MOVING INTERFACIAL GRIFFITH CRACKS
BETWEEN BONDED DISSIMILAR ORTHOTROPIC HALF PLANES

S. DAS

ABSTRACT. The plane strain problem of determining the dynamic stress intensity factors and strain energy release rates for moving collinear Griffith cracks at the interface of two dissimilar orthotropic half planes is considered. The problem is formulated in terms of a system of singular integral equations of the second kind which have finally been solved by using Jacobi polynomials. Graphical plots of the stress intensity factors and strain energy release rates for the problem of a pair of collinear interfacial Griffith cracks in different particular cases are presented.

1 Introduction
In recent years there is an increasing interest in studying the effect of imperfections or flaws at the interface of bonded dissimilar materials due to their practical importance in designing Engineering structures and machines. It is extremely significant matter that one would regard the flaws or cracks as harmless or dangerous with seriously affecting the corresponding structure integrity. Further, from mathematical point of view, since the nature of stress singularities for a crack embedded in a homogeneous medium and for an interface crack is different, there is no smooth transition from one solution to the other as the crack distance from the interface goes to zero. In the above context this area, therefore, requires a great deal of attention.

Problems of interfacial Griffith crack of bonded isotropic elastic materials considered by Rice and Sih [15], Dhaliwal and Singh [5], Lowengrub and Sneddon [13], Erdogan and Gupta [8], Dhaliwal, et al. [3] and many others. Interfacial crack problems in orthotropic media had been studied by Dhaliwal, et al. [4], He, et al. [11], Gupta, et al. [10], Erdogan and Wu [9], Das and Patra [2] and in anisotropic media by Wang and Choi [17], Qu and Basani [14], etc.

In this paper, the determination of dynamic stress intensity factors and strain energy release rates around collinear moving Griffith cracks
of finite lengths located at the interface of two bonded orthotropic half planes is considered. The problem is reduced to solving a system of singular integral equations of second kind which are ultimately solved by the technique described by Erdogan [7] and by Erdogan and Wu [9]. Expressions of the stress intensity factors and strain energy release rates at the crack tips of the problem of a pair of collinear interfacial Griffith cracks are also obtained as a particular case of the problem. Graphical plots of the stress intensity factors and strain energy release rates for the problem of a pair of cracks are presented for a particular pair of orthotropic materials.

2 Formulation of the problem

Consider the plane elastodynamic problem in orthotropic half plane \((-\infty < X < \infty, 0 \leq Y < \infty)\) bonded to a dissimilar half plane \((-\infty < X < \infty, -\infty < Y \leq 0)\) with \(n\) collinear moving cracks of finite length located at the interface of two materials whose principal axes coincide with the Cartesian coordinate axes as shown in Figure 1. As in Yoffe model [20], it is assumed that the cracks are propagating with the constant speed \(c\) and without change in length along the positive \(X\)-axis. In what follows and in the sequel the quantities with superscripts \(i = 1, 2\) refer to those for the half planes 1 and 2, respectively.

Under the assumption of plane strain in an orthotropic medium, the
displacement equations of motion are

\[
\begin{align*}
\frac{c_{11}^{(i)} \partial^2 U^{(i)}}{\partial X^2} + c_{66}^{(i)} \frac{\partial^2 U^{(i)}}{\partial Y^2} + \left( c_{12}^{(i)} + c_{66}^{(i)} \right) \frac{\partial^2 V^{(i)}}{\partial X \partial Y} &= \rho^{(i)} \frac{\partial^2 U^{(i)}}{\partial t^2}, \\
c_{22}^{(i)} \frac{\partial^2 V^{(i)}}{\partial Y^2} + c_{66}^{(i)} \frac{\partial^2 V^{(i)}}{\partial X^2} + \left( c_{12}^{(i)} + c_{66}^{(i)} \right) \frac{\partial^2 U^{(i)}}{\partial X \partial Y} &= \rho^{(i)} \frac{\partial^2 V^{(i)}}{\partial t^2},
\end{align*}
\]

where \( i = 1, 2 \), and \( t \) is the time, \( \rho^{(i)} \), \( c_{jk}^{(i)} \)'s are respectively densities and elastic constants of the materials. Applying Gallelian transformation \( x = X - ct \), \( y = Yt = \), equations (2.1) reduce to

\[
\begin{align*}
\left( c_{11}^{(i)} - c_{22}^{(i)} \rho^{(i)} \right) \frac{\partial^2 u^{(i)}}{\partial x^2} + c_{66}^{(i)} \frac{\partial^2 u^{(i)}}{\partial y^2} + \left( c_{12}^{(i)} + c_{66}^{(i)} \right) \frac{\partial^2 v^{(i)}}{\partial x \partial y} &= 0, \\
\left( c_{66}^{(i)} - c_{22}^{(i)} \rho^{(i)} \right) \frac{\partial^2 v^{(i)}}{\partial x^2} + c_{22}^{(i)} \frac{\partial^2 v^{(i)}}{\partial y^2} + \left( c_{12}^{(i)} + c_{66}^{(i)} \right) \frac{\partial^2 u^{(i)}}{\partial x \partial y} &= 0,
\end{align*}
\]

where \( u^{(i)}(x, y) = U^{(i)}(X, Y, t) \) and \( v^{(i)}(x, y) = V^{(i)}(X, Y, t) \). The stress displacement relations are

\[
\begin{align*}
\sigma_{xx}^{(i)} &= c_{11}^{(i)} \frac{\partial u^{(i)}}{\partial x} + c_{12}^{(i)} \frac{\partial v^{(i)}}{\partial y}, \\
\sigma_{yy}^{(i)} &= c_{22}^{(i)} \frac{\partial v^{(i)}}{\partial y} + c_{12}^{(i)} \frac{\partial u^{(i)}}{\partial x}, \\
\sigma_{xy}^{(i)} &= c_{66}^{(i)} \left( \frac{\partial u^{(i)}}{\partial y} + \frac{\partial v^{(i)}}{\partial x} \right),
\end{align*}
\]

It is assumed that the crack are opened by internal tractions \( p_1(x), p_2(x) \) on the crack faces. The boundary conditions at the interface \( y = 0 \) are

\[
\begin{align*}
\sigma_{yy}^{(1)}(x, 0) &= \sigma_{yy}^{(2)}(x, 0) = -p_1(x), & x &\in L, \\
\sigma_{xy}^{(1)}(x, 0) &= \sigma_{xy}^{(2)}(x, 0) = -p_2(x), & x &\in L, \\
u^{(1)}(x, 0) &= u^{(2)}(x, 0), & x &\in L', \\
v^{(1)}(x, 0) &= v^{(2)}(x, 0), & x &\in L', \\
\sigma_{yy}^{(1)}(x, 0) &= \sigma_{yy}^{(2)}(x, 0), & -\infty &< x < \infty,
\end{align*}
\]
For the half planes 1 and 2 equations (3.2) have the solutions

\[ \sigma_{xy}^{(1)}(x,0) = \sigma_{xy}^{(2)}(x,0), \quad -\infty < x < \infty, \]

\[ L = \sum_{j=1}^{n} L_j, \quad L_j = a_j < x < b_j, \quad j = 2, 3, \ldots, n. \]

In addition, all components of stress and displacement vanish at the remote distances from the cracks. For a subsonic propagation it is assumed that the Mach numbers \( M_{ij}^{(i)} = c/v_j^{(i)} \) \((i, j = 1, 2)\), in which \( v_1^{(i)} = \sqrt{c_{11}^{(i)}/\rho^{(i)}} \) and \( v_2^{(i)} = \sqrt{c_{66}^{(i)}/\rho^{(i)}} \) \((i = 1, 2)\) are less than one (Willis [18]). Also \( a_j \) and \( b_j \) \((i = 1, 2, \ldots, n)\) are the endpoints of the cracks and \( L' \) is the complement of \( L \) in \((-\infty, \infty)\).

3 Solution of the problem  The appropriate integral solution of equations (2.2) can be taken as

\[ u^{(i)}(x,y) = \int_{0}^{\infty} A^{(i)}(s,y) \sin sx \, ds, \]

\[ v^{(i)}(x,y) = \int_{0}^{\infty} B^{(i)}(s,y) \cos sx \, ds \]

where \( A^{(i)} \) and \( B^{(i)} \) satisfy the equations

\[ (c_{11}^{(i)} - c^2 \rho^{(i)}) s^2 A^{(i)} - c_{66}^{(i)} \frac{d^2 A^{(i)}}{dy^2} + (c_{12}^{(i)} + c_{66}^{(i)}) s \frac{d^2 B^{(i)}}{dy^2} = 0, \]

\[ (c_{66}^{(i)} - c^2 \rho^{(i)}) s^2 B^{(i)} - c_{22}^{(i)} \frac{d^2 B^{(i)}}{dy^2} - (c_{12}^{(i)} + c_{66}^{(i)}) s \frac{d^2 A^{(i)}}{dy^2} = 0. \]

For the half planes 1 and 2 equations (3.2) have the solutions

\[ A^{(i)}(s,y) = A_1^{(i)}(s) e^{(-1)^i \gamma_1^{(i)} sy} + A_2^{(i)}(s) e^{(-1)^i \gamma_2^{(i)} sy}, \]

\[ B^{(i)}(s,y) = (-1)^{i+1} \left[ \frac{\alpha_1^{(i)}}{\gamma_1^{(i)}} e^{(-1)^i \gamma_1^{(i)} sy} A_1^{(i)}(s) \right. \]

\[ + \left. \frac{\alpha_2^{(i)}}{\gamma_2^{(i)}} e^{(-1)^i \gamma_2^{(i)} sy} A_2^{(i)}(s) \right], \quad i = 2, 3, \]

in which \( \gamma_1^{(i)} \) and \( \gamma_2^{(i)} \) \((< \gamma_1^{(i)}) \) are positive roots of the equation

\[ c_{66}^{(i)} c_{22}^{(i)} \gamma^4 + [(c_{12}^{(i)} + c_{66}^{(i)})^2 - c_{22}^{(i)} (c_{11}^{(i)} - c^2 \rho^{(i)}) \]

\[ - c_{66}^{(i)} (c_{66}^{(i)} - c^2 \rho^{(i)})] \gamma^2 + (c_{11}^{(i)} - c^2 \rho^{(i)})^2 c_{66}^{(i)} - c^2 \rho^{(i)} = 0 \]
The expressions for stresses for the half planes 1 and 2 are

\[ \sigma_{x y}^{(i)}(x, y) = (-1)^i c^{(i)}_{66} \int_{0}^{\infty} \left[ \frac{\beta^{(i)}}{\gamma^{(i)}_1} A_1^{(i)}(s) e^{-(s-1)^i \gamma^{(i)}_1 s y}, \right. \]
\[ \left. + \frac{\beta^{(i)}}{\gamma^{(i)}_2} A_2^{(i)}(s) e^{-(s-1)^i \gamma^{(i)}_2 s y} \right] s \sin sx \; ds, \]
\[ \sigma_{y y}^{(i)}(x, y) = \int_{0}^{\infty} \left[ (c^{(i)}_{12} - c^{(i)}_{22} \alpha^{(i)}_1 \gamma^{(i)}_1) A_1^{(i)}(s) e^{-(s-1)^i \gamma^{(i)}_1 s y} + (c^{(i)}_{12} - c^{(i)}_{22} \alpha^{(i)}_2 \gamma^{(i)}_2) A_2^{(i)}(s) e^{-(s-1)^i \gamma^{(i)}_2 s y} \right] s \cos sx \; ds \]

with \( \beta^{(i)}_j = \alpha^{(i)}_j + [\gamma^{(i)}_j]^2, \; i, j = 1, 2 \). Boundary conditions (2.8) and (2.9) yield

\[ \eta^{(i)}_1 A_1^{(i)}(s) + \eta^{(i)}_2 A_2^{(i)}(s) = \eta^{(2)}_2 A_2^{(2)}(s) + \eta^{(2)}_2 A_2^{(2)}(s), \]
\[ -\mu^{(i)}_1 A_1^{(i)}(s) - \mu^{(i)}_2 A_2^{(i)}(s) = \mu^{(1)}_1 A_1^{(1)}(s) + \mu^{(1)}_2 A_1^{(1)}(s) \]

where

\[ \eta^{(i)}_j = c^{(i)}_{12} - c^{(i)}_{22} \alpha^{(i)}_j, \quad \mu^{(i)}_j = c^{(i)}_{66} \frac{\beta^{(i)}_j}{\gamma^{(i)}_j}, \quad i, j = 1, 2. \]

The boundary conditions (2.4)–(2.7) in conjunction with equations (3.8) and (3.9) give rise to

\[ \int_{0}^{\infty} \left[ \eta^{(i)}_1 A_1^{(i)}(s) + \eta^{(i)}_2 A_2^{(i)}(s) \right] s \cos sx \; ds = -p_1(x), \quad x \in L, \]
\[ \int_{0}^{\infty} \left[ \mu^{(i)}_1 A_1^{(i)}(s) + \mu^{(i)}_2 A_2^{(i)}(s) \right] s \sin sx \; ds = -p_2(x), \quad x \in L, \]
\[ \int_{0}^{\infty} [L_1 A_1^{(1)}(s) + L_2 A_2^{(1)}(s)] \sin sx \; ds = 0, \quad x \in L', \]
\[ \int_{0}^{\infty} [M_1 A_1^{(1)}(s) + M_2 A_2^{(1)}(s)] \cos sx \; ds = 0, \quad x \in L'. \]
where

\[ L_j = 1 + \frac{\eta_j^{(1)} (\mu_1^{(2)} - \mu_2^{(2)}) + \mu_j^{(1)} (\eta_1^{(2)} - \eta_2^{(2)})}{\mu_2^{(2)} \eta_1^{(2)} - \mu_1^{(2)} \eta_2^{(2)}}, \]

\[ M_j = \frac{\alpha_j^{(1)}}{\gamma_j^{(1)}} + \frac{\eta_j^{(1)} (\alpha_1^{(2)} \mu_2^{(2)} - \alpha_2^{(2)} \mu_1^{(2)}) + \mu_j^{(1)} (\alpha_1^{(2)} \eta_2^{(2)} - \alpha_2^{(2)} \eta_1^{(2)})}{\mu_2^{(2)} \eta_1^{(2)} - \mu_1^{(2)} \eta_2^{(2)}}, \]

where \( j = 1, 2 \). Setting

\[ L_1 A_1^{(1)}(s) + L_2 A_2^{(1)}(s) = \frac{1}{s} \int_{a_j}^{b_j} f_1(t) \cos st \, dt, \]

\[ M_1 A_1^{(1)}(s) + M_2 A_2^{(1)}(s) = \frac{1}{s} \int_{a_j}^{b_j} f_2(t) \sin st \, dt, \]

the equations (3.12) and (3.13) under the conditions

\[ \int_{L_j} f_k(x) \, dx = 0, \quad (k = 1, 2; \ j = 1, 2, \ldots, n) \]

are identically satisfied. Equations (3.10) and (3.11), after a little algebra, become

\[ a_{11} f_1(x) + \frac{1}{\pi b_{11}} \int_{L} f_2(t) \, dt = -\frac{2}{\pi} p_1(x), \]

\[ c_{11} f_2(x) + \frac{1}{\pi d_{11}} \int_{L} f_1(t) \, dt = -\frac{2}{\pi} p_2(x), \quad x \in L \]

where

\[ a_{11} = \frac{\eta_1^{(1)} M_2 - \eta_2^{(1)} M_1}{L_1 M_2 - L_2 M_1}, \quad \frac{1}{b_{11}} = \frac{\eta_1^{(1)} L_2 - \eta_2^{(1)} L_1}{L_1 M_2 - L_2 M_1}, \]

\[ c_{11} = \frac{\mu_1^{(1)} L_2 - \mu_2^{(1)} L_1}{L_1 M_2 - L_2 M_1}, \quad \frac{1}{d_{11}} = \frac{\mu_1^{(1)} M_2 - \mu_2^{(1)} M_1}{L_1 M_2 - L_2 M_1}. \]

As \( a_{11}, b_{11}, c_{11} \) and \( d_{11} \) depend on the material constants and the velocity of propagation \( c \), the signs of these quantities may be of any combination. Varying \( c \) such that the Mach numbers remain less than unity,
if the signs of these quantities are all positive, then equations (3.17) can be expressed as

\begin{equation}
(3.18) \quad \phi_k(x) + \frac{1}{\pi i \varepsilon \gamma_k} \int_L \frac{\phi_k(t)}{t-x} \, dt = -g_k(x), \quad x \in L
\end{equation}

where

\[ \phi_k(x) = \sqrt{a_{11} b_{11}} f_1(x) + i r_k \sqrt{c_{11} d_{11}} f_2(x), \]
\[ g_k(x) = \frac{1}{\pi} \sqrt{\frac{b_{11}}{a_{11}}} p_1(x) + i r_k \sqrt{\frac{d_{11}}{c_{11}}} p_2(x), \quad k = 1, 2, \]
\[ r_1 = 1, \quad r_2 = -1, \quad \varepsilon = \sqrt{a_{11} b_{11} c_{11} d_{11}}, \]

where \( a_{11}, c_{11} \) and \( d_{11} \) are positive and \( b_{11} = -b_{22} < 0 \). Equations (3.17) can be put into the form

\begin{equation}
(3.19) \quad \phi_k(x) + \frac{1}{\pi i \varepsilon \gamma_k} \int_L \frac{\phi_k(t)}{t-x} \, dt = -g_k(x), \quad x \in L
\end{equation}

where

\[ \phi_k(x) = \sqrt{a_{11} b_{11}} f_1(x) + i r_k \sqrt{c_{11} d_{11}} f_2(x), \]
\[ g_k(x) = \frac{1}{\pi} \sqrt{\frac{b_{22}}{a_{11}}} p_1(x) + i r_k \sqrt{\frac{d_{11}}{c_{11}}} p_2(x), \quad k = 1, 2, \]
\[ r_1 = -1, \quad r_2 = 1, \quad \varepsilon = \sqrt{a_{11} b_{22} c_{11} d_{11}}. \]

For other combination of signs of \( a_{11}, b_{11}, c_{11}, d_{11} \), equation (3.17) can similarly be handled.

### 4 Solution of the integral equation

Introducing

\[ t = \frac{b_j - a_j}{2} \tau + \frac{b_j + a_j}{2}, \quad a_j < t < b_j, \quad -1 < \tau < 1, \]
\[ \ell = \frac{b_\ell - a_\ell}{2} r + \frac{b_\ell + a_\ell}{2}, \quad a_\ell < r < b_\ell, \quad -1 < r < 1 \]

in the integral equation (3.18) and defining

\[ \phi_k(x) = \Psi_{k\ell}(r), \quad \phi_k(t) = \Psi_{k\ell}(\tau), \quad g_k(x) = g_{k\ell}(r) \]
\[ h_{k\ell}(r, \tau) = \frac{b_j - a_j}{(b_j - a_j) \tau - (b_\ell - a_\ell) r + b_j + a_\ell - b_\ell - a_\ell} \]
\[ = \frac{1}{\tau - Z_{k\ell}}, \]
equation (3.18) may also be expressed as

\begin{align}
(4.2) \quad \Psi_{k\ell}(r) + \frac{1}{\pi i r_k} \int_{-1}^{1} \frac{\Psi_{k\ell}(\tau)}{\tau - r} \, d\tau \\
+ \frac{1}{\pi i r_k} \sum_{j=1}^{n} \int_{-1}^{1} h_{\ell j}(r, \tau) \Psi_{kj}(\tau) \, d\tau = -q_{k\ell}(r),
\end{align}

\[ \ell = 1, 2, \ldots, n, \quad k = 1, 2, \quad -1 < r < 1 \]

where \( \sum' \) indicates that the summation does not include the term corresponding to \( j = \ell \). The solution of the integral equation (4.2) may be assumed as

\begin{align}
(4.3) \quad \Psi_{k\ell}(r) = \omega_k(r) \sum_{n=0}^{\infty} C_{kn} P_n^{(\alpha_k, \beta_k)}(r)
\end{align}

where \( \omega_k(r) = (1 - r)^{\alpha_k} (1 + r)^{\beta_k} \)

with

\[ \alpha_k = \frac{1}{2} + i \omega_k, \quad \beta_k = \frac{1}{2} - i \omega_k, \quad \omega_k = r_k \omega, \quad k = 1, 2, \]

\[ \omega = \frac{1}{2\pi} \ln \left| \frac{1 + \varepsilon}{1 - \varepsilon} \right| \]

and \( C_{kn} \) are unknown constants.

By virtue of (3.16),

\[ \int_{L_j} \phi_k(x) \, dx = 0, \]

and hence

\[ \int_{-1}^{1} \Psi_{k\ell}(\tau) \, d\tau = 0 \quad \text{yield} \quad C_{k0} = 0, \quad \ell = 1, 2, \ldots, n. \]

Using the result of Karpenko [12],

\[ \frac{1}{\pi i} \int_{-1}^{1} w_k(\tau) P_n^{(\alpha_k, \beta_k)}(\tau) \frac{d\tau}{\tau - r} \]

\[ = -\varepsilon r_k w_k P_n^{(\alpha_k, \beta_k)}(r) + \frac{\sqrt{1 - \varepsilon^2}}{2i} p_{n-1}^{(-\alpha_k, -\beta_k)}(r), \quad |r| < 1 \]

\[ = (1 - \varepsilon r_k)[w_k(r) P_n^{(\alpha_k, \beta_k)}(r) - G_{kn}^{\infty}(r)], \quad |r| > 1 \]
where $G_{kn}^\infty(r)$ is the principal part of $w_k(r)P_{\alpha_k,\beta_k}(r)$ at infinity; the integral equation (4.2) with the aid of (4.3) gives rise to

$$\sqrt{1 - \varepsilon^2} \sum_{n=0}^{\infty} C_{kn} P_n^{(-\alpha_k,\beta_k)}(r) + \frac{1}{\varepsilon r_k} \sum_{j=1}^{n'} L_{\ell j n}(r) = -q_\ell(r),$$

$$\ell = 1, 2, \ldots, n, \quad -1 < r < 1$$

where

$$L_{\ell j n}(r) = \frac{1}{\pi i} \int_{-1}^{1} h_{\ell j}(r, \tau) w_j(\tau) \sum_{n=0}^{\infty} C_{jn} P_n^{(\alpha_j,\beta_j)}(\tau) d\tau$$

$$= \frac{1}{\pi i} \int_{-1}^{1} w_j(\tau) \sum_{n=0}^{\infty} C_{jn} P_n^{(\alpha_j,\beta_j)}(r) \frac{d\tau}{r - z_{\ell j}}$$

$$= i(1 + \varepsilon r_k)e^{\pi r_k} \left[ \sum_{n=0}^{\infty} C_{jn} P_n^{(\alpha_j,\beta_j)}(Z_{\ell j}) X(Z_{\ell j}) - G_j(Z_{k j}) \right]$$

where $G_j(Z)$ is the principal part of $g_j(Z)X(Z)$ at $|Z| = \infty$.

Multiplying both sides of equation (4.5) by $w_{\ell}^{-1}(r) P_{\alpha_k,\beta_k}^{-1}(r)$ and integrating with respect to $x$ from $-1$ to $1$ and using the orthogonality relation

$$\int_{-1}^{1} \omega(\tau) P_n^{(\alpha,\beta)}(\tau) P_j^{(\alpha,\beta)}(\tau) d\tau$$

$$= 0, \quad \text{when } n \neq j,$$

$$= \theta_j^{(\alpha,\beta)}$$

$$= \frac{2^{\alpha+\beta+1} \Gamma(j + \alpha + 1) \Gamma(j + \beta + 1)}{j! (2j + \alpha + \beta + 1) \Gamma(j + \alpha + \beta + 1)},$$

when $n = j$.

The following system of simultaneous algebraic equations for the determination of $C_{\ell m}$ is obtained:

$$\sqrt{1 - \varepsilon^2} C_{\ell m+1} g_n^{(-\alpha_k,-\beta_k)} + \frac{1}{\varepsilon r_k} \sum_{j=1}^{n'} L_{\ell j n m}(r) = -F_{\ell m}$$
where

\[ L_{\ell jnm} = \int_{-1}^{1} L_{\ell jn}(r) w_{\ell k}^{-1}(r) P_{m}^{(-\alpha_k, -\beta_k)}(r) \, dr, \]

(4.7)

\[ F_{\ell m} = \int_{-1}^{1} q_{\ell}(r) w_{\ell}^{-1}(r) P_{m}^{(-\alpha_k, -\beta_k)}(r) \, dr; \]

\( \ell = 1, 2, \ldots, n, \quad m = 0, 1, 2, \ldots \)

In general case of the problem the stress intensity factors and strain energy release rates at the crack tips are calculated as (Erdogan and Wu [9])

(4.8) \( K(b_\ell) = \sqrt{\frac{b_{11}}{a_{11}}} K_I^{b_\ell} + i r_k \sqrt{\frac{d_{11}}{c_{11}}} K_{II}^{b_\ell} = \lim_{x \to -b_\ell} \sqrt{2} (x - b_\ell)^{-\alpha_k} \)

\[ \times \left[ \sqrt{\frac{b_{11}}{a_{11}}} \sigma_{yy}^{(1)}(x, 0) + i r_k \sqrt{\frac{d_{11}}{c_{11}}} \sigma_{xy}^{(1)}(x, 0) \right] \]

\[ = -\frac{i \pi \sqrt{1 - \varepsilon^2}}{2 \varepsilon r_k} (b_\ell - a_\ell)^{-iw_k} \sqrt{\frac{b_\ell - a_\ell}{2}} \sum_{n=\ell}^{\infty} C_{\ell n} P_n^{(\alpha_k, \beta_k)}(1), \]

(4.9) \( K(a_\ell) = \sqrt{\frac{b_{11}}{a_{11}}} K_I^{a_\ell} + i r_k \sqrt{\frac{d_{11}}{c_{11}}} K_{II}^{a_\ell} = \lim_{x \to a_\ell} \sqrt{2} (a_\ell - x)^{-\beta_k} \)

\[ \times \left[ \sqrt{\frac{b_{11}}{a_{11}}} \sigma_{yy}^{(1)}(x, 0) + i r_k \sqrt{\frac{d_{11}}{c_{11}}} \sigma_{xy}^{(1)}(x, 0) \right] \]

\[ = \frac{i \pi \sqrt{1 - \varepsilon^2}}{2 \varepsilon r_k} (a_\ell - b_\ell)^{\omega_k} \sqrt{\frac{b_\ell - a_\ell}{2}} \sum_{n=1}^{\infty} C_{\ell n} P_n^{(\alpha_k, \beta_k)}(-1), \]

and

(4.10) \( G^{b_\ell} = \varepsilon \left[ \frac{b_{11}}{a_{11}} (K_I^{b_\ell})^2 + \frac{d_{11}}{c_{11}} (K_{II}^{b_\ell})^2 \right], \)

(4.11) \( G^{a_\ell} = \varepsilon \left[ \frac{b_{11}}{a_{11}} (K_I^{a_\ell})^2 + \frac{d_{11}}{c_{11}} (K_{II}^{a_\ell})^2 \right]. \)

When \( a_{11}, c_{11} \) and \( d_{11} \) are positive and \( b_{11} = -b_{22} < 0 \), with the modified
values of
\[ \alpha_k = -\frac{1}{2} - \frac{r_k}{\pi} \tan^{-1} \varepsilon, \quad \beta_k = -\frac{1}{2} + \frac{r_k}{\pi} \tan^{-1} \varepsilon, \]
\[ w_k = r_k w, \quad w = \frac{i}{\pi} \tan^{-1} \varepsilon, \]
\[ \varepsilon = \sqrt{a_{11} b_{22} c_{11} d_{11}}, \quad k = 1, 2, \]

the corresponding system of linear algebraic equation for the determination of \( C_{\ell m} \) is obtained by
\[ (4.12) \quad \frac{1}{2 \varepsilon r_k} C_{\ell m+1} e^{-\alpha_k, -\beta_k} + \frac{1}{\varepsilon r_k} \sum_{j=1}^{n} r_j \frac{L_{\ell jnm}}{r_j} (r) = -F_{\ell n} \]

where \( L_{\ell jnm} \) and \( F_{\ell n} \) are the same as in (4.7).

In this case the stress intensity factors and strain energy release rates at the crack tips are calculated as
\[ (4.13) \quad \sqrt{\frac{b_{22}}{a_{11}} K_I^{b_k} - r_k \sqrt{\frac{d_{11}}{c_{11}} K_H^{b_k}}} \]
\[ = \frac{\pi \sqrt{1 + \varepsilon^2}}{2 \varepsilon r_k} (b_k - a_k) \frac{w_k}{\varepsilon} \sqrt{\frac{b_k - a_k}{2}} \sum_{n=1}^{\infty} C_{\ell n} P_{\ell n}^{(\alpha_k, \beta_k)} (1), \]
\[ (4.14) \quad \sqrt{\frac{b_{22}}{a_{11}} K_I^{a_k} - r_k \sqrt{\frac{d_{11}}{c_{11}} K_H^{a_k}}} \]
\[ = \frac{\pi \sqrt{1 + \varepsilon^2}}{2 \varepsilon r_k} (b_k - a_k) \frac{w_k}{\varepsilon} \sqrt{\frac{b_k - a_k}{2}} \sum_{n=1}^{\infty} C_{\ell n} P_{\ell n}^{(\alpha_k, \beta_k)} (-1), \]

and
\[ (4.15) \quad G^{b_k} = \frac{\varepsilon}{2} \left[ \frac{b_{22}}{a_{11}} (K_I^{b_k})^2 - \frac{d_{11}}{c_{11}} (K_H^{b_k})^2 \right], \]
\[ (4.16) \quad G^{a_k} = \frac{\varepsilon}{2} \left[ \frac{b_{22}}{a_{11}} (K_I^{a_k})^2 - \frac{d_{11}}{c_{11}} (K_H^{a_k})^2 \right]. \]

5 A particular case As a particular case of the problem we consider the problem of a pair of equal collinear moving Griffith cracks of finite length \((b - a)\) at the interface of two dissimilar orthotropic half planes as shown in Figure 2.
In this case equations (4.6) and (4.12) become

\[\frac{\sqrt{1 - \varepsilon^2}}{2i\pi r_k} C_{km+1} \theta_m^{(-\alpha_k, -\beta_k)} = -F_{km}, \quad k = 1, 2; \quad m = 0, 1, 2, \ldots,\]  

\[\frac{\sqrt{1 + \varepsilon^2}}{2\pi r_k} C_{km+1} \theta_m^{(-\alpha_k, -\beta_k)} = -F_{km}, \quad k = 1, 2; \quad m = 0, 1, 2, \ldots.\]

The corresponding stress intensity factors and strain energy release rates at the crack tips \(x = b\) and \(x = a\) are

\[b_{11} \sqrt{a_{11}} K_I^b + i r_k \sqrt{d_{11}} C_{11} K_{II}^b = -i \frac{\pi \sqrt{1 - \varepsilon^2}}{2\pi r_k} (b-a)^{i \omega_k} \sqrt{\frac{b-a}{2}} \sum_{n=1}^{\infty} C_{kn} P_n^{(\alpha_k, \beta_k)}(1),\]

\[b_{11} \sqrt{a_{11}} K_I^a + i r_k \sqrt{d_{11}} C_{11} K_{II}^a = -i \frac{\pi \sqrt{1 - \varepsilon^2}}{2\pi r_k} (b-a)^{-i \omega_k} \sqrt{\frac{b-a}{2}} \sum_{n=1}^{\infty} C_{kn} P_n^{(\alpha_k, \beta_k)}(-1),\]

and

\[G^b = \varepsilon \left[ \frac{b_{11}}{\theta_{11}} (K_I^b)^2 + \frac{d_{11}}{c_{11}} (K_{II}^b)^2 \right],\]
When $a_{11}, b_{11}, c_{11}, d_{11} > 0$ and $b_{11} = -b_{22} < 0$; the stress intensity factors and strain energy release rates at the crack tips are

\[
G^b = \frac{\varepsilon}{2} \left[ \frac{b_{22}}{a_{11}} (K_I^b)^2 + \frac{d_{11}}{c_{11}} (K_{II}^b)^2 \right],
\]

\[
(5.6)
\]

When the cracks are far apart ($b - a = \text{constant}$ and $(b + a) \to \infty$),

\[
\frac{G^b}{G^*} \to 1 \quad \text{and} \quad \frac{G^a}{G^*} \to 1
\]

where $G^*$ is the strain energy release rate for a single Griffith crack at the interface of two bonded dissimilar orthotropic half planes (Das and Debnath [1]).

Again when $b - a = \text{constant}$ and $a \to 0$,

\[
\frac{G^b}{G^*} \to \frac{1}{2}, \quad \frac{G^a}{G^*} \to \infty.
\]

Hence results are in complete agreement with the results obtained by Erdogan and Wu [9].
Over the cracks the distance apart of two surfaces is given by

\[
(5.11) \quad v^{(1)}(x, 0) - v^{(2)}(x, 0) = \text{Im} \frac{\pi i \varepsilon (b - a)}{2\sqrt{1 - \varepsilon^2}} \left( \frac{b - x}{b - a} \right)^{\frac{1}{2} + i\omega} \left[ 1 - \frac{3 - 2i\omega}{9 + 4\omega^2} \cdot \frac{b - x}{b - a} \right].
\]

For this problem the distances of two surfaces should be greater than or at least equal to zero, but near the ends of the cracks the sign changes infinitely indicating that the upper and lower surfaces of the cracks should wrinkle and overlap each other, which is physically impossible.

Since we are interested to evaluate the size of the region in which the overlapping occurs, we consider the crack surfaces first come to rest when

\[
(5.12) \quad \cos \left( \omega \ln \frac{b - x}{b - a} \right) = 0.
\]

Thus, contact first takes place at a distance \( \delta \) from the ends of the cracks where

\[
\omega \ln \left( \frac{\delta}{b - a} \right) = \pm \frac{\pi}{2}
\]

and hence \( \delta \) has a maximum value of

\[
(5.13) \quad \delta = (b - a)e^{-\pi/2\omega}.
\]

### 6 Numerical results and discussion

Numerical calculations are carried out for the problem of a pair of cracks when the cracks are opened by constant normal pressure \( p \), i.e., \( p_1(x) = p \) and \( p_2(x) = 0 \).

For a subsonic propagation, the stress intensity factors at the crack tips and the strain energy release rates are calculated for various values of crack speed \( c \) with \( a = 0.5 \) and \( b = 1 \) for \( \alpha \)-Uranium and Beryllium composite (Das and Patra [2]). Graphical plots of \( K^b_I/p \), \( K^b_{II}/p \), \( K^a_I/p \), \( K^a_{II}/p \), and \( G^b \), \( G^a \) vs. \( c \) are presented through Figs. 3–14. It is seen from the figures that \( K^b_I/p \), \( K^b_{II}/p \) decrease but \( K^a_I/p \), \( K^a_{II}/p \), \( G^b \) and \( G^a \) increase up to crack speed \( c = 0.575 \) and then these have oscillatory nature when \( 0.575 < c \leq 0.624 \) which is expected as there is a change of propagation phase from subsonic to supersonic one. Similar oscillation phenomena have been observed by England [6], Williams [19], Sneddon and Lowengrub [16], etc.
Acknowledgment  The author is grateful to the referee for the valuable comments for the improvement of the paper.

REFERENCES

1. D. Das and L. Debnath, 
2. S. Das and B. Patra, 
3. R. S. Dhaliwal, W. He and H. S. Saxena, 
4. R. S. Dhaliwal, H. S. Saxena and J. G. Rokne, 
5. R. S. Dhaliwal and B. M. Singh, 
6. A. H. England, 
7. F. Erdogan, 
8. F. Erdogan and G. Gupta, 
9. F. Erdogan and B. Wu, 
10. V. Gupta, A. S. Argon and Z. Suo, 
11. W. He, R. S. Dhaliwal and H. S. Saxena, 
12. L. N. Karpenko, 
    *Approximate solution of singular integral equation by means of Jacobi polynomials*, PMM 30 (1966), 688.
13. M. Lowengrub and I. N. Sneddon, 
14. J. Qu and J. L. Basani, 
15. J. R. Rice and G. C. Sih, 
16. I. N. Sneddon and M. Lowengrub, 
17. S. S. Wang and I. Choi, 
18. J. R. Willis, 

DEPARTMENT OF MATHEMATICS, B. P. PODDAR INSTITUTE OF MANAGEMENT AND TECHNOLOGY
PODDAR VIHAR, 137, V.I.P. ROAD, CALCUTTA–700 052, WEST BENGAL, INDIA