STATE-SPACE APPROACH ON GENERALIZED THERMOELASTICITY FOR AN INFINITE MATERIAL WITH A SPHERICAL CAVITY AND VARIABLE THERMAL CONDUCTIVITY SUBJECTED TO RAMP-TYPE HEATING

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ABSTRACT. The present paper is concerned with a homogeneous and isotropic unbounded body with a spherical cavity subjected to ramp-type heating. The Lord and Shulman model of thermoelasticity is employed with variable thermal conductivity. The closed form solutions for distributions of displacement, temperature, strain and stresses are obtained by using the Laplace transform and the state-space approach. Numerical results applicable to a copper-like material are also presented graphically and the nature of variations of the physical quantities with radial coordinate and with the thermal conductivity.

1 Introduction

The generalized theories of thermoelasticity, which admit the finite speed of the thermal signal, have been the center of interest of active research during last three decades. These theories remove the paradox of infinite speed of heat propagation inherent in the conventional coupled dynamical theory of thermoelasticity introduced by Biot [1]. Because of the inclusion of the thermal relaxation parameters, the basic governing equations involved in the generalized theories of thermoelasticity are all differential equations of hyperbolic type and these theories are also referred to as hyperbolic thermoelasticity theories [5]. The extended thermoelasticity theory proposed by Lord and Shulman [16], which is also known as thermoelasticity theory with one relaxation time, and the temperature-rate dependent theory of thermoelasticity developed by Green and Lindsay [9], which is also known as thermoelasticity theory with two relaxation times, are two important models of generalized theory of thermoelasticity. Moodie and Tait [19]
used the linear Gurtin-Pipkin theory of heat conduction to study the problem of an homogenous half-space whose boundary is subjected to step temperature and Sawatzky and Moodie [26] used the same theory in a deformable one dimensional material of homogeneous thermoelastic half-space subjected to thermal and mechanical disturbances at the boundary, while, Onçu and Moodie [22, 23], made an analysis of the thermal transient generated by nonuniform sources applied to circular cavities and circular hole in inhomogeneous conductor. McCarthy et al. [17] examined the propagation of the thermal and mechanical transients through general linear thermoviscoelastic medium. Oncu and Moodie [24] used the theory of Green and Lindsay to solve the one dimensional problem and studied the wave propagation of a plane wave in an isotropic homogeneous half-space and also they got the derivation of the constitutive relations of an elastic heat conductor for which the heat flux and the temperature obey a frame-invariant form of Cattaneo’s equation [25]. Sherief and Salah [28], Mukhopadhyay [20], Misra, et al. [18] Chandrasekharaihaiah and Murthy [2]) are relevant for the present work. Recently, Green and Nagdhi ([10, 11, 12]) have formulated three different models of thermoelasticity in an alternative way. Among these, in one of the models (Green and Nagdhi, [12]) the most significance difference is that the internal rate of production of entropy is identically zero, i.e., there is no dissipation of thermal energy. This theory (GN theory) is known as thermoelasticity without energy dissipation theory (TEWOEDT). In the development of this theory the thermal displacement gradient is considered as a constitutive variable, whereas in the conventional development of a thermoelasticity theory, the temperature gradient is taken as a constitutive variable. A uniqueness theorem in the case of linearized version of this theory has been given independently by Green and Nagdhi [12] and Chandrasekharaihaiah [3]. Later on, Chandrasekharaihaiah [4] studied free plane harmonic waves without energy dissipation in an unbounded body. Chandrasekharaihaiah and Srinath [6, 7] have studied cylindrical/spherical waves due to (i) a load applied to the boundary of the cylindrical/spherical cavity in an unbounded body, (ii) a line/point heat source in an unbounded body. Sharma and Chouhan [27] tackled a problem on thermoelastic interaction without energy dissipation due to body forces and heat sources. Mukhopadhyay [21] dealt with a problem concerning the thermoelastic interactions without energy dissipation in an unbounded medium with a spherical cavity subjected to thermal shock.

State space methods are the cornerstone of modern control theory. The essential feature of state space methods is the characterization of
the processes of interest by differential equations instead of transfer functions. This may seem like a throwback to the earlier, primitive, period where differential equations also constituted the means of representing the behavior of dynamic processes. But in the earlier period the processes were simple enough to be characterized by a single differential equation of fairly low order. In the modern approach the processes are characterized by systems of coupled, first order differential equations. In principle there is no limit to the order (i.e., the number of independent first order differential equations) and in practice the only limit to the order is the availability of computer software capable of performing the required calculations reliably.

The importance of state space analysis is recognized in fields where the time behavior of any physical process is of interest. The state space approach is more general than the classical Laplace and Fourier transform theory. Consequently, state space theory is applicable to all systems that can analyzed by integral transforms in time, and is applicable to many systems for which transform theory breaks down. Furthermore, state space theory gives a somewhat different insight into the time behavior of linear systems.

Almost all the problems, which have been solved, were in the context of the theory of Lord and Shulman; El-Maghraby and Youssef \cite{8} used the state space approach to solve a thermo-mechanical shock problem. Sherief and Youssef \cite{29} get the short time solution for a problem in magneto-thermoelasticity. Youssef \cite{30} constructed a model of the dependence of the modulus of elasticity and the thermal conductivity on the reference temperature and solved a problem of an infinite material with a spherical cavity.

It is usual to assume in thermal stress calculations that material properties are independent of temperature. Significant variations do however occur over the working temperature range of the “engineering ceramics”, particularly in the coefficient of thermal conductivity, K. Godfrey has reported decreases of up to 45 per cent in the thermal conductivity of various samples of silicon nitride between 1 and 400°C. The question arises: what are the effects of these variations on the stress and displacement distributions in metal components ?

Modern structural elements are often subjected to temperature changes of such magnitude that their material properties may no longer be regarded as having constant values even in an approximate sense. The thermal and mechanical properties of materials vary with temperature, so that the temperature dependence of material properties must be taken into consideration in the thermal stress analysis of these elements. In
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Suhara solved the thermoelastic problem of a hollow circular cylinder of which only the shear modulus was temperature dependent. Since his work, many investigators have attacked the thermal stress problems in elastic and inelastic materials with temperature dependent properties \[13\].

This paper is concerned with thermal stress problems in generalized thermoelasticity where thermal conductivity depends on the temperature applied to an infinite material with a spherical cavity subjected to ramp-type heating.

2 The governing equations Let us consider a perfectly conducting infinite homogeneous isotropic elastic medium free from any sources of heat. The heat equation in the form \[14\]:

\[
(K \theta_i)_i = \left(1 + \tau_0 \frac{\partial}{\partial t}\right) \left(\rho C_E \dot{\theta} + \gamma T_0 \dot{e}\right)
\]

where \(K\) is the thermal conductivity, \(\tau_0\) is the relaxation time, \(\rho\) is the density, \(C_E\) is the specific heat at constant strain, \(T_0\) is the reference temperature, \(\gamma = \alpha_T (3\lambda + 2\mu)\), \(\alpha_T\) is the thermal linear expansion, \(\lambda\) and \(\mu\) are Lame’s constants, \(\theta = (T - T_0)\) is the temperature increment such that \(\theta/T_0 \ll 1\) and \(e\) is the cubic dilatation.

In most materials, the dependence of \(K\) and \(C_E\) on \(\theta\) is a function in some range of the temperature \[14\], i.e.,

\[
(2) \quad K = K(\theta)
\]

and

\[
(3) \quad \rho C_E = \frac{K}{\kappa}
\]

where \(\kappa\) is the diffusivity.

Using equation (3) with equation (1), we get

\[
(K \theta_i)_i = \left(1 + \tau_0 \frac{\partial}{\partial t}\right) \left(\frac{K}{\kappa} \dot{\theta} + \gamma T_0 \dot{e}\right), \quad i = 1, 2, 3.
\]

We will use the mapping \[14\]:

\[
(5) \quad \dot{\theta} = \frac{1}{K_0} \int^{\theta}_{\theta_0} K(\theta') d\theta'
\]
where $K_0$ is the thermal conductivity when $K$ is independent of $\theta$.

Differentiating equation (5) with respect to the coordinates, we get

$$\partial_i \theta = \frac{1}{K_0} K(\theta) \theta_i.$$  

Differentiating again the above equation with respect to the coordinates, we obtain

$$K_0 \partial_{ii} \theta = [K(\theta) \partial_i \theta].$$  

Differentiating equation (5) with respect to time, we get

$$K_0 \dot{\theta} = K(\theta) \dot{\theta}.$$  

Substituting from equations (7) and (8) in the heat equation (4), we obtain

$$\partial_{ii} \theta = \left( \frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) \left( \frac{\partial}{\partial \kappa} + \gamma T_0 \frac{e}{K_0} \right).$$  

The equations of motion have the form [29]:

$$\rho \ddot{u}_i = (\lambda + \mu) u_{j,j,i} + \mu u_{i,j,j} - \gamma \theta_i + \rho F_i, \quad i, j = 1, 2, 3$$  

where $u_i$ are the displacement components and $F_i$ are the body force components.

By using the relation (2) and equation (6), we get

$$\rho \ddot{u}_i = (\lambda + \mu) u_{j,j,i} + \mu u_{i,j,j} - \frac{\gamma K_0}{K(\theta)} \delta_{ij} + \rho F_i.$$  

For linearity, we approximate the thermal conductivity $K(\theta) \approx K(T_0)$, which is constant depending on the reference temperature $T_0$.

Hence, we have

$$\rho \ddot{u}_i = (\lambda + \mu) u_{j,j,i} + \mu u_{i,j,j} - \frac{\gamma K_0}{T_0} \theta_i + \rho F_i.$$  

The constitutive equations take the form [14]:

$$\sigma_{ij} = 2 \mu e_{ij} + (\lambda \delta_{ij} - \gamma \theta) \delta_{ij}$$  

where $\sigma_{ij}$ are the stresses components, $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ are the strain components where $e_{11} + e_{22} + e_{33} = e$, and $\delta_{ij}$ is the Kronecker delta symbol.

By using the relation (2) and the same approximation that we have used above, we get

$$\sigma_{ij} = 2 \mu e_{ij} + \left( \frac{\lambda e_{kk}}{K(T_0)} \partial \right) \delta_{ij}.$$  


3 Formulation of the problem  Let \((r, \psi, \phi)\) denote the radial coordinate, the co-latitude, and longitude of a spherical coordinate system, respectively. We consider a homogeneous, isotropic, thermoelastic medium occupying the region \(R \leq r < \infty\) obeying equations (9), (11) and (13), without any body forces and initially quiescent where \(r\) is the radius of the shell.

We note that due to spherical symmetry the displacement components have the form

\[
  u_r = u(r, t), \quad u_{\psi} = u_{\phi} = 0.
\]

The strain tensor components also have the form

\[
  e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\phi\phi} = e_{\psi\psi} = \frac{u}{r},
\]

\[
  e_{r\phi} = e_{r\psi} = e_{\phi\psi} = 0.
\]

The cubic dilatation takes the form

\[
  e = \frac{\partial u}{\partial r} + \frac{2u}{r} = \frac{1}{r^2} \frac{\partial (r^2 u)}{\partial r}.
\]

The heat equation takes the form

\[
  \nabla^2 \psi = \left( \frac{\partial}{\partial r} + \tau_0 \frac{\partial^2}{\partial r^2} \right) \left( \frac{\vartheta}{\kappa} + \frac{\gamma T_0}{K_0} \right) e
\]

where \(\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right)\).

The equation of motion takes the form

\[
  \rho \ddot{\psi} = (\lambda + 2\mu) \frac{\partial e}{\partial r} - \frac{\gamma K_0}{K(T_0)} \frac{\partial \vartheta}{\partial r}.
\]

The constitutive equations are

\[
  \sigma_{rr} = \lambda e + 2\mu \frac{\partial u}{\partial r} - \frac{\gamma K_0}{K(T_0)} \vartheta,
\]

\[
  \sigma_{\psi\psi} = \sigma_{\phi\phi} = \lambda e + 2\mu \frac{r}{r^2} - \frac{\gamma K_0}{K(T_0)} \vartheta,
\]

\[
  \sigma_{r\phi} = \sigma_{r\psi} = \sigma_{\phi\psi} = 0.
\]
For simplicity, we use the following non-dimensional variables:

\[ r = \frac{c_0}{\kappa} r', \quad u = \frac{c_0}{\kappa} u', \quad t = \frac{c_0^2}{\kappa} t', \quad \tau_0 = \frac{c_0}{\kappa} \tau', \]

\[ \sigma = \frac{\sigma'}{\mu}, \quad \vartheta = \frac{\vartheta'}{T_0}, \quad \epsilon^2 = \frac{\lambda + 2\mu}{\rho}, \]

obtaining

\begin{align}
(23) \quad \ddot{u} &= \frac{\partial \epsilon}{\partial r} - a \frac{\partial \vartheta}{\partial r}, \\
(24) \quad \nabla^2 \vartheta &= \left( \frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) (\vartheta + \gamma e), \\
(25) \quad \sigma_{rr} &= \beta^2 e - 4 \frac{u}{r} - b \vartheta, \\
(26) \quad \sigma_{\psi \psi} &= \sigma_{\phi \phi} = (\beta^2 - 2) e + 2 \frac{u}{r} - b \vartheta,
\end{align}

where

\[ \beta = \left( \frac{\lambda + 2\mu}{\mu} \right)^{\frac{1}{2}}, \quad b = \frac{\gamma T_0}{\mu} \frac{K_0}{(T_0)}, \quad g = \frac{\gamma \kappa}{K_0}, \quad a = \frac{b}{\beta^2}. \]

For convenience, the prime has been omitted.

Equation (23) could be written in the form

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \ddot{u} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 e \right) - a \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \vartheta \right). \]

Using equation (17), we obtain

\[ (27) \quad \nabla^2 e - a \nabla^2 \vartheta = \ddot{e}. \]

Taking the Laplace transform, this is defined as the following:

\[ \mathcal{F}(s) = \int_0^\infty f(t) e^{-st} \, dt. \]

We get

\[ (28) \quad (\nabla^2 - s^2) \overline{e} = a \nabla^2 \overline{\vartheta}, \]
Equations (28) and (29) can be written in the forms

\begin{equation}
\nabla^2 \overline{\eta} = |s^2 a g(s + \tau_0 s^2)\overline{\eta} + a(s + \tau_0 s^2)\overline{\eta},
\end{equation}

\begin{equation}
\nabla^2 \overline{\psi} = \overline{\sigma}_{\psi \psi} = (\beta^2 - 2)\overline{\psi} + 2\frac{\overline{\psi}}{r} - b\overline{\eta}.
\end{equation}

Choosing as state variable the temperature increment, the strain component in the \(r\)-direction, equations (28) and (29) can be written in matrix form as

\begin{equation}
\begin{bmatrix} \overline{\sigma}(r,s) \\ \overline{\psi}(r,s) \end{bmatrix} = \begin{bmatrix} s^2 + ag(s + \tau_0 s^2) & a(s + \tau_0 s^2) \\ g(s + \tau_0 s^2) & s + \tau_0 s^2 \end{bmatrix} \begin{bmatrix} \overline{\sigma}(r,s) \\ \overline{\psi}(r,s) \end{bmatrix}.
\end{equation}

The formal solution of system (34) can be written in the form

\begin{equation}
V(r,s) = C_1 e^{-A(s)r} + C_2 e^{A(s)r}.
\end{equation}

where \(C_1\) and \(C_2\) are constants.

For a bounded solution as \(r \to \infty\), we have to choose \(C_2 = 0\), hence we have

\begin{equation}
V(r,s) = C_1 e^{-A(s)r}.
\end{equation}

Since \(r = R\) must satisfy the last equation, we can get the constant \(C_1\) in the form

\begin{equation}
C_1 = R e^{A(s)R} V(R,s).
\end{equation}

Hence, equation (36) will take the form

\begin{equation}
V(r,s) = \frac{R}{r} e^{-A(s)(r-R)} V(R,s), \quad r \geq R.
\end{equation}
where

\[ \nabla(R, s) = \begin{bmatrix} \tau(R, s) \\ \vartheta(R, s) \end{bmatrix}. \]

We will use the well-known Cayley-Hamilton theorem to find the form of the matrix of \( \exp(-\sqrt{A(s)}(r - R)) \). The characteristic equation of the matrix \( A(s) \) can be written as

\[ \lambda^2 - \ell \lambda + m = 0 \]

where \( \ell = s^2 + (s + \tau_0 s^2)(1 + \varepsilon) \) and \( m = s^2(s + \tau_0 s^2) \).

The roots of this equation, namely, \( \lambda_1 \) and \( \lambda_2 \), satisfy the relations

\[ \lambda_1 + \lambda_2 = s^2 + (s + \tau_0 s^2)(1 + \varepsilon) \]

where \( \varepsilon = ag \) and

\[ \lambda_1 \lambda_2 = s^2(s + \tau_0 s^2). \]

The Taylor series expansion for the matrix exponential in (36) is given by

\[ \exp(-\sqrt{A(s)}(r - R)) = \sum_{n=0}^{\infty} \frac{[-\sqrt{A(s)}(r - R)]^n}{n!}. \]

Using Cayley-Hamilton theorem, we can express \( A^2 \) and higher orders of the matrix \( A \) in terms of \( A \) and \( I \) where \( I \) is the unit matrix of second order.

Thus, the infinite series in equation (28) can be reduced to

\[ \exp(-\sqrt{A(s)}(r - R)) = a_0 I + a_1 A \]

where \( a_0 \) and \( a_1 \) are some coefficients depending on \( s \) and \( r \).

By Cayley-Hamilton theorem, the characteristic roots \( \lambda_1 \) and \( \lambda_2 \) of the matrix \( A \) must satisfy equation (44), thus we have

\[ \exp(-\sqrt{\lambda_1}(r - R)) = a_0 + a_1 \lambda_1, \]

\[ \exp(-\sqrt{\lambda_2}(r - R)) = a_0 + a_1 \lambda_2. \]
Solving the above linear system of equations, we get

\begin{align}
(47a) \quad a_0 &= \frac{\lambda_1 e^{-\sqrt{\lambda_2}(r-R)} - \lambda_2 e^{-\sqrt{\lambda_1}(r-R)}}{\lambda_1 - \lambda_2}, \\
(47b) \quad a_1 &= \frac{e^{-\sqrt{\lambda_2}(r-R)} - e^{-\sqrt{\lambda_1}(r-R)}}{\lambda_2 - \lambda_1}.
\end{align}

Hence, we have the following matrix

\begin{align}
(48) \quad \exp(-\sqrt{A(s)}(r-R)) &= [L_{ij}], \quad i, j = 1, 2
\end{align}

where

\begin{align}
L_{11} &= \frac{1}{\lambda_1 - \lambda_2} \left( \left[ \lambda_1 - (s + \tau_0 s^2) \right] e^{-\sqrt{\lambda_1}(r-R)} - \left[ \lambda_2 - (s + \tau_0 s^2) \right] e^{-\sqrt{\lambda_2}(r-R)} \right), \\
L_{12} &= \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2} \left[ e^{-\sqrt{\lambda_1}(r-R)} - e^{-\sqrt{\lambda_2}(r-R)} \right], \\
L_{21} &= \frac{\lambda_1 - \lambda_2}{\lambda_2 - \lambda_1} \left[ \left[ \lambda_2 - (s + \tau_0 s^2) \right] e^{-\sqrt{\lambda_2}(r-R)} - \left[ \lambda_1 - (s + \tau_0 s^2) \right] e^{-\sqrt{\lambda_1}(r-R)} \right], \\
L_{22} &= \frac{1}{\lambda_2 - \lambda_1} \left( \left[ \lambda_2 - (s + \tau_0 s^2) \right] e^{-\sqrt{\lambda_2}(r-R)} - \left[ \lambda_1 - (s + \tau_0 s^2) \right] e^{-\sqrt{\lambda_1}(r-R)} \right).
\end{align}

We can write the solution in the form

\begin{align}
(50) \quad V(r, s) &= \frac{R}{r} [L_{ij}] V(R, s).
\end{align}

Finally, we have

\begin{align}
(51) \quad \overline{\sigma}(r, s) &= \frac{R}{(\lambda_1 - \lambda_2)r} \left[ L_{11} e^{-\sqrt{\lambda_1}(r-R)} - L_{22} e^{-\sqrt{\lambda_2}(r-R)} \right], \\
(52) \quad \overline{\eta}(r, s) &= \frac{R}{(\lambda_1 - \lambda_2)r} \left[ M_{11} e^{-\sqrt{\lambda_1}(r-R)} - M_{22} e^{-\sqrt{\lambda_2}(r-R)} \right],
\end{align}
(53) $u(r, s) = -\frac{R}{(\lambda_1 - \lambda_2)r^2} \left[ \frac{L_1}{\lambda_1} e^{-\sqrt{\lambda_1}(r-R)} - \frac{L_2}{\lambda_2} e^{-\sqrt{\lambda_2}(r-R)} \right],$

(54) $\sigma_{rr}(r, s) = \frac{R}{(\lambda_1 - \lambda_2)r^2} \left\{ \left[ (\beta^2 + \frac{4}{\lambda_1 r^2})L_2 - bM_1 \right] e^{-\sqrt{\lambda_1}(r-R)} \right. $

$\left. - \left[ (\beta^2 + \frac{4}{\lambda_2 r^2})L_2 - bM_2 \right] e^{-\sqrt{\lambda_2}(r-R)} \right\},$

(55) $\sigma_{\phi\phi}(r, s) = \frac{R}{(\lambda_1 - \lambda_2)r^2} \left\{ \left[ (\beta^2 - 2 + \frac{2}{\lambda_1 r^2})L_1 - bM_1 \right] e^{-\sqrt{\lambda_1}(r-R)} \right. $

$\left. - \left[ (\beta^2 - 2 + \frac{2}{\lambda_2 r^2})L_1 - bM_2 \right] e^{-\sqrt{\lambda_2}(r-R)} \right\},$

where

$L_1 = [a(s + \tau_0 s^2)] \overline{\sigma}(R, s) + [\lambda_1 - (s + \tau_0 s^2)] \overline{\sigma}(R, s),$

$L_2 = [a(s + \tau_0 s^2)] \overline{\sigma}(R, s) + [\lambda_2 - (s + \tau_0 s^2)] \overline{\sigma}(R, s),$

$M_1 = [\lambda_2 - (s + \tau_0 s^2)] \overline{\sigma}(R, s) + [g(s + \tau_0 s^2)] \overline{\sigma}(R, s),$

$M_2 = [\lambda_1 - (s + \tau_0 s^2)] \overline{\sigma}(R, s) + [g(s + \tau_0 s^2)] \overline{\sigma}(R, s).$

4 Application  In order to evaluate the unknown parameters $L_1$, $L_2$, $M_1$ and $M_2$, we will first consider that, the dependence of the thermal conductivity $K$ and $C_E$ on $\theta$ is linear function in some range of the temperature, i.e.,

(56) $K(\theta) = K_0(1 + K_1 \theta)$

where $K_1$ is a small negative constant [14].

When we apply the mapping in (5), we get

(57) $\vartheta = \theta + \frac{K_1}{2} \theta^2.$

Now, we will use the boundary conditions on the internal surface of the shell, $r = R$ which are given by:
(1) Thermal boundary condition

The internal surface \( r = R \) is subjected to ramp-type heating in the form

\[
\theta(R, t) = \begin{cases} 
0, & t \leq 0 \\
\frac{\theta_1}{t_0} t, & 0 < t \leq t_0 \\
\theta_1, & b > t_0 
\end{cases}
\]  

(58)

where \( \theta_1 \) is constant and \( t_0 \) is called the ramping parameter.

Applying the Laplace transforms, we get

\[
\overline{\theta}(R, s) = \frac{\theta_1 (1 - e^{-st_0})}{t_0 s^2},
\]

and

\[
\overline{\theta}^2(R, s) = \frac{\theta_1^2}{t_0^2 s^3} \left( 2 - 2e^{-st_0}(st_0 + 1) \right).
\]

Hence, we have from equation (58) the following condition

\[
\overline{\theta}(R, s) = \overline{\theta} = \frac{\theta_1 (1 - e^{-st_0})}{t_0 s^2} + \frac{k_1 \theta_1^2}{t_0^2 s^3} \left[ 1 - e^{-st_0}(st_0 + 1) \right].
\]

(2) Mechanical boundary condition

The internal surface \( r = R \) has a rigid foundation, which is rigid enough to prevent any strain, i.e.,

\[
e(R, t) = 0.
\]

Applying the Laplace transforms, we get

\[
\overline{\sigma}(R, s) = 0.
\]

Using the conditions (62) and (64) into equations (52)–(56), we get

\[
\overline{\varphi}(r, s) = \frac{R}{(\lambda_1 - \lambda_2) r} \left[ L_1 e^{-\lambda_1(r-R)} - L_2 e^{-\lambda_2(r-R)} \right],
\]

\[
\overline{\varphi}'(r, s) = \frac{R}{(\lambda_1 - \lambda_2) r} \left[ M_1' e^{-\sqrt{\lambda_1}(r-R)} - M_2' e^{-\sqrt{\lambda_2}(r-R)} \right],
\]
\[ \mathbf{u}(r,s) = \frac{-R}{(\lambda_1 - \lambda_2)r^2} \left[ \frac{L_1'}{\lambda_1} e^{-\sqrt{\lambda_1} (r-R)} - \frac{L_2'}{\lambda_2} e^{-\sqrt{\lambda_2} (r-R)} \right], \]

\[ \mathbf{\sigma}_{rr}(r,s) = \frac{R}{(\lambda_1 - \lambda_2)r} \left\{ \left[ \left( \beta^2 + \frac{4}{\lambda_1 r^2} \right) L_1' - b M_1' \right] e^{-\sqrt{\lambda_1} (r-R)} - \left[ \left( \beta^2 + \frac{4}{\lambda_2 r^2} \right) L_2' - b M_2' \right] e^{-\sqrt{\lambda_2} (r-R)} \right\}, \]

\[ \overline{\tau}_{\phi\phi}(r,s) = \frac{R}{(\lambda_1 - \lambda_2)r} \left\{ \left[ \left( \beta^2 - 2 + \frac{2}{\lambda_2 r^2} \right) L_2' - b M_2' \right] e^{-\sqrt{\lambda_2} (r-R)} - \left[ \left( \beta^2 - 2 + \frac{2}{\lambda_1 r^2} \right) L_1' - b M_1' \right] e^{-\sqrt{\lambda_1} (r-R)} \right\}, \]

where

\[ L_1' = [a(s + \tau_0 s^2)] \vartheta_R, \]
\[ L_2' = [a(s + \tau_0 s^2)] \vartheta_R, \]
\[ M_1' = [\lambda_2 - (s + \tau_0 s^2)] \vartheta_R, \]
\[ M_2' = [\lambda_1 - (s + \tau_0 s^2)] \vartheta_R. \]

By obtaining \( \vartheta \), the temperature increment \( \theta \) can be obtained by solving equation (58) to give

\[ \theta = \frac{-1 + \sqrt{1 + 2k_1 \vartheta}}{k_1}. \]

They complete the solution on the Laplace domain.

5 Inversion of the Laplace transform In order to invert the Laplace transform, we adopt a numerical inversion method based on a Fourier series expansion [15].

By this method the inverse \( f(t) \) of the Laplace transform \( \mathcal{F}(s) \) is approximated by

\[ f(t) = \frac{e^{6c t}}{t_1} \left[ \frac{1}{2} \mathcal{F}(c) + R_1 \sum_{k=1}^{N} \mathcal{F} \left( c + \frac{i k \pi}{t_1} \right) \exp \left( \frac{i k \pi t}{t_1} \right) \right], \quad 0 < t_1 < 2t \]
where \( N \) is a sufficiently large integer representing the number of terms in the truncated Fourier series, chosen such that

\[
\exp(ckt)R1 \left[ \mathcal{F} \left( c + \frac{iN\pi}{t1} \right) \exp \left( \frac{iN\pi t}{t1} \right) \right] \leq \varepsilon_1
\]

where \( \varepsilon_1 \) is a prescribed small positive number that corresponds to the degree of accuracy required. The parameter \( c \) is a positive free parameter that must be greater than the real part of all the singularities of \( \mathcal{F}(s) \). The optimal choice of \( c \) was obtained according to the criteria described in [15].

6 Numerical results and discussion  

The copper material was chosen for purposes of numerical evaluations and the constants of the problem were taken as follows:

\[
K_0 = 386N/K \text{ sec.}, \quad \alpha_T = 1.78(10^{-5})K^{-1}, \quad C_E = 383.1m^2/K
\]

\[
\mu = 3.86(10^{10}) \text{ N/m}^2, \quad \lambda = 7.76(10^{10}) \text{ N/m}^2, \quad \rho = 8954 \text{ kg/m}^3
\]

\[
\tau_0 = 0.02, \quad T_0 = 293 \text{ K}, \quad \varepsilon = 1.68
\]

\[
\beta^2 = 4, \quad R = 1.
\]

The computations were carried out for \( \theta_1 = 1 \). The temperature, stresses, displacement and strain distributions are represented graphically at different value of \( K_1 \) and time \( t \).

The field quantities, temperature, stresses, displacement and strain depend not only on the state and space variables \( t \) and \( r \), but also depend on \( k_1 \) and \( t_0 \). It has been observed that, \( k_1 \) has significant effect on the temperature, stresses, displacement and strain distributions quantities. Here all the variables/parameters are taken in nondimensional forms.

Figure 1 exhibits the space variation of temperature, and we observe the following:

(i) Some differences in the values of temperature are noticed for different values of \( K_1 \). We can see that, the value of the temperature decreases when the absolute value of the parameter \( K_1 \) increases. Physically, when the absolute value of \( K_1 \) increases the value of the thermal conductivity decreases, which mean that, the ability of the particles to transport the heat through the medium will decrease,
so we see the curve of the case of $K_1 = -0.1$ appears below the curve of the case $K_1 = 0.0$.

(ii) The ramping parameter $t_0$ has a clear effect on the values of the temperature; actually, the value of the temperature increases when $t \geq t_0$ and decreases when $t < t_0$ where larger $t$ with respect to $t_0$ means larger heating on the boundary.

Figure 2 exhibits the space variation of strain, and we observe the following:

(i) Some differences in the value of strain are noticed for different values of $K_1$. We can see that, the absolute value of the maximum point of the strain decreases when the thermal conductivity is variable.

(ii) The ramping parameter $t_0$ has a clear effect on the values of the strain; actually the absolute value of the strain increases when $t \geq t_0$ and decreases when $t < t_0$ for the same reason that shown previously.
(iii) The position of the maximum point of the strain with respect to $r$ axis increases when the time increases.

(iv) The difference between the values of the maximum points of the strain for the two cases of the thermal conductivity increases when the time increases, i.e., the effect of the thermal conductivity is more clear in larger time.

Figure 2: The strain distribution at $t_0 = 0.2$.

Figure 3 exhibits the space variation of displacement, and we observe the following:

(i) Small difference in the value of displacement is noticed for different values of $K_1$, however we can say that the value of the displacement decreases when the value of the parameter $K_1$ decreases.

(ii) The ramping parameter $t_0$ has a clear effect on the values of the displacement; actually the value of the displacement increases when $t \geq t_0$ and decreases when $t < t_0$ for the same reason that was shown previously.
Figure 4 exhibits the space variation of the stresses; we observe the following:

(i) Small difference in the value of the stress is noticed for different values of $K_1$.

(ii) The ramping parameter $t_0$ has a clear effect on the values of the displacement. The value of the displacement increases when $t \geq t_0$ and decreases when $t < t_0$ for the same reason that was shown previously.

(iii) The position of the maximum point of the stress with respect to $r$ axis increases when the time increases.

Figures 5–8 exhibit the space variation of all the fields at constant value of $r = 1.2$, and with different values of time, we observe a clear difference in the value of temperature distribution for the two cases of the thermal conductivity but small differences in the value of the strain, displacement and stress.

One can see that, although the curves are smooth and do not have any discontinuous points as in $[23, 17]$, they take the same behavior in some range of time where the boundary conditions somehow are different.
FIGURE 4: The radial stress distribution at $t_0 = 0.2$.

FIGURE 5: The temperature distribution at $r = 1.2$. 
FIGURE 6: The strain distribution at $r = 1.2$.

FIGURE 7: The displacement distribution at $r = 1.2$. 
7 Conclusion  Temperature, strain, stress, and displacements fields in a perfectly conducting infinite homogeneous isotropic elastic medium free from any sources heat due to linear temperature ramping have been examined within the framework of the generalized thermoelasticity theory of Lord and Shulman taking into account the dependence of the thermal conductivity on temperature increment. We have found that, the values of all the fields have been affected by the changing of the thermal conductivity. Mathematically and physically, the ramp-type heating is more realistic than the thermal shock that expressed by the unite step function where the ramping parameter of heating has a clear affect all the fields.

REFERENCES


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