar{E}_2) \cdot (\bar{\sigma}_h^i - \nabla u^i) \, dx \\
&\quad - \frac{1}{2|K_1|} \int_{K_1} (\bar{x} - \bar{E}_1) \cdot (\bar{\sigma}_h^i - \nabla u^i) \, dx.
\end{align*}
\]

The result follows from inequality (65) applied to each of the two terms of the right member of (78). \qed

### 3 Numerical implementation

To solve numerically, the hybrid formulation (57), we first rewrite it in the form

\[
\begin{cases}
\int_K \sigma_h \cdot \tau_K \, dx + \int_K \nabla \cdot (\bar{\sigma}_h) \cdot \bar{u}_h \, dx \\
\quad - \int_{\partial K} \bar{\lambda}_h \tau_K \cdot \bar{n} \, ds = 0, \quad \forall \tau_K \in RT_0(K)^2, \\
\nu \int_K \text{div}(\sigma_h) \, dx = \nu \int_K \bar{f} \, dx, \quad \forall K \in T_h, \\
\int_{\partial K} \bar{n} \cdot \bar{\lambda}_h \, ds = 0, \quad \forall K \in T_h, \\
\nu [\sigma_h \cdot \bar{n}] = [p_h] \cdot \bar{n} \quad \text{on } e, \quad \forall e \subset \partial \Omega.
\end{cases}
\]

(79)
We first derive explicit formulas for $\sigma_{h|K}$ and $\tilde{s}_{h|K}$ on each triangle $K$ of the triangulation $T_h$ in terms of the Lagrange multiplier $\tilde{\lambda}_{h|K}$. In what follows, we denote by $K$ a triangle of $T_h$ and by $\overrightarrow{A}, \overrightarrow{B}, \overrightarrow{C}$, its three vertices.

**Proposition 3.1.**

\[
\sigma_{h|K} = \frac{|a|}{|K|} \tilde{\lambda}_{h|a} \cdot \vec{n}_{t,a} + \frac{|b|}{|K|} \tilde{\lambda}_{h|b} \cdot \vec{n}_{t,b} + \frac{|c|}{|K|} \tilde{\lambda}_{h|c} \cdot \vec{n}_{t,c} - \frac{1}{2\nu} \overline{\int_K} (\vec{x} - \vec{x}_K)^t
\]

\[
\sigma_{h|K} \cdot \vec{n}_{t,a} = \frac{|a|^2 + |b|^2 - |c|^2}{2|a||K|} \tilde{\lambda}_{h|b} \vec{n}_{t,b} - \frac{|a|^2 + |c|^2 - |b|^2}{2|a||K|} \tilde{\lambda}_{h|c} \vec{n}_{t,c} - \frac{|K|}{3|a|\nu} \overline{\int_K} f K(\overrightarrow{x})
\]

where $\vec{x}_K$ is the barycenter of the triangle $K$ ($\vec{x}_K = \frac{1}{3}(\overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C})$) and $\overline{\int_K} f = \frac{1}{|K|} \int_K f(x)dx$.

**Proof.** In view of (79)\((ii)\), let us set $\tilde{s}_{1,K} = \tilde{s}_{1,K} = \frac{1}{|K|} \int_K f_1(x)dx$, taking $\tau_K = (\tilde{s}_{1,K}, \tilde{s}_{2,K})^T$ being unknown. In the first equation of the system (79), taking $\tau_K = (\tilde{s}_{1,K}, \tilde{s}_{2,K})^T$, where $c_1$ and $c_2$ are constants, we obtain the equation

\[
|K| \tilde{s}_{1,K} = \lambda_{h|a}^1 |a| \vec{n}_{t,a} + \lambda_{h|b}^1 |b| \vec{n}_{t,b} + \lambda_{h|c}^1 |c| \vec{n}_{t,c}.
\]

Thus $\tilde{s}_{1,K}$ is of the form

\[
\tilde{s}_{1,K} = \frac{1}{|K|} \lambda_{h|a}^1 |a| \vec{n}_{t,a} + \frac{1}{|K|} \lambda_{h|b}^1 |b| \vec{n}_{t,b} + \frac{1}{|K|} \lambda_{h|c}^1 |c| \vec{n}_{t,c} - \frac{1}{2\nu} \overline{\int_K} (\vec{x} - \vec{x}_K)^t.
\]

In the same way, one proves that $\tilde{s}_{1,K}^2$ is of the form

\[
\tilde{s}_{2,K}^2 = \frac{1}{|K|} \lambda_{h|a}^2 |a| \vec{n}_{t,a} + \frac{1}{|K|} \lambda_{h|b}^2 |b| \vec{n}_{t,b}
\]
\[ + \frac{1}{|K|} \lambda_{h,c}^2 |\bar{u}_c|^2 = \frac{1}{2\nu} \int_K (\bar{x} - \tilde{x}_K)^t. \]

(82) and (83) imply (80). From (82) follows that the normal component of \( \bar{n}_h^1 \) along the side \( a \) of \( K \) is given by

\[ \bar{n}_h^1 \cdot \bar{n}_a = \frac{|a|}{|K|} \lambda_h^1 + \frac{|b|}{|K|} \bar{n}_a \cdot \bar{n}_b \lambda_h^1 + \frac{|c|}{|K|} \bar{n}_a \cdot \bar{n}_c \lambda_h^1 - \frac{1}{2\nu} \int_K (a^* - \tilde{x}_K) \cdot \bar{n}_a, \]

where \( a^* \) denotes the midpoint of the side \( a \). But

\[ 2|K| = 3|a|(|a^* - \tilde{x}_K) \cdot \bar{n}_a, \]

so that

\[ \bar{n}_h^1 \cdot \bar{n}_a = \frac{|a|}{|K|} \lambda_h^1 + \frac{|b|}{|K|} \bar{n}_a \cdot \bar{n}_b \lambda_h^1 + \frac{|c|}{|K|} \bar{n}_a \cdot \bar{n}_c \lambda_h^1 - \frac{|K|}{3|a|\nu} \int_K. \]

Finally, by the law of cosines: \( |c|^2 = |a|^2 + |b|^2 - 2|a||b| \cos C \), we obtain

\[ \bar{n}_h^1 \cdot \bar{n}_a = \frac{|a|}{|K|} \lambda_h^1 - \frac{|a|^2 + |b|^2 - |c|^2}{2|a||b|} \lambda_h^1 - \frac{|a|^2 + |c|^2 - |b|^2}{2|a||b|} \lambda_h^1 - \frac{|K|}{3|a|\nu} \int_K. \]

In the same way, we show that

\[ \lambda_{h,b}^2 = \frac{|a|}{|K|} \lambda_h^2 + \frac{|b|^2 + |c|^2}{2|a||b|} \lambda_h^2 - \frac{|a|^2 + |c|^2 - |b|^2}{2|a||b|} \lambda_h^2 - \frac{|K|}{3|a|\nu} \int_K. \]

\[ \bar{u}_{h,K} = \frac{1}{3} (\bar{\lambda}_{h,a} + \bar{\lambda}_{h,b} + \bar{\lambda}_{h,c}) + \frac{|a|^2 + |b|^2 + |c|^2}{144\nu|K|} \int_K \bar{f} \, dx. \]
Proof. In the first equation of system (79), if we take
\[ \gamma : K \to [\mathbb{R}]^{2 \times 2} : \vec{x} \mapsto \begin{pmatrix} \vec{x} - (\vec{x}_K) \\ 0 \end{pmatrix} \equiv \begin{pmatrix} x_1 - (\vec{x}_K)_1 & x_2 - (\vec{x}_K)_2 \\ 0 & 0 \end{pmatrix}, \]
we obtain
\[ \int_K \overline{\sigma}_h \cdot (\vec{x} - \vec{x}_K) \, dx + 2|K|\lambda^1_h (\vec{x} - \vec{x}_K) \cdot \vec{n}_K \, ds = 0. \]

But
\[ \int_K \overline{\sigma}_h \cdot (\vec{x} - \vec{x}_K) \, dx = - \frac{1}{2\nu} \overline{\gamma}_K \int_K |\vec{x} - \vec{x}_K|^2 \, dx \]

because \( \overline{\sigma}_h = \overline{\sigma}_h (\vec{x}_K) - \frac{1}{2\nu} \overline{\gamma}_K (\vec{x} - \vec{x}_K) \)
\[ = - \frac{|K|}{2\nu} \overline{\gamma}_K \left( |\vec{a} - \vec{x}_K|^2 + |\vec{b} - \vec{x}_K|^2 + |\vec{c} - \vec{x}_K|^2 \right) \]

where \( \vec{a}, \vec{b}, \text{ and } \vec{c} \) denote, respectively, the midpoints of the three sides \( a, b, \text{ and } c \) of the triangle \( K \). Since
\[ |\vec{e} - \vec{x}_K| = \frac{1}{3} |\vec{e} - \vec{E}| \quad \text{for } \vec{e} \in \{ \vec{a}, \vec{b}, \vec{c} \}, \]
\( \vec{E} \in \{ \vec{A}, \vec{B}, \vec{C} \} \) opposite to \( e \), and the median theorem
\[ |\vec{a} - \vec{A}|^2 = \frac{|b|^2}{2} + \frac{|c|^2}{2} - \frac{|a|^2}{4} \]
(with similar formulas for \( |\vec{b} - \vec{B}|^2 \) and \( |\vec{c} - \vec{C}|^2 \)), we obtain
\[ \int_K \overline{\sigma}_h \cdot (\vec{x} - \vec{x}_K) \, dx = - \frac{\overline{\gamma}_K}{2\nu}|K| \left( |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 \right). \]

On the other hand
\[ \int_{\partial K} \lambda^1_h (\vec{x} - \vec{x}_K) \cdot \vec{n}_K \, ds \]
\begin{equation}
\begin{aligned}
&= \lambda_{h|a}^1 \int_a (\bar{x} - \bar{x}_K) \cdot \bar{n}_a \, ds + \lambda_{h|b}^1 \int_b (\bar{x} - \bar{x}_K) \cdot \bar{n}_b \, ds \\
&\quad + \lambda_{h|c}^1 \int_c (\bar{x} - \bar{x}_K) \cdot \bar{n}_c \, ds.
\end{aligned}
\end{equation}

Observing that \( \int_e (\bar{x} - \bar{x}_K) \cdot \bar{n}_e \, ds = \frac{\sqrt{3}}{4} |K| \) for \( e \in \{a, b, c\} \) in (86), we obtain
\begin{equation}
\int_{\partial K} \lambda^1_h (\bar{x} - \bar{x}_K) \cdot \bar{n}_K \, ds = \frac{2|K|}{3} (\lambda^1_{h|a} + \lambda^1_{h|b} + \lambda^1_{h|c}).
\end{equation}

By the above formula at the beginning of the proof, we have
\begin{equation}
u^1_{h,K} = \frac{1}{2|K|} \left[ \int_{\partial K} \lambda^1_h (\bar{x} - \bar{x}_K) \cdot \bar{n}_K \, ds - \int_K \sigma^1_h \cdot (\bar{x} - \bar{x}_K) \, dx \right].
\end{equation}

Using (87) and (85), we obtain
\begin{equation}
u^1_{h,K} = \frac{1}{3} (\lambda^1_{h|a} + \lambda^1_{h|b} + \lambda^1_{h|c}) + \frac{|a|^2 + |b|^2 + |c|^2}{144|K|^2} \int_K f_1 \, dx.
\end{equation}

In the same way, one proves that
\begin{equation}
u^2_{h,K} = \frac{1}{3} (\lambda^2_{h|a} + \lambda^2_{h|b} + \lambda^2_{h|c}) + \frac{|a|^2 + |b|^2 + |c|^2}{144|K|^2} \int_K f_2 \, dx.
\end{equation}

These two equalities together give us the result. \( \square \)

We now reduce our problem to the resolution of a linear system with explicit coefficients in terms of the geometry of the triangulation with only the Lagrange multiplier \( \lambda_h \) on each edge and the discrete pressure \( p_h \) on each triangle as unknowns.

**Proposition 3.3.** Let \( e \) be any edge of the triangulation \( T_h \) such that \( e \subset \Omega \) and \( K_1, K_2 \) two triangles of \( T_h \) such that \( e = K_1 \cap K_2 \). Let us call the two other sides of \( K_i \) \( (i = 1, 2) \) than \( e : d_i \) and \( g_i \). Then

\begin{equation}
\begin{aligned}
\frac{|e|^2}{|K_1|} + \frac{1}{|K_2|} \lambda_{h,e} - \frac{|e|^2}{2 |K_1|} \lambda_{h,d_1} \\
- \frac{|e|^2}{2 |K_1|} \lambda_{h,g_1} - \frac{|e|^2}{2 |K_2|} \lambda_{h,d_2}
\end{aligned}
\end{equation}
\[-\frac{|e|^2 + |g_2|^2 - |d_2|^2}{2|K_2|} \lambda_{h,g_2} + \frac{|e|}{\nu} (p_{h,K_2} - p_{h,K_1}) \bar{n}_{K_1}\]
\[= \frac{1}{3\nu} \int_{K_1 \cup K_2} f \, dx.\]

**Proof.** We have, denoting by \(\bar{n}, \bar{n}_{K_1}\) or \(\bar{n}_{K_2}\),
\[\nu(\sigma_{h,K_2} - \sigma_{h,K_1}) \bar{n} = (p_{h,K_2} - p_{h,K_1}) \bar{n} \quad \text{on} \ e.\]

So, by (81) and while considering for example \(\bar{n} = \bar{n}_{K_2}\) we obtain
\[
\frac{1}{\nu} (p_{h,K_2} - p_{h,K_1}) \cdot \bar{n} = \frac{|e|}{|K_2|} \lambda_{h,e} - \frac{|e|^2 + |d_2|^2 - |g_2|^2}{2|e| |K_2|} \lambda_{h,d_2}
\[- \frac{|e|^2 + |g_2|^2 - |d_2|^2}{2|e| |K_2|} \lambda_{h,g_2}
\[+ \frac{|e|}{|K_1|} \lambda_{h,e} - \frac{|e|^2 + |d_1|^2 - |g_1|^2}{2|e| |K_1|} \lambda_{h,d_1}
\[- \frac{|e|^2 + |g_1|^2 - |d_1|^2}{2|e| |K_1|} \lambda_{h,g_1}
\[- \frac{1}{3\nu|e|} \int_{K_1 \cup K_2} f \, dx.\]

from which, we obtain the result. \(\square\)

Additionally to the equations (88), one for each interior edge \(e\), we must also add the equation
\[(89) \sum_{K \in T_h} \text{aire}(K) p_{h|K} = 0\]
which expresses the fact that the discrete pressure \(p_h\) is of mean zero.
We must also express the third equation of system (79)
\[(90) \int_{\partial K} \bar{n}_K \cdot \bar{\lambda}_h \, ds = 0\]
for each triangle \(K\) of the triangulation \(T_h\) except one due to the fact that
\[\sum_{K \in T_h} \int_{\partial K} \bar{n}_K \cdot \bar{\lambda}_h \, ds = \int_{\Gamma} \bar{n} \cdot \bar{\lambda}_h \, ds = 0,\]
as the Lagrange multiplier $\tilde{\lambda}_h$ is identically 0 on the boundary $\Gamma$ of the polygonal domain $\Omega$. Rewriting,
\[
\int_{\partial K} \tilde{n}_K \cdot \tilde{\lambda}_h \, ds = \int_a \tilde{n}_{ia} \cdot \tilde{\lambda}_{h|a} \, ds + \int_b \tilde{n}_{ib} \cdot \tilde{\lambda}_{h|b} \, ds + \int_c \tilde{n}_{ic} \cdot \tilde{\lambda}_{h|c} \, ds
= |a| \tilde{n}_{ia} \cdot \tilde{\lambda}_{h|a} + |b| \tilde{n}_{ib} \cdot \tilde{\lambda}_{h|b} + |c| \tilde{n}_{ic} \cdot \tilde{\lambda}_{h|c}.
\]
Equation (90) becomes
\[
(91) \quad |a| \tilde{n}_{ia} \cdot \tilde{\lambda}_{h|a} + |b| \tilde{n}_{ib} \cdot \tilde{\lambda}_{h|b} + |c| \tilde{n}_{ic} \cdot \tilde{\lambda}_{h|c} = 0.
\]
Solving the linear system formed by the equations (88), (89) and (91), we obtain the Lagrange multiplier $\tilde{\lambda}_h$ and the discrete pressure $p_h$. Then, applying formulas (80) and (84), we derive from the knowledge of the Lagrange multiplier $\tilde{\lambda}_h$, the discrete tensor field $\sigma_h$ and the discrete velocity field $\tilde{u}_h$.

REFERENCES


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