SIMILARITY SOLUTIONS AND EVOLUTION OF WEAK DISCONTINUITIES IN A VAN DER WAALS GAS

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ABSTRACT. The basic equations, governing the unsteady one-dimensional axisymmetric motion in a Van der Waals gas, are considered. Using the Invariance group properties of the original system, the new autonomous system is found. The propagation of weak discontinuities is considered in the known particular solution of the autonomous system. The critical time, when these weak discontinuities culminate into a shock, is also determined for an ideal gas.

1 Introduction

The study of shock waves generated by spherical and cylindrical pistons in a gas has received much attention in the past four decades. The prime impetus has come from plasma fusion research. This work has been motivated by the fact that such shocks are an essential part of the mechanism responsible for sonoluminescence, that is, the light which under certain conditions is emitted from a bubble of gas trapped in a liquid and compressed by incident sound waves.

Jena and Sharma [5] have found out the infinitesimals of the group of transformations admitted by the system of basic equations governing the unsteady axi-symmetric motion in a dusty gas with state of equation of Mie-Gruneisen type [9]. It is interesting to note that these infinitesimals are the same for our system of basic equations governing the unsteady axi-symmetric flow in a Van der Waals gas. We also mention the work of Moodie, et al. [8] and He and Moodie [4] who have contributed much in the field of nonlinear hyperbolic waves. Moodie, et al. [8] uses asymptotic wavefront expansions in the study of nonlinear hyperbolic waves.

Donato and Oliveri [3] have shown that if any two of the infinitesimal operators of the invariance group commute, then the system of PDEs can be transformed into an autonomous one. We apply this technique to obtain the particular exact solution to our system. The evolution of

Keywords: Similarity solutions, wave propagation, Van der Waals gas.
weak discontinuities propagating into the above known solution is also discussed. The method given by Boillat (see [1] and [2]) is used for finding the amplitude of the weak discontinuity.

This paper deals with a problem in gas dynamics which in essence uses acceleration waves to come up with shock-initiation time. The critical time, when these weak discontinuities culminate into a shock (see [8], [4], [6] and [10]), is also calculated.

2 Basic equations and their infinitesimal operators  The basic equations describing the one-dimensional unsteady flow of a Van der Waals gas, can be written as [12]:

\[
\begin{align*}
\rho_t + \rho u_x + u \rho_x + m \rho \frac{u^2}{x} &= 0, \\
\rho (u_t + uu_x) + p_x &= 0, \\
p_t + up_x + a^2 \left( u_x + m \frac{u}{x} \right) &= 0,
\end{align*}
\]

where \( x \) is the spatial coordinate being either axial in flows with planar geometry or radial in cylindrically and spherically symmetric flows, \( t \) the time, \( \rho \) the density, \( p \) the pressure, \( b \) the Van der Waals excluded volume and \( m = 0, m = 1, m = 2 \) correspond, respectively, to planar, cylindrical and spherical motion; the entity \( a = \left\{ \frac{\gamma - 1}{\gamma} \right\} \frac{1}{(p/p)} \) is the equilibrium speed of sound with \( \gamma = c_p/c_v \) where \( c_p \) is the specific heat of the gas at constant pressure, and \( c_v \) the specific heat of the gas at constant volume. It may be noticed that the case \( b = 0 \) corresponds to the ideal gas. The non-numeric subscripts will denote the partial differentiation with respect to the indicated variable. The equation of state, for motion in a Van der Waals gas, is of the form: \( p(1 - bp) = \rho RT \), where \( R \) is the gas constant and \( T \) is the temperature.

Jena and Sharma [5] found that the system (1) admits the following infinitesimal operators:

**Case I.**  \( b \neq 0 \):

(i)  \( m \neq 0 \),

\[
Y_1 = 2x \partial_x + t \partial_t + u \partial_u + 2p \partial_p, \quad Y_2 = x \partial_x + u \partial_u + 2p \partial_p, \quad Y_3 = \partial_t;
\]

(ii)  \( m = 0 \),

\[
Y_1, Y_2, Y_3, Y_4 = \partial_x \text{ and } Y_5 = t \partial_x + \partial_u;
\]

**Case II.**  \( b = 0 \):

(i)  \( m \neq 0 \),

\[
Y_2, Y_3, Y_6 = x \partial_x + t \partial_t + \rho \partial_\rho + p \partial_p \text{ and } Y_7 = -x \partial_x + \rho \partial_\rho - u \partial_u - p \partial_p;
\]
(ii) $m = 0,$
$Y_2, Y_3, Y_4, Y_5, Y_6$ and $Y_7$.

3 Similarity analysis and autonomous system

Case I. $b \neq 0$ and $m \neq 0$.

(A) It can be verified that the infinitesimal operators $Y_2$ and $Y_3$ commute, that is, $[Y_2, Y_3] = Y_2Y_3 - Y_3Y_2 = 0$. Hence, using the method given by Donato and Oliveri [3], we introduce a set of canonical variables $\mu, \omega, V, F$ and $G$ such that

(2) $Y_3 \mu = 1, \quad Y_3 \omega = 0, \quad Y_3 V = 0, \quad Y_3 F = 0, \quad Y_3 G = 0.$

The characteristic equations corresponding to equations (2) are

$$\frac{dx}{0} = \frac{dt}{1} = \frac{dp}{0} = \frac{du}{0} = \frac{dp}{1},$$

which yield the following transformation of variables

(3) $\mu = t, \quad \omega = x, \quad u = V, \quad p = F, \quad \rho = G.$

Expressing $Y_2$ in terms of the new variables we obtain: $\bar{Y}_2 = \omega \partial_\omega + V \partial_V + 2F \partial_F$. Similarly, we choose a second set of canonical variables $\nu, U$ and $P$ such that

(4) $\bar{Y}_2 \nu = 1, \quad \bar{Y}_2 U = 0, \quad \bar{Y}_2 P = 0.$

The characteristic equations corresponding to equations (4) are

$$\frac{d\omega}{\omega} = \frac{dV}{V} = \frac{dF}{2F} = \frac{d\nu}{1},$$

which upon integration, yield

(5) $\nu = \ln \omega, \quad V = \omega U, \quad F = \omega^2 P.$

Combination of equations (3) and (5) yields the following transformation of variables:

(6) $\mu = t, \quad \nu = \ln x, \quad \rho = G, \quad u = xU, \quad p = x^2 P.$
Hence, using $\partial/\partial x = (1/x)\partial/\partial v$ and $\partial/\partial t = \partial/\partial \mu$, system (1) takes the following autonomous form:

\begin{align}
G_\mu + UG_\nu + GU_\nu &= -(m + 1)GU, \\
G(U_\mu + UU_\nu) + P_\nu &= -(GU^2 + 2P), \\
P_\mu + \frac{\gamma P}{1-bG}U_\nu + UP_\nu &= -2UP - (m + 1)\frac{\gamma UP}{1-bG}.
\end{align}

System (7) can be written in the matrix form:

\begin{align}
\partial_\mu \mathbf{U} + \mathbf{A}(\mathbf{U}) \partial_\nu \mathbf{U} = \mathbf{B}(\mathbf{U}),
\end{align}

where

\begin{align}
\mathbf{U} &= \begin{pmatrix} G \\ U \\ P \end{pmatrix}, \\
\mathbf{A}(\mathbf{U}) &= \begin{pmatrix} U & G & 0 \\ 0 & U & 1/G \\ 0 & \gamma P/(1-bG) & U \end{pmatrix}, \\
\mathbf{B}(\mathbf{U}) &= \begin{pmatrix} -(m + 1)GU \\ -U^2 - 2P/G \\ -2UP - (m + 1)\gamma UP/(1-bG) \end{pmatrix}.
\end{align}

System (8), (9) has a particular solution given by

\begin{align}
U = U_0, \quad G = G_0 \exp[-(m + 1)v], \quad P = \frac{U_0^2 G(1-bG)}{\gamma(m + 1)},
\end{align}

where $U_0$ and $G_0$ are constants. Hence, using (6) and (10), the particular exact solution of the system (1) is given by

\begin{align}
\rho = G_0 x^{-(m+1)}, \quad u = U_0 x, \quad p = x^2 P.
\end{align}

\textbf{(B)} The infinitesimal operators $Y_1$ and $Y_2$ commute, i.e., $[Y_1, Y_2] = Y_1 Y_2 - Y_2 Y_1 = 0$. Hence, we introduce a set of canonical variables $\mu, \omega, V, F$ and $G$ defined by:

\begin{align}
Y_1 \mu = 1, \quad Y_1 \omega = 0, \quad Y_1 V = 0, \quad Y_1 F = 0, \quad Y_1 G = 0.
\end{align}

The characteristic equations corresponding to eqs. (11) are:

\begin{align}
\frac{dx}{2x} = \frac{dt}{t} = \frac{du}{u} = \frac{dp}{2p} = d\mu.
\end{align}
which upon integration yield the following transformation of variables

\begin{align*}
\mu &= \ln t, \quad \omega = \frac{x}{t^2}, \quad u = tV, \quad p = t^2F, \quad \rho = G.
\end{align*}

Expressing \( Y_2 \) in terms of the new variables we obtain:

\begin{align*}
\tilde{Y}_2 \nu &= 1, \quad \tilde{Y}_2 U = 0, \quad \tilde{Y}_2 P = 0.
\end{align*}

The characteristic equations corresponding to equations (13) are

\begin{align*}
\frac{d\omega}{\omega} &= \frac{dV}{V} = \frac{dF}{2F} = d\nu,
\end{align*}

which upon integration yield

\begin{align*}
\nu &= \ln \omega, \quad V = \omega U, \quad F = \omega^2 P.
\end{align*}

Combination of equations (12) and (14), yields the following transformation of variables:

\begin{align*}
\mu &= \ln t, \quad \nu = \ln \frac{x}{t^2}, \quad \rho = G(\nu, \mu),
\end{align*}

\begin{align*}
u = \frac{x}{t} U (\nu, \mu), \quad p = \frac{x^2}{t^2} P (\nu, \mu).
\end{align*}

Using \( \partial_t = (1/t)\partial_\mu - (2/t)\partial_\nu \) and \( \partial_x = (1/x)\partial_\nu \), system (1) yields the following autonomous form:

\begin{align*}
G_{\mu} + (U - 2) G_\nu + G U_\nu + (m + 1) G U &= 0, \\
G U_{\mu} + G (U - 2) U_\nu + P_\nu + 2P + GU^2 - GU &= 0, \\
P_{\mu} + (U - 2) P_\nu + \frac{\gamma P}{1 - bG} U_\nu \\
+ 2UP - 2P + (m + 1) \frac{\gamma UP}{1 - bG} &= 0.
\end{align*}

System (16) can be written in the matrix form (8), where

\begin{align*}
U = \begin{pmatrix} G \\ U \\ P \end{pmatrix}, \quad A (U) = \begin{pmatrix}
U - 2 & G & 0 \\
0 & U - 2 & 1/G \\
0 & \gamma P/(1 - bG) & U - 2
\end{pmatrix},
\end{align*}

\begin{align*}
B (U) = \begin{pmatrix}
-(m + 1) G U - U^2 - 2P/G \\
2P - 2UP - (m + 1) \gamma UP/(1 - bG)
\end{pmatrix}.
\end{align*}
System (8), (17) has a particular solution given by

\[
U = U_0, \quad G = G_0 \exp(\alpha \nu), \quad P = \frac{\alpha G (1 - bG)}{(m + 1) \gamma U_0 + 2(1 - bG)},
\]

where \(G_0\) and \(U_0\) are constants, \(c = (m + 1)U_0/(2 - U_0), U_0 \neq 2, \quad \alpha = U_0(U_0 - 1)(U_0 - 2)\). Using (15) and (18), the following particular exact solution of system (1) is obtained:

\[
\rho = G_0 \left(\frac{x}{t^2}\right)^c, \quad u = U_0 \frac{x}{t}, \quad p = \frac{x^2}{t^2} P.
\]

**Case II.** \(b = 0\) and \(m \neq 0\),

(A) The infinitesimal operators \(Y_6\) and \(Y_7\) commute, that is, \([Y_6, Y_7] = 0\); hence, using the same method [3], we introduce a set of canonical variables \(\mu, \omega, V, F\) and \(G\) such that

\[
Y_6 \mu = 1, \quad Y_6 \omega = 0, \quad Y_6 V = 0, \quad Y_6 F = 0, \quad Y_6 G = 0.
\]

The characteristic equations corresponding to equations (20) are

\[
\frac{dx}{x} = \frac{dt}{t} = \frac{d\rho}{\rho} = \frac{du}{0} = \frac{dp}{\rho} = \frac{d\mu}{1},
\]

which yield the following transformation of variables

\[
\mu = \ln t, \quad \omega = x t, \quad \rho = tG, \quad u = V, \quad p = tF.
\]

Expressing \(Y_7\) in terms of the new variables we obtain: \(\tilde{Y}_7 = -\omega \partial_\omega + G \partial_G - V \partial_V - F \partial_F\). Similarly, we obtain a second set of canonical variables \(\nu, S, U\) and \(P\) such that

\[
\tilde{Y}_7 \nu = 1, \quad \tilde{Y}_7 S = \tilde{Y}_7 U = 0, \quad \tilde{Y}_7 P = 0.
\]

The characteristic equations corresponding to eqs. (22) are

\[
\frac{-d\nu}{\omega} = \frac{dG}{G} = -\frac{dV}{V} = \frac{-dF}{F} = \frac{d\nu}{1},
\]

which upon integration yields,

\[
\nu = -\ln \omega, \quad G = \frac{S}{\omega}, \quad V = \omega U, \quad F = \omega P.
\]
Combination of equations (21) and (23), yields the following transformation of variables:

\begin{equation}
\mu = \ln t, \quad \nu = \ln \left( \frac{t}{x} \right), \quad \rho = -\frac{t^2}{x} S, \quad u = \frac{x}{t} U, \quad p = x P.
\end{equation}

Using \( \frac{\partial}{\partial \nu} = -\frac{1}{x} \frac{\partial}{\partial \tau} \) and \( \frac{\partial}{\partial \tau} = \frac{1}{t} \left( \frac{\partial}{\partial \nu} + \frac{\partial}{\partial \tau} \right) \), system (1) takes the following autonomous form

\begin{align*}
S_{\mu} + (1 - U)S_{\nu} - SU_{\nu} &= -S(mU + 2), \\
U_{\mu} + (1 - U)U_{\nu} - \left( \frac{1}{S} \right) P_{\nu} &= U - U^2 - \frac{P}{S}, \\
P_{\mu} - \gamma P_{U\nu} + (1 - U)P_{\nu} &= -(\gamma(m + 1) + 1)UP.
\end{align*}

System (25) can be written in the matrix form (8), where

\begin{equation}
U = \begin{pmatrix} S \\ U \\ P \end{pmatrix}, \quad A(U) = \begin{pmatrix} 1 - U & -S & 0 \\ 0 & 1 - U & -1/S \\ 0 & -\gamma P & 1 - U \end{pmatrix},
\end{equation}

\begin{equation}
B(U) = \begin{pmatrix} -S(mU + 2) \\ U - U^2 - P/S \\ -(\gamma + \gamma + 1)UP \end{pmatrix}.
\end{equation}

System (8), (26) has a particular solution given by

\begin{equation}
U = U_0, \quad S = S_0 \exp(c\nu), \quad P = P_0 \exp(c\nu),
\end{equation}

where \( U_0, \ S_0, \ P_0 \) and \( c \) are constants, which are related by the following equations:

\begin{align*}
U_0 &= \frac{2}{(m + 1)\gamma - (m - 1)}, \quad P_0(U_0(m - 1) + 3) = U_0(U_0 - 1)^2S_0, \\
\gamma &= \frac{2U_0 + 2}{U_0 - 1},
\end{align*}

where \( U_0 \neq 1 \). Hence, using (24) and (27), the particular exact solution of the system (1) is given by

\begin{equation}
\rho = S_0 \frac{t^{\gamma+2}}{x^{\gamma+1}}, \quad u = U_0 \frac{x}{t}, \quad p = P_0 \frac{t^\gamma}{x^{\gamma-1}}.
\end{equation}
(B) The infinitesimal operators $Y_2$ and $Y_6$ commute, i.e., $[Y_2, Y_6] = Y_2 Y_6 - Y_6 Y_2 = 0$, hence using the same method [3], we introduce a set of canonical variables $\mu, \nu, V, F$ and $G$ such that

\begin{align*}
Y_6 \mu &= 1, & Y_6 \omega &= 0, & Y_6 V &= 0, & Y_6 F &= 0, & Y_6 G &= 0.
\end{align*}

The characteristic equations corresponding to eqs. (29) are

\begin{align*}
\frac{dx}{x} = \frac{dt}{t} = \frac{d\rho}{\rho} = \frac{du}{0} = \frac{dp}{p} = \frac{d\mu}{1},
\end{align*}

which yield the following transformation of variables

\begin{align*}
\mu &= \ln t, & \omega &= \frac{x}{t}, & \rho &= tG, & u &= V, & p &= tF.
\end{align*}

Expressing $Y_2$ in terms of the new variables we obtain: $\tilde{Y}_2 = \omega \partial_\omega + V \partial_V + 2F \partial_F$. Similarly, we choose a second set of canonical variables $\nu, U$ and $P$ such that

\begin{align*}
\tilde{Y}_2 \nu &= 1, & \tilde{Y}_2 U &= 0, & \tilde{Y}_2 P &= 0.
\end{align*}

The characteristic equations corresponding to equation (31) are

\begin{align*}
\frac{d\nu}{\omega} = \frac{dV}{V} = \frac{dF}{2F} = \frac{d\nu}{1},
\end{align*}

which upon integration, yield

\begin{align*}

\nu &= \ln \omega, & V &= \omega U, & F &= \omega^2 P.
\end{align*}

Combination of equations (30) and (32) yields the following transformation of variables:

\begin{align*}
\mu &= \ln t, & \nu &= \ln \left(\frac{x}{t}\right), & \rho &= tG, & u &= \left(\frac{x}{t}\right) U, & p &= \frac{x^2}{t} P.
\end{align*}

Hence using $\frac{\partial}{\partial x} = \frac{1}{x} \frac{\partial}{\partial \nu}$ and $\frac{\partial}{\partial t} = \frac{1}{t} (\frac{\partial}{\partial \mu} - \frac{\partial}{\partial \nu})$, system (1) takes the following form:

\begin{align*}
G_\mu + (U - 1)G_\nu + GU_\nu &= -G - (m + 1)GU, \\
G \left( U_\mu + (U - 1)U_\nu \right) + P_\nu &= -GU(U - 1) - 2P, \\
P_\mu + \gamma PU_\nu + (U - 1)P_\nu &= -P(2U - 1 + \gamma(m + 1)U).
\end{align*}
System (34), which is in the autonomous form, has the following particular solution:

\[ U = U_0, \quad G = G_0 \exp(c\nu), \quad P = P_0 \exp(c\nu), \]

where \( U_0, G_0, P_0 \) and \( c \) are the constants, where

\[ c = \frac{U_0(m + 1) + 1}{1 - U_0}, \quad P_0 = \frac{U_0(U_0 - 1)^2 G_0}{\gamma(m + 1)U_0 + 1}, \quad U_0 \neq 1. \]

Hence, using (33) and (35), the particular exact solution of the system (1) is given by

\[ \rho = G_0 \frac{x^c}{t^{c+1}}, \quad u = U_0 \frac{x}{t}, \quad p = P_0 \frac{x^{2+c}}{t^{1+c}}. \]

4 Evolution of weak discontinuities

Let \( \varphi(x, t) = 0 \) be the curve across which the first derivatives of the flow variables \( \rho, u, p \) suffer a jump, \( \Omega \) the speed of propagation of the weak discontinuity defined by \( \Omega = -\varphi_t/\varphi_x \), and \( \delta \) the jump of the first order derivative of a given flow variable defined by

\[ \left( \frac{\partial}{\partial \varphi} \right)_{\varphi=0^+} - \left( \frac{\partial}{\partial \varphi} \right)_{\varphi=0^-} = \delta. \]

The eigenvalues of the matrix \( \mathbf{A}(\mathbf{U}) \) in equation (9) are:

\[ \lambda_0 = U, \quad \lambda_+ = U + M, \quad \lambda_- = U - M, \]

where \( M^2 = \gamma P/[G(1-bG)] \), and the corresponding left and right eigenvectors can be written as:

\[ \mathbf{l}_0 = \begin{pmatrix} -M^2 & 0 & 1 \end{pmatrix}, \quad \mathbf{r}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \]

\[ \mathbf{l}_+ = \begin{pmatrix} 0 & MG & 1 \end{pmatrix}, \quad \mathbf{r}_+ = \begin{pmatrix} G \\ M \\ M^2G \end{pmatrix}; \]

\[ \mathbf{l}_- = \begin{pmatrix} 0 & -MG & 1 \end{pmatrix}, \quad \mathbf{r}_- = \begin{pmatrix} G \\ -M \\ M^2G \end{pmatrix}. \]
The eigenvalues of the matrix $A(U)$ in equation (17) are:

\[(38)\quad \lambda_0 = U - 2, \quad \lambda_+ = U - 2 + M, \quad \lambda_- = U - 2 - M,\]

where $M^2 = \gamma P/[G(1-bG)]$, and the corresponding left and right eigenvectors are:

\[
\mathbf{l}_0 = (-M^2 \quad 0 \quad 1), \quad \mathbf{r}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix};
\]

\[
\mathbf{l}_+ = (0 \quad MG \quad 1), \quad \mathbf{r}_+ = \begin{pmatrix} G \\ M \\ M^2G \end{pmatrix};
\]

\[
\mathbf{l}_- = (0 \quad -MG \quad 1), \quad \mathbf{r}_- = \begin{pmatrix} G \\ -M \\ M^2G \end{pmatrix}.
\]

The eigenvalues of the matrix $A(U)$ in equation (26) are:

\[(40)\quad \lambda_0 = 1 - U, \quad \lambda_+ = 1 - U + M, \quad \lambda_- = 1 - U - M,\]

where $M^2 = \gamma P/S$, with the following left and right eigenvectors:

\[
\mathbf{l}_0 = (-\gamma P \quad 0 \quad S), \quad \mathbf{r}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix};
\]

\[
\mathbf{l}_+ = (0 \quad -MS \quad 1), \quad \mathbf{r}_+ = \begin{pmatrix} S \\ -M \\ M^2S \end{pmatrix};
\]

\[
\mathbf{l}_- = (0 \quad MS \quad 1), \quad \mathbf{r}_- = \begin{pmatrix} S \\ M \\ M^2S \end{pmatrix}.
\]

It was checked that in the above cases (equations (36)–(41)), the characteristic fields corresponding to the eigenvalues $\lambda_+$ and $\lambda_-$ are not locally linearly degenerate (see [8] and [4]) about $U = U_0$ ($U_0$ being a known solution characterizing the state where the discontinuities propagate), i.e., $(\nabla_U \lambda_+ \cdot \mathbf{r}_+ \neq 0$ and $(\nabla_U \lambda_- \cdot \mathbf{r}_- \neq 0$, when evaluated at $U = U_0$; hence no local linear degeneracies are there.
However, the characteristic fields corresponding to the eigenvalue \( \lambda_0 \), in the above cases (equations (36)–(41)), are locally linearly degenerate about \( \mathbf{U} = \mathbf{U}_0 \), i.e., \( \nabla_{\mathbf{U}} \lambda_0 \cdot \mathbf{r}_0 \equiv 0 \) when evaluated at \( \mathbf{U} = \mathbf{U}_0 \); hence discontinuities in any derivative of the solution \( \mathbf{U} \) will not develop into shocks along the wavefront (see [8] and [4]).

Denoting by \( \mathbf{\tilde{\pi}} = (\delta G \delta \mathbf{U} \delta P) (\top \text{ is the transposition}) \) the discontinuity vector corresponding to the fast speed \( \lambda_+ \), we follow the procedure [1], to obtain:

\[
(42) \quad \mathbf{\tilde{\pi}} = \pi \mathbf{r}_{+0},
\]

where \( \pi \) is the amplitude of the discontinuity vector and \( \mathbf{r}_{+0} \) is the value of \( \mathbf{r}_+ \) evaluated at \( \mathbf{U} = \mathbf{U}_0 \).

The amplitude of the discontinuity wave satisfies a Bernoulli equation [2]

\[
(43) \quad \frac{d\pi}{d\sigma} + \Gamma (\sigma) \pi^2 + \Lambda (\sigma) \pi = 0,
\]

where

\[
(44) \quad \Gamma (\sigma) = [\nabla \mathbf{U} \lambda_+)_0 \cdot \mathbf{r}_{+0}] \varphi_\nu,
\]

\[
\Lambda (\sigma) = \frac{1}{\mathbf{1}_{+0} \cdot \mathbf{r}_{+0}} \left\{ \mathbf{1}_{+0} \cdot \mathbf{r}_{+0} + \left\{ \left( \frac{\partial P}{\partial \mathbf{U}^j} \right)_0 \right\}_{\mathbf{r}_{+0}} \mathbf{U}_{0\sigma}^j \right\} + (\mathbf{1}_{+0} \cdot \mathbf{U}_{0\nu}) \left[ (\nabla \mathbf{U} \lambda_+)_0 \cdot \mathbf{r}_{+0} - [\nabla \mathbf{U} (\mathbf{1}_+ \cdot \mathbf{B})]_0 \cdot \mathbf{r}_{+0} \right],
\]

where \( l^i, r^j \) and \( U^i \) are respectively the components of \( \mathbf{1}, \mathbf{r} \) and \( \mathbf{U} \); the summation indices \( i \) and \( j \) take values from 1 to 3; \( d/d\sigma \) denotes the derivative along the discontinuity curve such that \( d/d\sigma = \partial/\partial \mu + \lambda_0 (d/d\nu) \), and \( \nabla \mathbf{U} \) is the gradient operator with respect to the flow variables. If we consider the particular exact solution depending only on the variable \( \nu \), then all the coefficients in equation (43) depend on \( \nu \) and \( d/d\sigma \equiv \lambda_0 (d/d\nu) \). Therefore, the solution of equation (43) is given by

\[
(45) \quad \pi = \frac{\pi_0 \exp \left( -\int_0^\nu \Lambda (z) \, dz \right)}{1 + \pi_0 \int_0^\nu \Gamma (z) \exp \left( -\int_0^\zeta \Lambda (\xi) \, d\xi \right) \, d\zeta},
\]

where \( \pi_0 \) is constant.
For $\lambda_+ = U + M$ in system (8), (9), we calculated the values of $\Gamma(\nu)$ and $\Lambda(\nu)$ as:

\begin{align*}
\Gamma(\nu) &= \frac{(\gamma + 1) M \varphi_0}{2 (1 - bG)}, \\
\Lambda(\nu) &= \frac{U_0 + M}{2} \left( \frac{G'}{G} + \frac{3M'}{M} \right) + \frac{\gamma + 1}{4M (1 - bG)} \left( \frac{P'}{G} + U_0^2 \right) \\
&\quad + \frac{(m + 1) U_0 (\gamma + bG)}{2 (1 - bG)} + \frac{M}{2\gamma} + \frac{(m}{2} + 2) M + 2U_0,
\end{align*}

where the prime denotes the derivative with respect to $\nu$. The expression of $\pi$ may be obtained from equation (45) by integrating with respect to $\nu$ after substituting $d\sigma = (1/\lambda_+ \nu) d\nu$. A very thorough treatment of both the local and global properties of the amplitude function $\pi$, within a general framework, has been given in [7] and [11]. For an ideal gas, i.e., $b = 0$, $M^2 = M_0^2 = U_0^2/(m + 1)$, which does not depend on $\nu$. In this case, $\lambda_+ = U_0 + M_0 = \text{constant}$, so we obtain

\begin{align*}
\Gamma &= \Gamma_0 = \frac{(\gamma + 1) M_0 \varphi_0}{2}, \\
\Lambda &= \Lambda_0 = \frac{c}{2} (U_0 + M_0) + \frac{\gamma + 1}{4M_0} \left[ \frac{cU_0^2}{(m + 1)\gamma} + U_0^2 \right] \\
&\quad + \frac{m + 1}{2} \gamma U_0 + \frac{M_0}{2\gamma} + \left( \frac{m}{2} + 2 \right) M_0 + 2U_0,
\end{align*}

and the expressions for $\pi$, and the critical time $t_c$, when the wave amplitude $\pi$ becomes unbounded, and consequently the weak discontinuities culminate into shock waves, are calculated from equation (45) as:

\begin{align*}
\pi &= \frac{\pi_0 \left( \frac{1}{v_0} \right)^{-\Lambda_0} \left[ \left( \frac{1}{v_0} \right)^{-\Lambda_0} - 1 \right]}{1 - \pi_0 \Lambda_0} \left( \frac{\pi_0 \Gamma_0 + \Lambda_0}{\pi_0 \Gamma_0} \right)^{-\Lambda_0}, \\
t_c &= t_0 \left( \frac{\pi_0 \Gamma_0 + \Lambda_0}{\pi_0 \Gamma_0} \right)^{-\Lambda_0}.
\end{align*}

For $\lambda_+ = U - 2 + M$ in system (8), (17), we calculated the value of
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\[ \Gamma(\nu) = \left( \frac{\gamma + 1}{1 - bG} \right) M \varphi_{\nu}, \]
\[ \Lambda(\nu) = \frac{U_0 - 2 + M}{2} \left( \frac{G''}{G} + \frac{3M'}{M} \right) + \frac{\gamma + 1}{4M(1 - bG)} \times \left( \frac{P'}{G} - U_0 + U_0^2 \right) \]
\[ \quad + \frac{M}{2\gamma} + \left( \frac{m}{2} + 2 \right) M + 2U_0, \]

where the prime denotes the derivative with respect to \( \nu \). For an ideal gas, i.e., \( b = 0 \), \( M^2 = M_0^2 = \frac{\alpha\gamma}{(m + 1)\gamma U_0 + 2} \), which does not depend on \( \nu \). In this case, \( \lambda_{+0} = U_0 - 2 + M_0 = \) constant, so we obtain
\[ \Gamma \equiv \Gamma_0 = \frac{(\gamma + 1)M_0 \varphi_{\nu}}{2}, \]
\[ \Lambda \equiv \Lambda_0 = \frac{c}{2} (U_0 - 2 + M_0) \]
\[ \quad + \frac{\gamma + 1}{4M_0} \left( \frac{\alpha c}{(m + 1)\gamma U_0 + 2} - U_0 + U_0^2 \right) \]
\[ \quad + \frac{m + 1}{2} \gamma U_0 + \frac{M_0}{2\gamma} + \left( \frac{m}{2} + 2 \right) M_0 + 2U_0, \]

and the expressions for \( \pi \), and the critical time \( t_c \), when the weak discontinuities culminate into shock waves, are given in equations (47).

Figure 1 shows that the amplitude \( \pi \) increases as time \( t \) increases, and becomes unbounded at the critical time. If we increase the Van der Waals gas volume \( b \), the rate of growth in \( \pi \) increases. Also the critical time decreases as \( b \) increases.
FIGURE 1: The amplitude $\pi$ versus time $t$ for different values of $b$ corresponding to equations (43) and (48).

REFERENCES
