MODELLING AND PRICING OF VARIANCE SWAPS FOR MULTI-FACtor STOCHASTIC VOLATILITIES WITH DELAY

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ABSTRACT. Variance swaps for financial markets with underlying asset and multi-factor stochastic volatilities with delay are modelled and priced in this paper. We obtain some analytical closed forms for the expectation and variance of the realized continuously sampled variances for multi-factor stochastic volatilities with delay. As applications, we provide numerical examples using the S&P 60 Canada Index (1998–2002) to price variance swaps with delay for all these models.

1 Introduction

1.1 Variance swaps. A stock’s variance is the square of a stock’s volatility (or standard deviation) and the stock’s volatility is the simplest measure of a stock’s risk or uncertainty. Formally, the volatility \( \sigma_R \) is the annualized standard deviation of the stock’s returns during the period of interest, where the subscript R denotes the observed or realized volatility, and \( \sigma_R^2 \) is the realized variance.

The easy way to trade variance, the square of volatility, is to use variance swaps, sometimes called realized variance forward contracts, (see [7]).

Variance swaps are forward contracts on future realized stock variance, the square of the future volatility. This instrument provides an easy way for investors to gain exposure to the future level of variance.

Demeterfi et al. [9] explained the properties and the theory of both variance and volatility swaps. They derived an analytical formula for the theoretical fair value in the presence of realistic volatility skews, and pointed out that volatility swaps can be replicated by dynamically trading the more straightforward variance swap.

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Javaheri et al. [23] discussed the valuation and hedging of a GARCH(1,1) stochastic volatility model.

The working paper by Théoret et al. [40] presented an analytical solution for pricing of volatility swaps, as proposed by Javaheri et al. [23]. They priced the volatility swaps within the framework of a GARCH(1,1) stochastic volatility model and applied the analytical solution to price a swap on the volatility of the S&P 60 Canada Index (for the 5-year historical period: 1997–2002).

Brockhaus and Long [5] provided an analytical approximation for the valuation of volatility swaps and analyzed other options with volatility exposure.

In the paper by Swishchuk [38], we found the values of variance and volatility swaps for financial markets when the underlying asset and variance follow the Heston [19] model. We also studied covariance and correlation swaps for the financial markets. As an application, we provided a numerical example using the S&P60 Canada Index to price swap on the volatility.


The paper by Swishchuk [39] studies the modeling and pricing of variance swaps for the one-factor stochastic volatility with delay. As an application, we provide a numerical example using the S&P60 Canada Index to price variance swaps.

1.2 Volatility It is known that the probability distribution of an equity has a fatter left tail and thinner right tail than the lognormal distribution (see [41, 42] and [21]), and the assumption of constant volatility $\sigma$ in a financial model, such as the original Black-Scholes model, (see [3]), is incompatible with the derivative prices observed in the market.

This issue has been addressed and studied in several ways, such as:

(i) Volatility is assumed to be a deterministic function of the time: $\sigma \equiv \sigma(t)$, (see [43]); Merton [29] extended the term structure of volatility to $\sigma := \sigma_t$ (deterministic function of time), with the implied volatility for an option of maturity $T$ given by $\tilde{\sigma}_T^2 = \frac{T}{T} \int_0^T \sigma^2_u \, du$;

(ii) Volatility is assumed to be a function of the time and the current level of the stock price $S(t)$: $\sigma \equiv \sigma(t, S(t))$, (see [9]);

(iii) Volatility is described by a stochastic differential equation with the same source of randomness as the stock’s price, (see [23]);

(iv) The time variation of the volatility involves an additional source
of randomness, besides $W_1(t)$, represented by $W_2(t)$, and is given by

$$d\sigma(t) = a(t, \sigma(t))dt + b(t, \sigma(t))dW_2(t),$$

where $W_2(t)$ and $W_1(t)$, (the initial Wiener process that governs the price process), may be correlated (see [6], [22], [19], [12]):

(v) The volatility depends on a random parameter $x$ such as $\sigma(t) \equiv \sigma(x(t))$, where $x(t)$ is some random process, (see [11, 17, 37, 36]);

(vi) Another approach is connected with the so-called uncertain volatility scenario (see [1, 6]);

(vii) The volatility $\sigma(t, S_t)$ depends on $S_t := S(t + \theta)$ for $\theta \in [-\tau, 0]$, namely, stochastic volatility with delay (see [27]).

In this paper, we are going to incorporate the case (vii) above to price variance swaps for multi-factor stochastic volatility with delay, namely, for three two-factor and one three-factor stochastic volatility models with delay.

1.3 Multi-factor models Eydeland and Geman [14] proposed extending the Heston [19] stochastic volatility model to gas or electricity prices by introducing mean-reversion in the spot price and keeping the CIR model for the variance.

Geman et al. [15] proposed a three-state variable for oil prices by introducing mean-reversion in the spot price, geometric Brownian motion in the equilibrium, (or mean-reverting), price and the CIR model for the variance.

Gibson and Schwartzl [16] note that the convenience yield $y(t)$ has been shown to be a key factor driving the relationship between spot and futures prices, and they proposed the two-state variable model for oil-contingent claim pricing.

To go to the more complex level of two factor models, keeping the same mean, the SDEs for spot price and mean process can be generalized by either allowing the long run mean or the volatility to be governed by an SDE. This leads to two distinct two factor models, with different dynamics. The first model assumes a stochastic long run mean and was introduced by Pilipovic [33].

The second generalization is the two-factor model where volatility is allowed to be stochastic (see [28]).

Pilipovich [33] describes a two-factor mean-reverting model where spot prices revert to a long term equilibrium level which is itself a random variable. Pilipovich derives a closed-form solution for forward prices to
her model when the spot and long term prices are uncorrelated, but however does not discuss option pricing in her two-factor model.

Gibson and Schwartz [16], Schwartz [35] and Hilliard and Reis [20] all analyze versions of the same two-factor model that allows for a stochastic convenience yield and permits a high level of analytical tractability. The first factor is the spot price process which is assumed to follow the geometric Brownian motion, (GBM), and the second factor is the instantaneous convenience yield of the spot energy. This is assumed to follow the mean reverting process.

Schwartz [35] extends his two-factor model to include stochastic interest rates. In this three-factor model the short term rate is assumed to follow the Vasicek mean-reverting process.

Two different types of three-factor mean-reverting models have been introduced in [28]. The first is a three-factor mean-reverting model with an intermediate level of stochastic mean governing growth or decay of spot price, and a second level of stochastic mean-reversion, inducing higher local shocks to the stock price via mean-reverting process. This system may be a suitable for modeling the spikes of electricity prices.

Fouque et al. [12] considered the following multi-factor stochastic volatility model

\[ dS_t = \mu S_t \, dt + \sigma_t S_t \, dW_t, \]
\[ \sigma_t = f(Y_t, Z_t), \]
\[ dY_t = ac_1(Y_t) \, dt + \sqrt{a} \, g_1(Z_t) \, dW^1_t, \]
\[ dZ_t = bc_2(Z_t) \, dt + \sqrt{b} \, g_2(Z_t) \, dW^2_t, \]

where \( S_t \) is the underlying asset price with a constant rate of return \( \mu \) and a stochastic volatility \( \sigma_t \) driven by the stochastic processes \( Y_t \) and \( Z_t \) varying on the respective time scales \( 1/a \) and \( 1/b \). The standard Brownian motions \( (W_t, W^1_t, W^2_t) \) are correlated.

Fouque and Han [13] found that two-factor SV models provide a better fit for the term structure of implied volatility than one factor SV models by capturing the behavior at short and long maturities.

Chernov et al. [8] used two-factor SV family models to obtain comparable empirical goodness-of-fit.

Molina et al. [32] found a strong evidence of two-factor SV models with well-separated time scales in foreign exchange data.

2 Pricing of variance swaps for one-factor stochastic volatility model with delay

As indicated in the introduction, variance
swaps are forward contracts on future realized stock variance, the square of the future volatility.

A variance swap is a forward contract on annualized variance, the square of the realized volatility. Its payoff at expiration is equal to

$$N(\sigma_{R}^{2}(S) - K_{var}),$$

where $\sigma_{R}^{2}(S)$ is the realized stock variance (quoted in annual terms) over the life of the contract

$$\sigma_{R}^{2}(S) := \frac{1}{T} \int_{0}^{T} \sigma^{2}(s) \, ds,$$

$K_{var}$ is the delivery price for variance, and $N$ is the notional amount of the swap in dollars per annualized volatility point squared. The holder of variance swap at expiration receives $N$ dollars for every point by which the stock’s realized variance $\sigma_{R}^{2}(S)$ has exceeded the variance delivery price $K_{var}$. We note that usually $N = \alpha I$, where $\alpha$ is a converting parameter such as 1 per volatility-square, and $I$ is a long-short index (+1 for long and −1 for short).

The value of a forward contract $P$ on future realized variance with strike price $K_{var}$ is the expected present value of the future payoff in the risk-neutral world

$$P^{*} = E_{P^{*}}\{e^{-rT}(\sigma_{R}^{2}(S) - K_{var})\},$$

where $r$ is the risk-free discount rate corresponding to the expiration date $T$ and $E_{P^{*}}$ denotes the expectation under the risk-neutral measure $P^{*}$.

Thus, for calculating variance swaps we need to know only $E\{\sigma_{R}^{2}(S)\}$, namely, mean value of the underlying variance.

In this way, a variance swap for stochastic volatility with delay is a forward contract on annualized variance $\sigma_{R}^{2}(t, S_{t})$. Its payoff at expiration equals

$$N(\sigma_{R}^{2}(S) - K_{var}),$$

where $\sigma_{R}^{2}(S)$ is the realized stock variance (quoted in annual terms) over the life of the contract,

$$\sigma_{R}^{2}(S) := \frac{1}{T} \int_{0}^{T} \sigma^{2}(u, S(u - \tau)) \, du, \quad \tau > 0.$$
2.1 Key features of one-factor stochastic volatility model with delay

We assume that the underlying asset \( S(t) \) follows the process

\[
dS(t) = \mu S(t) \, dt + \sigma(t, S_t) S_t \, dW(t)
\]

and the asset volatility is defined as the solution of the following equation:

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) \, dW(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t).
\]

The initial data for (1) is defined by \( S(t) = \varphi(t) \) is deterministic function, \( t \in [-\tau, 0], \tau > 0 \).

Throughout the paper we note

\[ S_t := S(t - \tau). \]

The key features of the stochastic volatility model with delay in (11) are the following:

(i) it is a continuous-time analogue of discrete-time GARCH model;
(ii) it has mean-reversion;
(iii) it does not contain another Wiener process;
(iv) market is complete;
(v) it incorporates the expectation of log-return.

In the risk-neutral world, the underlying asset \( S(t) \) follows the process

\[
dS(t) = r S(t) \, dt + \sigma(t, S_t) \, dW^*(t),
\]

and the asset volatility is defined then as follows or

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) \, dW^*(s) - \int_{t-\tau}^{t} \lambda(u)\sigma(u, S_u) \, du \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t)
\]

\[
= \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) \, dW^*(s) + (\mu - r)\tau \right]^2
\]

\[- (\alpha + \gamma)\sigma^2(t, S_t).\]
where \( W^*(t) \) is defined as follows:

\[
W^*(t) := \int_0^t \frac{\mu - r}{\sigma(s, S_s)} \, ds + W(t)
\]

with risk-neutral probability:

\[
\frac{dP^*}{dP} := \eta(T) := \exp \left\{ \int_0^T \frac{r - \mu}{\sigma(s, S_s)} \, dW(s) - \frac{1}{2} \int_0^T \left( \frac{r - \mu}{\sigma(s, S_s)} \right)^2 \, ds \right\}
\]

upon which \( W^* \) is a standard Wiener process.

In this way, we may suppose that we are in the risk-neutral world, and the underlying asset \( S(t) \) and the asset variance \( \sigma^2(t, S_t) \) follow the following equations

\[
dS(t) = rS(t) \, dt + \sigma(t, S_t) \, dW^*(t),
\]

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^t \sigma(s, S_s) \, dW^*(s) + (\mu - r) \tau \right] ^2 - (\alpha + \gamma) \sigma^2(t, S_t).
\]

Taking the expectations under risk-neutral measure \( P^* \) of both sides of the second equation (4) and denoting \( v(t) := E_{P^*}[\sigma^2(t, S_t)] \), we obtain the following deterministic delay differential equation

\[
\frac{dv(t)}{dt} = \gamma V + \alpha \tau (\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^t v(s) \, ds - (\alpha + \gamma) v(t).
\]

Note that (5) has a stationary solution

\[
v(t) \equiv X = V + \frac{\alpha \tau (\mu - r)^2}{\gamma}.
\]

### 2.2 Pricing of variance swaps with delay in stationary regime under risk-neutral measure

In the case of risk-neutral measure \( P^* \) we have

\[
v(t) = E_{P^*}[\sigma^2(t, S_t)] = V + \frac{\alpha \tau (\mu - r)^2}{\gamma}.
\]

Hence,

\[
E_{P^*}[\text{Var}(S)] = \frac{1}{T} \int_0^T E_{P^*}[\sigma^2(t, S_t)] \, dt = V + \frac{\alpha \tau (\mu - r)^2}{\gamma}.
\]
Therefore, from (6) and (7) it follows that the price $P^*$ of variance swap for stochastic volatility with delay in stationary regime under risk-neutral measure $P^*$ equals

$$P^* = e^{-rT} \left[ V - K + \frac{\alpha \tau (\mu - r)^2}{\gamma} \right].$$

It is interesting to note that (7) contains parameter $\mu$ even after risk-neutral valuation. This is because of the delay $\tau$: if $\tau = 0$, then

$$E_{P^*}[\text{Var}(S)] = V$$

and

$$P^* = e^{-rT}[V - K].$$

### 2.3 Pricing of variance swaps with delay in general case

There is no way to write a solution in an explicit form for arbitrarily given initial data. However, we can understand an approximate behavior of solutions of (5) by looking at its eigenvalues (see [18]). Substitute $v(t) = X + Ce^{\rho t}$ into (5), where $X$ is defined in (6). Then, the characteristic equation for $\rho$ is:

$$\rho = \frac{\alpha}{\rho \tau} (1 - e^{-\rho \tau}) - (\alpha + \gamma).$$

This is equivalent to (when $\rho \neq 0$):

$$\rho^2 = \frac{\alpha}{\tau} - \frac{\alpha}{\tau} e^{-\rho \tau} - (\alpha + \gamma)\rho.$$ 

The only solution to this equation is $\rho \approx -\gamma$, assuming that $\gamma$ is sufficiently small.

Then, the behavior of any solution is stable near $X$, and

$$v(t) \approx X + Ce^{-\gamma t}$$

for large values of $t$.

We shall consider the variance swap for stochastic volatility with delay in the case

$$v(t) \approx X + Ce^{-\gamma t} = V + \frac{\alpha \tau (\mu - r)^2}{\gamma} + Ce^{-\gamma t}.$$
Since
\[ v(0) = \sigma(0, S(0 - \tau)) = \sigma(0, \phi(-\tau)) := \sigma_0, \]
we can find the value of \( C \) from (9)
\[ C = v(0) - X = \sigma_0^2 - V - \frac{\alpha \tau (\mu - r)^2}{\gamma}. \]

In this way, from (10) and (11) we obtain
\[ v(t) = E_{P^*}[\sigma^2(t, S_t)] \approx V + \frac{\alpha \tau (\mu - r)^2}{\gamma} + \left( \sigma_0^2 - V - \frac{\alpha \tau (\mu - r)^2}{\gamma} \right) e^{-\gamma t}. \]

Hence,
\[
E_{P^*}[\text{Var}(S)] = \frac{1}{T} \int_0^T E_{P^*}[\sigma^2(t, S_t)] dt \\
\approx \frac{1}{T} \int_0^T \left[ V + \frac{\alpha \tau (\mu - r)^2}{\gamma} + \left( \sigma_0^2 - V - \frac{\alpha \tau (\mu - r)^2}{\gamma} \right) e^{-\gamma t} \right] dt \\
= V + \frac{\alpha \tau (\mu - r)^2}{\gamma} + \left( \sigma_0^2 - V - \frac{\alpha \tau (\mu - r)^2}{\gamma} \right) \frac{1 - e^{-\gamma T}}{T \gamma}.
\]

Therefore, from (12) and last expression it follows that the price \( P^* \) of variance swap for stochastic volatility with delay in the case in risk-neutral measure \( P^* \) is approximately
\[ P^* \approx e^{-\gamma T} \left[ V - K + \frac{\alpha \tau (\mu - r)^2}{\gamma} \right] + \left( \sigma_0^2 - V - \frac{\alpha \tau (\mu - r)^2}{\gamma} \right) \frac{1 - e^{-\gamma T}}{T \gamma}. \]

3 Pricing of variance swaps for multi-factor stochastic volatility models with delay  In this section, we calculate the variance swaps for four multi-factor stochastic volatility models with delay.
3.1 Pricing of variance swap for two-factor stochastic volatility model with delay and with geometric Brownian motion mean-reversion

The two-factor stochastic volatility model with delay and with geometric Brownian motion mean-reversion is defined in the following way.

\[
\begin{align*}
\frac{d\sigma^2(t, S_t)}{dt} &= \gamma V_t + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 \\
&\quad - (\alpha + \gamma)\sigma^2(t, S_t), \\
\frac{dV_t}{V_t} &= \xi dt + \beta dW_1(t).
\end{align*}
\]

(14)

Here \( S_t \) is defined as \( S_t := S(t - \tau) \) and

\[dS(t) = \mu S(t) dt + \sigma(t, S_t) dW(t).\]

The Wiener processes \( W(t) \) and \( W_1(t) \) may be correlated.

In order to incorporate a correlation between the Brownian motions \((W(t), W_1(t))\), we set

\[
\begin{align*}
W(t) &= W^*(t), \\
W_1(t) &= \rho W^*(t) + \sqrt{1 - \rho^2} W_1^*(t),
\end{align*}
\]

where \((W^*(t), W_1^*(t))\) are independent standard Brownian motions, and the correlation coefficient \(\rho\) satisfies \(|\rho| < 1\).

We note that the market is incomplete. A second martingale (or risk-neutral) measure \(\tilde{P}\) may be defined, in addition to \(P^*\) in (3), for example, in the following way (see, for example, [25]):

\[
\frac{d\tilde{P}}{dP} := \eta(T)\tilde{\eta}(T),
\]

where

\[
\tilde{\eta}(T) := \exp\left(W_1(T) - \frac{1}{2} T\right),
\]

and \(\eta(T)\) is defined in (3).

The measure \(P^*\) is the minimal martingale probability measure associated with \(P\) (see [25]).
Under the risk-neutral probability measure $\mathbb{P}^*$, a family of two-factor stochastic volatility models with delay and with GBM mean-reversion can be described as follows:

\begin{align}
\begin{cases}
\frac{d^2 \gamma^2(t, S_t)}{dt^2} = \gamma V_t + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW^*(s) + (\mu - r)\tau \right]^2 \\
\frac{dV_t}{V_t} = \left( \xi - \lambda \beta \right) dt + \beta (\rho dW^*(t) + \sqrt{1 - \rho^2} dW^*_1(t)).
\end{cases}
\end{align}

Here $(W^*(t), W^*_1(t))$ are independent standard Brownian motions,

\begin{align*}
dS(t) = rS(t) dt + \sigma(t, S_t) dW^*(t),
\end{align*}

where $W^*(t)$ is defined in (3). The constant $\lambda$ is the market price of volatility risk.

Since $E_{\mathbb{P}^*} V_t = V_0 e^{(\xi - \lambda \beta) t}$, the first equation takes the following form

\begin{align*}
\frac{dv(t)}{dt} = \gamma V_0 e^{(\xi - \lambda \beta) t} + \alpha \tau (\mu - r)^2 \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) ds - (\alpha + \gamma) v(t),
\end{align*}

where $v(t) := E_{\mathbb{P}^*} \sigma^2(t, S_t)$.

To value the variance swap we have to find a solution for the last equation, a nonhomogeneous integro-differential equation with delay.

After taking the first derivative of this equation we obtain

\begin{align*}
v''(t) = (\xi - \lambda \beta) \gamma V_0 e^{(\xi - \lambda \beta) t} + \frac{\alpha}{\tau} [v(t) - v(t - \tau)] - (\alpha + \gamma) v'(t).
\end{align*}

To solve this equation we rewrite it in vector form:

\begin{align*}
\tilde{v}''(t) = Av(t) + B\tilde{v}(t - \tau) + \tilde{f}(t),
\end{align*}

where

\begin{align*}
\tilde{v}(t) := \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}, \quad A := \begin{pmatrix} 0 & 0 \\ \frac{\alpha}{\tau} & -\frac{(\alpha + \gamma)}{\tau} \end{pmatrix}, \\
\tilde{f}(t) := \begin{pmatrix} 0 \\ (\xi - \lambda \beta) \gamma V_0 e^{(\xi - \lambda \beta) t} \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\alpha}{\tau} \end{pmatrix}.
\end{align*}
This is a nonhomogeneous first-order differential equation with delay. A complementary homogeneous equation has the following look:

\[ \ddot{v}_h(t) = A\dot{v}(t) + B\dot{v}_h(t - \tau), \]

and has the following approximate solution (see [18] and Section 2.3)

\[ v_h(t) \approx X + Ce^{-\gamma t}, \]

where

\begin{equation}
X := V_0 + \frac{\alpha \tau (\mu - \eta)^2}{\gamma}, \quad C := \sigma_0^2 - V_0 - \frac{\alpha \tau (\mu - \eta)^2}{\gamma}.
\end{equation}

Therefore, the nonhomogeneous equation has the following approximate solution (see [18])

\[ v(t) \approx X + Ce^{-\gamma t} + (\xi - \lambda \beta)\gamma V_0 \int_0^t (X + C e^{\gamma (t-s)}) e^{(\xi - \lambda \beta)s} ds. \]

After calculations, we see

\begin{equation}
v(t) \approx X + Ce^{-\gamma t} + (\xi - \lambda \beta)\gamma V_0 \left( \frac{X}{\xi - \lambda \beta} (e^{(\xi - \lambda \beta)t} - 1) + \frac{C}{\xi - \lambda \beta - \gamma} (e^{(\xi - \lambda \beta)t} - e^{\gamma t}) \right).
\end{equation}

We note that

\[ v(t) := E_{P^*}[\sigma^2(t, S_t)], \]

and to value the swap we have to calculate

\begin{equation}
E_{P^*}[Var(S)] := \frac{1}{T} \int_0^T E_{P^*}[\sigma^2(t, S_t)] dt = \frac{1}{T} \int_0^T v(t) dt.
\end{equation}

After substitution of \( v(t) \) in (17) into (18) we obtain the \( \mathcal{P}^* \) price of
variance swap for two-factor stochastic volatility with delay

\[ P^* \approx e^{-rT} \left\{ V - K + \frac{\alpha \tau (\mu - r)^2}{\gamma} + \left( \sigma_0^2 - V - \frac{\alpha \tau (\mu - r)^2}{\gamma} \right) \frac{1 - e^{-\gamma T}}{T \gamma} \right. \]

\[ + \frac{(\xi - \lambda \beta) \gamma V_0}{T} \left[ \frac{X}{(\xi - \lambda \beta)} \left( \frac{e^{(\xi - \lambda \beta)T} - 1}{(\xi - \lambda \beta) - \gamma} \right) - \frac{C(e^{\gamma T} - 1)}{\gamma((\xi - \lambda \beta) - \gamma)} \right] \}

\[ = e^{-rT} \left\{ X - K + C \frac{1 - e^{-\gamma T}}{T \gamma} \right. \]

\[ + \frac{(\xi - \lambda \beta) \gamma V_0}{T} \left[ \frac{X}{(\xi - \lambda \beta)} \left( \frac{e^{(\xi - \lambda \beta)T} - 1}{(\xi - \lambda \beta) - \gamma} \right) - \frac{C(e^{\gamma T} - 1)}{\gamma((\xi - \lambda \beta) - \gamma)} \right] \}

or, finally,

\[ P^* \approx e^{-rT} \left\{ X - K + C \frac{1 - e^{-\gamma T}}{T \gamma} \right. \]

\[ + \frac{(\xi - \lambda \beta) \gamma V_0}{T} \left[ \frac{X}{(\xi - \lambda \beta)} \left( \frac{e^{(\xi - \lambda \beta)T} - 1}{(\xi - \lambda \beta) - \gamma} \right) - \frac{C(e^{\gamma T} - 1)}{\gamma((\xi - \lambda \beta) - \gamma)} \right] \}

The constants \( X \) and \( C \) are defined in (16).

3.2 Pricing of the variance swap for the two-factor stochastic volatility model with delay and with Ornstein-Uhlenbeck mean-reversion

The two-factor stochastic volatility model with elay and with Ornstein-Uhlenbeck (OU) mean-reversion is defined in the following way.

\[ \frac{d\sigma^2(t, S_t)}{dt} = \gamma V_0 + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 \]

\[ - (\alpha + \gamma) \sigma^2(t, S_t). \]

\[ dV_t = \xi (L - V_t) dt + \beta dW_1(t). \]
The Wiener processes $W(t)$ and $W^1(t)$ may be correlated, $S_t$ is defined as $S_t := S(t - \tau)$, and
\[ dS(t) = \mu S(t) dt + \sigma(t, S_t) dW(t). \]

Taking into account the same reasonings as in Section 3.1, a family of two-factor stochastic volatility model with delay and with OU mean-reversion under the risk-neutral probability measure $P^*$ can be described as follows.
\[
\begin{aligned}
\frac{d\sigma^2(t, S_t)}{dt} &= \gamma V_t + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW^*(s) + (\mu - r)\tau \right]^2 \\
&- (\alpha + \gamma)\sigma^2(t, S_t), \\
dV_t &= \xi((L - \frac{\lambda\beta}{\xi}) - V_t) dt + \beta(\rho dW^*(t) \\
&+ \sqrt{1 - \rho^2} dW^*_1(t)).
\end{aligned}
\]

Here $(W^*(t), W^*_1(t))$ are independent standard Brownian motion
\[ dS(t) = r S(t) dt + \sigma(t, S_t) dW^*(t), \]
$W^*(t)$ is defined in (3), and the correlation coefficient $|\rho| < 1$.

We note that
\[ E_{P^*} V_t = e^{-\xi t} \left( V_0 - \left( L - \frac{\lambda\beta}{\xi} \right) \right) + \left( L - \frac{\lambda\beta}{\xi} \right). \]

Thus, after taking the expectation $E_{P^*}$, the first equation finally takes the following form
\[
\frac{dv(t)}{dt} = \gamma \left( e^{-\xi t} \left( \left( V_0 - \left( L - \frac{\lambda\beta}{\xi} \right) \right) + \left( L - \frac{\lambda\beta}{\xi} \right) \right) \\
+ \alpha\tau(\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) ds - (\alpha + \gamma)v(t). \]

We shall proceed with the same steps as in Section 3.1.

To value the variance swap we have to find a solution for the last equation, a nonhomogeneous integro-differential equation with delay.
The nonhomogeneous equation has the following approximate solution (see [18])

\[ v(t) \approx X + C e^{-\gamma t} - \xi \gamma \left( V_0 - \left( L - \frac{\lambda \beta}{\xi} \right) \right) \int_0^t (X + C e^{\gamma (t-s)}) e^{-\xi s} ds. \]

After calculations, we see

\[ v(t) \approx X + C e^{-\gamma t} + \xi \gamma \left( V_0 - \left( L - \frac{\lambda \beta}{\xi} \right) \right) \times \left[ \frac{X}{\xi} (e^{-\xi t} - 1) + \frac{C}{\xi + \gamma} (e^{-\xi t} - e^{-\xi t}) \right]. \]

We note that

\[ v(t) := E_{P^*}[\sigma^2(t, S_t)]. \]

To find the swap we have to calculate

\[ E_{P^*}[\text{Var}(S)] := \frac{1}{T} \int_0^T E_{P^*}[\sigma^2(t, S_t)] dt = \frac{1}{T} \int_0^T v(t) dt. \]

After calculation, we obtain finally,

\[ P^* \approx e^{-\gamma T} \left\{ X - K + C \frac{1 - e^{-\gamma T}}{T \gamma} \right\} + \frac{\xi \gamma (V_0 - (L - \frac{\lambda \beta}{\xi}))}{T} \times \left[ \frac{X}{\xi} \left( \frac{e^{-\xi T} - 1}{\xi} + T \right) + \frac{C (e^{-\xi T} - 1)}{\xi (\xi + \gamma)} + \frac{C (e^T - 1)}{\gamma (\gamma + \xi)} \right], \]

where the constants \( X \) and \( C \) are defined in (16).

### 3.3 Pricing of the variance swap for the two-factor stochastic volatility model with delay and with Pilipovich one-factor mean-reversion

The two-factor stochastic volatility model with delay and with Pilipovich one-factor mean-reversion is defined in the following way:

\[
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V_t + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^t \sigma(s, S_s) dW(s) \right]^2 - (\alpha + \gamma)\sigma^2(t, S_t),
\]

\[ dV_t = \xi (L - V_t) dt + \beta V_t dW_1(t). \]
Here Wiener processes $W(t)$ and $W_1(t)$ may be correlated, $S_t$ is defined as $S_t := S(t - \tau)$, where

$$dS(t) = \mu S(t) \, dt + \sigma(t, S_t) \, dW(t).$$

Taking into account the same reasoning as in Section 3.1, a family of two-factor stochastic volatility model with delay and with Pilipovich one-factor mean-reversion under risk-neutral probability measure $P^*$ can be define as follows:

\begin{align*}
\begin{cases}
\frac{d\sigma^2(t, S_t)}{dt} = \gamma V_t + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) \, dW^*(s) + (\mu - r) \tau \right]^2 \\
- (\alpha + \gamma)\sigma^2(t, S_t). \\
dV_t = (\xi + \lambda \beta) \left( L \frac{\xi}{\xi + \lambda \beta} - V_t \right) \, dt \\
+ \beta V_t \left( \rho \, dW^*(t) + \sqrt{1 - \rho^2} \, dW_1^*(t) \right).
\end{cases}
\end{align*}

(26)

Here $(W^*(t), W_1^*(t))$ are independent standard Brownian motion, $\lambda$ is the market price of volatility risk,

$$dS(t) = r S(t) \, dt + \sigma(t, S_t) \, dW^*(t),$$

$W^*(t)$ is defined in (3), the correlation coefficient $|\rho| < 1$.

We note that

$$E_{P^*} V_t = e^{-(\xi + \lambda \beta) t} \left( V_0 - L \frac{\xi}{\xi + \lambda \beta} \right) + \frac{\xi}{\xi + \lambda \beta}. $$

Thus, after taking the expectation $E_{P^*}$, the first equation finally takes the following form:

\begin{align*}
\frac{dv(t)}{dt} &= \gamma \left( e^{-(\xi + \lambda \beta) t} \left( V_0 - L \frac{\xi}{\xi + \lambda \beta} \right) + \frac{\xi}{\xi + \lambda \beta} \right) + \alpha \tau (\mu - r)^2 \\
&\quad + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) \, ds - (\alpha + \gamma) v(t).
\end{align*}
Proceeding with the similar calculations as in Section 3.2, we can derive the following expression for function \( v(t) \):

\[
(27) \quad v(t) \approx X + C e^{-\gamma t} + \frac{\gamma \xi}{\xi + \lambda \beta} \left[ X \left( \frac{V_0(\xi + \lambda \beta)}{\xi} - L \right) \right. \\
\times \left. \left( 1 - e^{-\left(\xi + \lambda \beta\right)t} \right) + X L t \right] \\
+ \frac{C(\xi V_0(\xi + \lambda \beta) - L)}{\xi + \lambda \beta + \gamma} + \frac{C L (e^{\gamma t} - 1)}{\gamma}.
\]

The price of a variance swap for two-factor stochastic volatility with delay and with Pilipovich one-factor mean-reversion has the following form:

\[
(28) \quad P^* \approx e^{-rT} \left\{ X - K + C \frac{1 - e^{-\gamma T}}{T \gamma} + \frac{\gamma \xi}{\xi + \lambda \beta} \left[ X \left( \frac{V_0(\xi + \lambda \beta)}{\xi} - L \right) \left( e^{-\left(\xi + \lambda \beta\right)T} - 1 \right) \right] \right. \\
+ \left. \frac{X L T^2}{2} + \frac{C(\xi V_0(\xi + \lambda \beta) - L)}{\xi + \lambda \beta + \gamma} \left( \frac{e^{\gamma T} - 1}{\gamma} + \frac{e^{-\left(\xi + \lambda \beta\right)T} - 1}{\xi + \lambda \beta} \right) + \frac{C L (e^{\gamma T} - 1)}{\gamma} - T \right\}.
\]

The constants \( X \) and \( C \) are defined in (16).

3.4 The variance swap for a three-factor stochastic volatility model with delay and with Pilipovic mean-reversion

The three-factor stochastic volatility model with delay and with Pilipovic mean-reversion is defined in the following way:

\[
(29) \begin{cases} \\
\frac{d \sigma^2(t, S_t)}{dt} = \gamma V_t + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW(s) \right]^2 - (\alpha + \gamma) \sigma^2(t, S_t), \\
dV_t = \xi (L_t - V_t) \ dt + \beta V_t \ dW_1(t), \\
dL_t = \beta_1 L_t \ dt + \eta L_t \ dW_2(t).
\end{cases}
\]
Here the Wiener processes $W(t), W^1(t)$ and $W^2(t)$ may be correlated, $S_t$ is defined as $S_t := S(t - \tau)$,

$$dS(t) = \mu S(t) dt + \sigma(t, S_t) dW(t).$$

In order to incorporate a correlation between the Brownian motions $(W(t), W^1(t)), W^2(t)$, we set

$$\begin{cases}
W(t) = W^*(t), \\
W^1(t) = \rho_1 W^*(t) + \sqrt{1 - \rho^2_1} W^*_1(t), \\
W^2(t) = \rho_2 W^*(t) + \rho_{12} W^*_1(t) + \sqrt{1 - \rho^2_2 - \rho^2_{12}} W^*_2(t).
\end{cases}$$

Here $(W^*(t), W^*_1(t)), W^*_2(t)$ are independent standard Brownian motions, and the correlation coefficients $\rho_1, \rho_2$ and $\rho_{12}$ satisfy

$$|\rho_1| < 1$$

and

$$|\rho^2_2 + \rho^2_{12}| < 1.$$ 

Taking into account the same reasoning as in Section 3.1, a family of three-factor stochastic volatility model with delay and with Pilipovich mean-reversion under the risk-neutral probability measure $P^*$ can be described as follows:

$$\begin{align*}
\frac{d\sigma^2(t, S_t)}{dt} &= \gamma V_t + \frac{\alpha}{\tau} \left[ \int_{t-\tau}^{t} \sigma(s, S_s) dW^*(s) + (\mu - r)\tau \right]^2 \\
&\quad - (\alpha + \gamma)\sigma^2(t, S_t), \\
\frac{dV_t}{dt} &= (\xi + \lambda\beta) \left( L_t \frac{\xi}{\xi + \lambda\beta} - V_t \right) dt + \beta V_t \left( \rho_1 dW^*(t) \\
&\quad + \sqrt{1 - \rho^2_1} dW^*_1(t) \right), \\
\frac{dL_t}{dt} &= (\beta_1 - \lambda_1 \eta) L_t dt + \eta L_t \left( \rho_2 dW^*(t) + \rho_{12} W^*_1(t) \\
&\quad + \sqrt{1 - \rho^2_2 - \rho^2_{12}} dW^*_2(t) \right).
\end{align*}$$

Here $(W^*(t), W^*_1(t)), W^*_2(t)$ are independent standard Brownian motions, $\lambda$ and $\lambda_1$ are the market prices of volatility risk.
We note, that (see [33])

\[ E_P \cdot V_t = e^{-(\xi + \lambda \beta)t}V_0 + \frac{\xi + \lambda \beta}{\xi + \lambda \beta + \beta_1}L_0(e^{(\beta_1 - \lambda_1 \eta)t} - e^{-(\xi + \lambda \beta)t}). \]

After taking the expectation \( E_P \), the first equation finally takes the following form

\[
\frac{dv(t)}{dt} = \gamma(e^{-((\xi + \lambda \beta)t}V_0 + \frac{\xi + \lambda \beta}{\xi + \lambda \beta + \beta_1}L_0(e^{(\beta_1 - \lambda_1 \eta)t} - e^{-(\xi + \lambda \beta)t})) \\
+ \alpha \tau(\mu - r)^2 + \frac{\alpha}{\tau} \int_{t-\tau}^{t} v(s) \, ds - (\alpha + \gamma)v(t).
\]

We are going to proceed with the same steps as in Section 3.2.

To derive the value of the variance swap we have to find a solution for the last equation, namely, nonhomogeneous integro-differential equation with delay.

After calculations, the function \( v(t) \) in (30) has the following form

\[
(31) \quad v(t) \approx X + Ce^{-(\gamma t)} - (\xi + \lambda \beta)\gamma V_0 \left[ \frac{X}{\xi + \lambda \beta} - e^{-(\xi + \lambda \beta)t} \right] \\
+ \frac{C}{\xi + \lambda \beta + \gamma} \left( e^{\gamma t} - e^{-(\xi + \lambda \beta)t} \right) + L_0 \frac{\xi + \lambda \beta}{\xi + \lambda \beta + \beta_1} \\
\times \left[ X(e^{(\beta_1 - \lambda_1 \eta)t} - e^{-(\xi + \lambda \beta)t}) + \frac{C(\beta_1 - \lambda_1 \eta)}{(\beta_1 - \lambda_1 \eta - \gamma)} \\
\times (e^{(\beta_1 - \lambda_1 \eta)t} - e^{\gamma t}) + \frac{C(\xi + \lambda \beta)}{(\xi + \lambda \beta + \gamma)}(e^{\gamma t} - e^{-(\xi + \lambda \beta)t}) \right].
\]

We note that \( v(t) := E_{P^*}[\sigma^2(t, S_t)]. \)

To value the swap we have to calculate

\[
(32) \quad E_{P^*}[\text{Var}(S)] := \frac{1}{T} \int_0^T E_{P^*}[\sigma^2(t, S_t)] \, dt = \frac{1}{T} \int_0^T v(t) \, dt.
\]

After substitution of \( v(t) \) into (32) we obtain the \( P^* \), price of the variance swap for a three-factor stochastic volatility with delay and with
Pilipovich mean-reversion:

\[
\mathcal{P}^* \approx e^{-\gamma T} \left\{ \left[ X - K + C \frac{1 - e^{-\gamma T}}{T \gamma} \right] - \frac{(\xi + \lambda \beta)\gamma V_0}{T} \right. \\
\times \left[ \frac{X}{(\xi + \lambda \beta)} \left( \frac{e^{-\gamma (\xi + \lambda \beta) T} - 1}{(\xi + \lambda \beta)} + T \right) \\
+ \frac{C(e^{-\gamma (\xi + \lambda \beta) T} - 1)}{(\xi + \lambda \beta)(\xi + \lambda \beta + \gamma)} + \frac{C(e^{\gamma T} - 1)}{\gamma (\xi + \lambda \beta)} \right] \\
+ \frac{(\xi + \lambda \beta) L_0}{(\xi + \lambda \beta)(\beta_1 - \lambda_1 \eta)} \left[ X(e^{\beta_1 - \lambda_1 \eta T} - 1 - (\beta_1 - \lambda_1 \eta)T) \right. \\
\times \left. \beta_1 - \lambda_1 \eta \right] \\
+ C \left( \frac{\beta_1 - \lambda_1 \eta}{\beta_1 - \lambda_1 \eta - \gamma} \left( \frac{e^{(\beta_1 - \lambda_1 \eta T) - 1}}{\beta_1 - \lambda_1 \eta} - \frac{e^{\gamma T} - 1}{\gamma} \right) \\
+ \frac{\xi + \lambda \beta}{\xi + \lambda \beta + \gamma} \left( \frac{e^{-\gamma (\xi + \lambda \beta) T} - 1}{(\xi + \lambda \beta)} + \frac{e^{\gamma T} - 1}{\gamma} \right) \right\}. 
\]

The constants \(X\) and \(C\) are defined in (16).

4 Numerical example 1: S&P60 Canada Index In this section, we apply the analytical solutions from Section 4.3 to price the variance swap of the S&P60 Canada index for five years (January 1998–February 2002), (see [40]).

Suppose at the end of February 2002, we wish to price the fixed leg of a variance swap based on the S&P60 Canada index. The statistics of log returns S&P60 Canada Index for 5 years (January 1997–February 2002) are presented in Table 1.

From the histogram of the S&P60 Canada index log returns on a 5-year historical period, (1,300 observations from January 1998 to February 2002) we observe leptokurtosis. If we look at the graph of the S&P60 Canada index log returns on a 5-year historical period we may see volatility clustering in the returns series. These facts indicate the conditional heteroscedasticity.

A GARCH(1,1) regression is applied to the series and the results are obtained in Table 2.

This table allows us to generate different input variables for the volatility swap model.
Statistics on Log Returns S&P60 Canada Index

<table>
<thead>
<tr>
<th>Series:</th>
<th>LOG RETURNS S&amp;P60 CANADA INDEX</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample:</td>
<td>1  1300</td>
</tr>
<tr>
<td>Observations:</td>
<td>1300</td>
</tr>
<tr>
<td>Mean</td>
<td>0.000235</td>
</tr>
<tr>
<td>Median</td>
<td>0.000593</td>
</tr>
<tr>
<td>Maximum</td>
<td>0.051983</td>
</tr>
<tr>
<td>Minimum</td>
<td>-0.101108</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>0.013567</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.665741</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>7.787327</td>
</tr>
</tbody>
</table>

TABLE 1

Estimation of the GARCH(1,1) process

<table>
<thead>
<tr>
<th>Dependent Variable: Log returns of S&amp;P60 Canada Index Prices</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method: ML-ARCH</td>
</tr>
<tr>
<td>Included Observations: 1300</td>
</tr>
<tr>
<td>Convergence achieved after 28 observations</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Std. error:</th>
<th>z-statistic:</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>0.000617</td>
<td>0.000338</td>
<td>1.824378</td>
</tr>
</tbody>
</table>

Variance Equation

| C                     | 2.58E-06  |
| ARCH(1)               | 0.060445  |
| GARCH(1)              | 0.927264  |
| R-squared             | -0.000791 |
| Adjusted R-squared    | -0.003108 |
| S.E. of regression    | 0.013588  |
| Sum squared resid     | 0.239283  |
| Log likelihood        | -3857.508 |

TABLE 2

We use the following relationship

$$\theta = \frac{V}{dt}, \quad k = \frac{1 - \alpha - \beta}{dt},$$
to calculate the following discrete GARCH(1,1) parameters:

- ARCH(1,1) coefficient $\alpha = 0.060445$;
- GARCH(1,1) coefficient $\beta = 0.927264$;
- GARCH(1,1) coefficient $\gamma = 0.012391$;

the Pearson kurtosis (fourth moment of the drift-adjusted stock return) $\xi = 7.787327$;

- long volatility $\theta = 0.05289724$;
- $k = 3.09733$;
- a short volatility $\sigma_0$ equals to 0.01;
- $\mu = 0.000235$,
- $r = 0.02 \text{ and } \tau = 1(\text{day})$.

Parameter $V$ may be found from the expression $V = \frac{C}{1-\alpha-\beta}$, where $C = 2.58 \times 10^{-6}$ is defined in Table 2. Thus, $V = 0.00020991$;

$dt = 1/252 = 0.003968254$.

Applying the expression (16) for our data, we find the following values

$$X = V + \frac{\alpha \tau (\mu - r)^2}{\gamma} = 0.0002,$$

$$C = \sigma_0^2 - V - \frac{\alpha \tau (\mu - r)^2}{\gamma} = 0.007.$$  

For series of maturities up to 100 years we obtain the following plots (see Figures 1–10) for variances and prices of variance swaps for four multi-factor stochastic volatilities with delay. Figures 1–10 depict the dependence of the different variance swaps with delay on maturity. Here $V_0 = 0.0002$, $\xi = 0.02$, $\gamma = 0.01$, $X = 0.0002$, $C = 0.007$ and delay $\tau = 1$.

5 Conclusions In the paper we studied multi-factor, i.e., two- and three-factors, stochastic volatility models with delay to model and price variance swaps. We found some analytical closed forms for expectation and variance of the realized continuously sampled variances for multi-factor stochastic volatilities with delay. As an application of our analytical solutions, we provided numerical examples using S&P 60 Canada Index (1998–2002) to price variance swaps for four different multi-factor stochastic volatility models with delay.

Acknowledgements This research is supported by NSERC grant. The author thanks Professor Robert Elliott very much for many valuable suggestions, corrections and remarks that improved the present paper. The author remains responsible for any errors in this paper.
FIGURE 1: Variance of one-factor SV with delay (formula (10)).

FIGURE 2: The price of variance swap for one-factor SV with delay (formula (13)).
FIGURE 3: Variance of two-factor SV with delay and with GBM mean-reversion (formula (17)).

FIGURE 4: Price of variance swap for two-factor SV with delay and with GBM mean-reversion (formula (19)).
FIGURE 5: Variance of two-factor SV with delay and with OU mean-reversion (formula (22)).

FIGURE 6: Price of variance swap for two-factor SV with delay and with OU mean-reversion (formula (24)).
FIGURE 7: Variance of two-factor SV with delay and with Pilipovich one-factor mean-reversion (formula (27)).

FIGURE 8: Price of variance swap for two-factor SV with delay and with Pilipovich one-factor mean-reversion (formula (28)).
FIGURE 9: Variance of three-factor SV with delay and with Pilipovich two-factor mean-reversion (formula (31)).

FIGURE 10: Price of variance swap for three-factor SV with delay and with Pilipovich two-factor mean-reversion (formula (33)).
REFERENCES
