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FROM BOUSSINESQ SYSTEMS TO KP-TYPE EQUATIONS

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ABSTRACT. This short note is to demonstrate that the famous Kadomtsev-Petviashvilli II-type equation

$$\left(\eta_t + \eta_x + \frac{3\epsilon}{2}\eta\eta_x - \frac{\epsilon}{6}\eta_{xxt}\right)_x + \frac{\epsilon}{2}\eta_{\widehat{y}\widehat{y}} = 0, \tag{1}$$

for water waves which are weakly three-dimensional and propagating predominantly in one-direction can be derived formally from the three-dimensional Boussinesq system

$$\begin{aligned} &\eta_t + \nabla\cdot\boldsymbol{v} + \epsilon\nabla\cdot\eta\boldsymbol{v} - \frac{1}{6}\epsilon\Delta\eta_t = 0,\\ &\boldsymbol{v}_t + \nabla\eta + \epsilon\frac{1}{2}\nabla|\boldsymbol{v}|^2 - \frac{1}{6}\epsilon\Delta\boldsymbol{v}_t = 0, \end{aligned} \tag{2}$$

where $\boldsymbol{v} = (u, v)$ and ϵ is proportional to the typical wave height of the wave (see [**2, 3**]). The relationship between the dispersion relations of Euler equations, Boussinesq systems and KP equations are also analyzed.

1 From Boussinesq system to KP-II equations System (2) is one of a large class of Boussinesq systems which are derived for the small amplitude and long waves and which are all equivalent and approximate Euler equations up to the order ϵ, where ϵ is the ratio between the wave height and the still water depth (see e.g., [**2**] for derivation, [**4, 3, 5**] for analysis, justification and properties). It is written in non-dimensional variables which are order-one quantities, namely η is scaled by a_0, the typical height of the waves being modelled, and $\boldsymbol{v}$ is scaled by $a_0 g/c_0$, where $c_0 = \sqrt{gh_0}$ with g being the acceleration of gravity and h_0 the depth of water in its quiescent state. The coordinate $\boldsymbol{x} = (x, y)$ is scaled by λ_0, a representative wave length, and time t is scaled by λ_0/c_0. In equation (2), the Stokes number $S = a_0\lambda^2/h_0^3$ is taken to be exactly 1 for notational simplicity. One replaces the constant $1/6$ by $1/(6S)$ for general values of S. In any case, S is of order one which is one of the assumptions leading from the Euler equations to (2).

In summary, the following scaling was used to express the scales explicitly in the derivation, tilded variables are used to denote physical variables,

$$
\begin{aligned}
x &= \frac{\widetilde{x}}{\lambda_0} \sim \epsilon^{\frac{1}{2}}\widetilde{x}, \qquad y = \frac{\widetilde{y}}{\lambda_0} \sim \epsilon^{\frac{1}{2}}\widetilde{y}, \quad z = \frac{\widetilde{z}}{h_0} \sim \widetilde{z}, \\
t &= \frac{c_0\widetilde{t}}{\lambda_0} \sim \epsilon^{\frac{1}{2}}\widetilde{t}, \qquad \eta = \frac{\widetilde{\eta}}{a_0} \sim \epsilon^{-1}\widetilde{\eta}, \\
\boldsymbol{v} &= \frac{h_0\widetilde{\boldsymbol{v}}}{a_0c_0} \sim \epsilon^{-1}\widetilde{\boldsymbol{v}}, \quad w = \frac{\widetilde{w}\lambda_0}{a_0c_0} \sim \epsilon^{-3/2}\widetilde{w},
\end{aligned}
\tag{3}
$$

where $\widetilde{w}$ is the vertical velocity. Substitute these into the Euler equation and neglect the terms of order $O(\epsilon^2)$, one obtains the scaled Boussinesq systems such as (2). The details of the derivation can be found in [**2**].

The Kadomtsev-Petviashvili (K-P) equation is derived under further assumption that the wave is weakly three-dimensional, propagates predominantly in one direction (e.g., the positive x-direction) and the nonlinear, dispersive and three-dimensional effects are of equal importance. Specifically, it implies that

$$
\widehat{y} = \epsilon^{\frac{1}{2}}y \quad \text{and} \quad \widehat{v} = \epsilon^{-\frac{1}{2}}v
\tag{4}
$$

are $O(1)$ quantities. Substitute scaling (4) into (2) and drop the terms of order $O(\epsilon^2)$, then one finds

$$
\begin{aligned}
\eta_t + u_x + \epsilon\widehat{v}_{\widehat{y}} + \epsilon(\eta u)_x - \frac{\epsilon}{6}\eta_{xxt} &= 0, \\
u_t + \eta_x + \frac{\epsilon}{2}(u^2)_x - \frac{\epsilon}{6}u_{xxt} &= 0, \\
\widehat{v}_t + \eta_{\widehat{y}} + \frac{\epsilon}{2}(u^2)_{\widehat{y}} - \frac{\epsilon}{6}\widehat{v}_{xxt} &= 0.
\end{aligned}
\tag{5}
$$

Now, consider the $O(1)$ terms in the first two equations with initial data

$$
\begin{aligned}
\eta_t + u_x &= 0, \qquad \eta(x,\widehat{y},0) = f(x,\widehat{y}), \\
u_t + \eta_x &= 0, \qquad u(x,\widehat{y},0) = g(x,\widehat{y}).
\end{aligned}
$$

The solution reads

$$
\begin{aligned}
\eta(x,\widehat{y},t) &= \frac{1}{2}(f(x+t,\widehat{y}) + f(x-t,\widehat{y})) + \frac{1}{2}(-g(x+t,\widehat{y}) + g(x-t,\widehat{y})), \\
u(x,\widehat{y},t) &= \frac{1}{2}(g(x+t,\widehat{y}) + g(x-t,\widehat{y})) + \frac{1}{2}(-f(x+t,\widehat{y}) + f(x-t,\widehat{y})).
\end{aligned}
$$

Now by using the assumption that the wave is moving only to the right, one obtains $f = g$ and $u(x, \widehat{y}, t) = \eta(x, \widehat{y}, t) = f(x - t, \widehat{y})$ at the leading order. Therefore, one-directional wave satisfies

$$u(x, \widehat{y}, t) = \eta(x, \widehat{y}, t) + O(\epsilon) = f(x - t, \widehat{y}) + O(\epsilon). \tag{6}$$

Using the system (5) and the assumption that wave moves to right again,

$$\widehat{v} = \widehat{v}_0(x - t, \widehat{y}) + O(\epsilon)$$

which together with (6) implies

$$\partial_x = -\partial_t + O(\epsilon). \tag{7}$$

For the next order of approximation on u, it is natural to assume

$$u = \eta + \epsilon A(\eta, \widehat{v}) + O(\epsilon^2).$$

Substitute it into the first two equations in (5), one obtains

$$\begin{aligned} \eta_t + \eta_x + \epsilon A_x + \epsilon \widehat{v}_{\widehat{y}} + \epsilon(\eta^2)_x - \frac{\epsilon}{6}\eta_{xxt} &= O(\epsilon^2), \\ \eta_t + \eta_x + \epsilon A_t + \frac{1}{2}\epsilon(\eta^2)_x - \frac{\epsilon}{6}\eta_{xxt} &= O(\epsilon^2). \end{aligned} \tag{8}$$

For this pair of equations to be consistent,

$$A_x + \widehat{v}_{\widehat{y}} + \frac{1}{2}(\eta^2)_x = A_t + O(\epsilon).$$

By using (7), $A_t = -A_x + O(\epsilon)$, one can choose

$$A = -\frac{1}{2}\int \widehat{v}_{\widehat{y}}\, dx - \frac{1}{4}\eta^2 \tag{9}$$

and hence

$$u = \eta - \frac{\epsilon}{4}\eta^2 - \frac{\epsilon}{2}\int \widehat{v}_{\widehat{y}}\, dx + O(\epsilon^2).$$

Substituting (9) into the first equation of (8), one has

$$\eta_t + \eta_x + \frac{3}{2}\epsilon\, \eta\eta_x - \frac{1}{6}\epsilon\, \eta_{xxt} = -\frac{\epsilon}{2}\, \widehat{v}_{\widehat{y}} + O(\epsilon^2). \tag{10}$$

Using (7) and the leading order relation from the third equation in (5), it is obtained

$$(\widehat{v}_{\widehat{y}})_x = -(\widehat{v}_{\widehat{y}})_t + O(\epsilon) = \eta_{\widehat{y}\widehat{y}} + O(\epsilon). \tag{11}$$

Differentiating (10) with respect to x and using (11) yield the KP-type equation (KP II) [**6**]

$$\left(\eta_t + \eta_x + \frac{3}{2}\epsilon\,\eta\eta_x - \frac{\epsilon}{6}\eta_{xxt}\right)_x + \frac{\epsilon}{2}\eta_{\widehat{y}\widehat{y}} = 0. \tag{12}$$

A more familiar form of the KP-II equation

$$\left(\eta_t + \eta_x + \frac{3}{2}\epsilon\,\eta\eta_x + \frac{\epsilon}{6}\eta_{xxx}\right)_x + \frac{\epsilon}{2}\eta_{\widehat{y}\widehat{y}} = 0 \tag{13}$$

is obtained by another use of (7).

In summery, it is shown that KP-II type equations can be easily obtained from a Boussinesq system under the KP assumptions. It is worth noting that the argument presented above is purely a formal derivation. We assume that after the proper scaling, all terms keep their order of magnitude under derivation and integration. A rigorous justification along with numerical comparisons similar to [**1**], where the KdV-type equations and Boussinesq systems in one space dimension is compared, will be carried out in a separate paper.

This formal derivation is equivalent to say that the expansions of u and $\widehat{v}$ in terms of η are

$$\begin{aligned} u &= \eta - \frac{\epsilon}{4}\eta^2 - \frac{\epsilon}{2}\int\int^x \eta_{\widehat{y}\widehat{y}}(s,\widehat{y},t)\,ds\,dx + O(\epsilon^2),\\ \widehat{v} &= \int \eta_{\widehat{y}}\,dx + O(\epsilon). \end{aligned}$$

Substituting these into the first equation of (5) and using (7), one obtains a KP II-type equation by dropping out the higher order terms.

The next order of expansion for $\widehat{v}$ can be useful when the information on $\widehat{v}$ is required with η given. It is obtained with a similar argument that

$$\widehat{v} = \int \eta_{\widehat{y}}\,dx + \epsilon\int \eta\eta_{\widehat{y}}\,dx + \frac{\epsilon}{6}\eta_{x\widehat{y}} + O(\epsilon^2). \tag{14}$$

Remark 1.1. In fact, since all of the Boussinesq systems, such as classical Boussinesq system and the Bona-Smith system, are formally equivalent to each other to the order of $O(\epsilon)$, one can start from any one and derive the KP-type equations (12) or (13).

Remark 1.2. One can get rid of the ϵ's in (12) (and in (2)) by using the scaled physical variables (tilded variables)

$$\eta = \epsilon\widetilde{\eta}, \quad x = \epsilon^{-\frac{1}{2}}\widetilde{x}, \quad t = \epsilon^{-\frac{1}{2}}\widetilde{t}, \quad \widehat{y} = \epsilon^{-1}\widetilde{y}.$$

2 Dispersion relations It is also interesting to observe the relationship between the Euler equation, the Boussinesq system and the KP equation from their corresponding dispersive relations.

By assuming the solution has the form

$$\eta = \eta_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \qquad \boldsymbol{v} = \boldsymbol{v}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)},$$

where $\boldsymbol{k} = (k_1, k_2)$ and $\boldsymbol{x} = (x, y)$, one finds the dispersion relation for the two-dimensional linearized Euler equation is

$$\omega^2_{Euler} = \frac{|\boldsymbol{k}|}{\sqrt{\epsilon}}\tanh(\sqrt{\epsilon}|\boldsymbol{k}|) = |\boldsymbol{k}|^2 - \frac{1}{3}\epsilon|\boldsymbol{k}|^4 + \cdots \tag{15}$$

under the scaled variables (3) and $|\boldsymbol{k}| = \sqrt{k_1^2 + k_2^2}$.

Similarly, simple computations shows the two-dimensional linearized BBM^2 system (2) has the dispersion relation

$$\omega^2_{BBM} = \frac{|\boldsymbol{k}|^2}{(1 + \frac{\epsilon}{6}|\boldsymbol{k}|^2)^2} = |\boldsymbol{k}|^2 - \frac{\epsilon}{3}|\boldsymbol{k}|^4 + \cdots .$$

For all other abcd-systems in [**2**], the dispersion relations also have the same leading terms

$$\omega^2 = |\boldsymbol{k}|^2 - \frac{\epsilon}{3}|\boldsymbol{k}|^4 + \cdots .$$

Therefore, the difference between the dispersion relations of Euler equations and (2) (or any of the Boussinesq-type systems in [**2**]) is of order $O(\epsilon^2)$.

Remark 2.1. The dispersion relation for the BBM^2-system, ω^2_{BBM}, is positive for all $|\boldsymbol{k}|$, just as the Euler equations. This is one of the main differences between a BBM-type equation and a KdV-type equation.

The linearized KP-II equation (1) has the dispersion relation

$$\omega_{KP} = \frac{k_1^2 + \epsilon\frac{k_2^2}{2}}{k_1 + \frac{\epsilon}{6}k_1^3} = k_1 + \epsilon\frac{k_2^2}{2k_1} - \frac{1}{6}\epsilon k_1^3 + O(\epsilon^2). \tag{16}$$

It is worth noting that (16) is on ω_{KP}, instead of on ω_{KP}^2 because the KP assumption that the wave is traveling predominantly in one direction.

To compare ω_{KP} with ω_{euler}, we take the square root of (15), take the positive sign (right moving) and keep the leading orders

$$\omega_{euler} = |\boldsymbol{k}|\left(1 - \frac{1}{6}\epsilon|\boldsymbol{k}|^2\right) + O(\epsilon^2). \tag{17}$$

For (17) and (16) to agree up to the order ϵ, it is necessary to have $(k_2/k_1)^2 = O(\epsilon)$, the meaning of KP assumption that the wave is weakly three-dimensional. Therefore, it is observed that the dispersion relation of KP-II (1) approximates the dispersion relation of Euler equations up to the order $O(\epsilon)$ only under the assumption $(k_2/k_1)^2 = O(\epsilon)$.

3 Summary In this short note, we derived the famous KP-II type equation from Boussinesq systems, which are $O(\epsilon)$ approximations to Euler equations, under KP assumptions. Since there are less assumptions on waves with Boussinesq systems, it is expected that more interesting and physically relevant problems can be studied with the use of Boussinesq systems. On the other hand, the established, physically verified results from KP-II type equations should hold with Boussinesq systems.

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