ON AN OSCILLATION THEOREM OF WALTMAN

Dedicated to Paul Waltman on his 70th birthday

JAMES S. W. WONG

ABSTRACT. This is an expository article on oscillation theorems for second order nonlinear ordinary differential equations, more specifically the Emden-Fowler equations. This paper surveys a special area of second order nonlinear oscillation theory tracing its origin to the pioneering work of P. Waltman and subsequent improvements by G. J. Butler.

1 Introduction

We are here concerned with the study of oscillatory behaviour of solutions of second order nonlinear ordinary differential equation:

\[(1.1) \quad x'' + a(t)f(x) = 0,\]

where \(a(t)\) is continuous on \([t_0, \infty), \; t_0 > 0;\) \(f(x)\) is continuous on \(\mathbb{R} = (-\infty, \infty),\) \(f'(x)\) exists and is continuous on \(\mathbb{R} - \{0\}\) and \(f(x)\) also satisfies the nonlinear condition:

\[(F_0) \quad xf(x) > 0 \quad \text{and} \quad f'(x) > 0 \quad \text{where} \quad x \neq 0.\]

A typical example of (1.1) is the Emden-Fowler equation

\[(1.2) \quad x'' + a(t)|x|^\gamma \text{sgn} x = 0, \quad \gamma > 0,\]

where \(\text{sgn} x\) denotes the sign of the solution \(x(t)\) at a point \(t.\)

Our interest here is to study the oscillation and nonoscillation of solutions of (1.1). A solution \(x(t)\) is said to be oscillatory if it has arbitrarily large zeros, i.e., for any \(t_1 \geq t_0\) there exists a \(t_2 \geq t_1\) such that \(x(t_2) = 0.\)
A solution is said to be nonoscillatory if it is not oscillatory, i.e., it has only finitely many zeros. Equation (1.1) is said to be oscillatory if all of its solutions are oscillatory and it is called nonoscillatory if all its solutions are nonoscillatory.

When $\gamma = 1$, equation (1.2) becomes the linear equation of harmonic oscillators

\begin{equation}
(1.3) \quad x'' + a(t) x = 0,
\end{equation}

which is probably the most studied differential equation in literature. Sturm [55] showed in 1836 that equation (1.3) is either oscillatory or nonoscillatory. This follows from the Sturm Separation Theorem. We refer the reader to Leighton [32, Chapter 10, 175–198] for an elementary discussion on Sturm’s theory and the books of Hartman [18], Swanson [56], Coppel [11], and Reid [54], for more advanced discussion on the subject of second order linear oscillations. In the nonlinear case $\gamma \neq 1$, equation (1.2) can possess both oscillatory and nonoscillatory solutions simultaneously. We refer the reader to Bellman [3, Chapter 7, 143–164] for an introduction on Emden-Fowler equations and the books by Kiguradze and Chanturia [26] and Agarwal, Grace and O’Regan [1] for more advanced discussion on the subject of second order nonlinear oscillations. For the linear theory there are also survey papers by Rab [53] and Willett [59] and for the nonlinear equations (1.1) and (1.2) we refer the reader to Wong [64], [66], [67], Kiguradze [25] and Kartsatos [23].

This paper is confined to the discussion of oscillation theorems of nonlinear differential equations (1.1) and (1.2) when the coefficient $a(t)$ is allowed to take both positive and negative values for arbitrarily large values of $t \in [t_0, \infty)$. Existence of oscillatory and nonoscillatory solutions and also nonoscillation theorems are not discussed here. There is substantial research on the oscillatory theory of equations (1.1) and (1.2) when $a(t)$ is assumed in addition to be nonnegative, particularly in situations when both oscillatory and nonoscillatory solutions co-exist. These results are also not discussed here and we refer the interested readers to the references mentioned above.

2 Second order oscillation theory The subject of oscillation of solutions of the linear equation (1.3) has a long history. It was firmly established by the classical work of Sturm [55] in his celebrated Separation and Comparison Theorems. Using the Euler equation $x'' + kt^{-2}x = 0$ as comparison, Sturm proved that if $a(t) \geq kt^{-2}$ with $k > 1/4$ for all large
values of $t$, then equation (1.3) is oscillatory. Sturm’s oscillation criterion is a pointwise condition on the coefficient function $a(t)$ and it would seem desirable to provide an integral oscillation criterion involving the integral of $a(t)$. In 1918, Fite [15] proved that if $a(t) \geq 0$, $t \geq t_1 \geq t_0$ for some fixed $t_1$ and $\int_{t_0}^{\infty} a(t) \, dt = \infty$, then equation (1.2) is oscillatory. In 1949, Wintner [61] removed the superfluous assumption that $a(t)$ be nonnegative and proved the following oscillation criterion:

(A1) \[ \lim_{T \to -\infty} \int_{t_0}^{T} a(t) \, dt = +\infty. \]

In fact, Wintner proved a better result which is implied by (A1), namely,

(A2) \[ \lim_{T \to -\infty} \frac{1}{T} \int_{t_0}^{T} \int_{t_0}^{t} a(s) \, ds \, dt = +\infty. \]

The subject of second order nonlinear oscillation began with finding similar oscillation criteria for the more general nonlinear equations (1.1) and (1.2). In particular, one asks whether oscillation criteria (A1) and (A2) for the linear equation (1.3) remain valid for the Emden-Fowler equation (1.2). The term “second order nonlinear oscillations” was first coined by Atkinson [2] and again used by Wong [64] in a survey article.

In 1965, Waltman [58] proved that the Fite-Wintner criterion (A1) was also valid for the Emden-Fowler equation (1.2) when $\gamma = 2n - 1$ where $n$ is a positive integer. His proof was in fact valid for all $\gamma > 0$ as pointed out by Wong [63] and also by Bhatia [4] who proved that (A1) is an oscillation criterion for the more general equation (1.1) with $f(x)$ satisfying only condition [F0], (in fact, $f'(x) > 0$ for $x \neq 0$ can be relaxed to $f'(x) \geq 0$ for all $x$). It remained a challenge to prove that the weaker condition (A2) is also sufficient for the oscillation of equation (1.2).

In 1980 Butler proved in a substantive paper [7] that (A2) is indeed an oscillation criterion for equation (1.2) valid for all $\gamma > 0$. Butler’s results were proved for the more general equation (1.1) but the conditions imposed on $f(x)$ are somewhat unnatural and his proof was rather complicated. A simpler version of his proof was given in [69].

Returning to the linear equation (1.3), Hartman [17] in 1952 proved
an extension of Wintner’s oscillation criterion \((A_2)\) as follows:

\[(A_3) \quad -\infty < \lim_{T \to 1} \inf \frac{1}{T} \int_{t_0}^{T} \int_{t_0}^{t} a(s) \, ds \, dt \]

\[< \lim_{T \to 1} \sup \frac{1}{T} \int_{t_0}^{T} \int_{t_0}^{t} a(s) \, ds \, dt \leq +\infty.\]

Hartman’s result is significant in that it deals with the case when the integral \(\int_{t_0}^{T} a(t) \, dt\) is finite for all \(T\) and in particular, it applies to the case when \(a(t)\) is periodic with mean value zero. Thus \((A_3)\) shows that the equation \(x'' + (\sin t)x = 0\) is oscillatory, a result first proved by Yelchin [76] in 1946.

There is another important extension of Wintner’s condition \((A_2)\). In 1978 Kamenev [22] proved the following oscillation criterion for equation (1.3):

\[(A_4) \quad \text{For any } \alpha > 1, \quad \lim_{T \to 1} \sup \frac{1}{T^\alpha} \int_{t_0}^{T} (T - t)^\alpha a(t) \, dt = +\infty.\]

Note that condition \((A_2)\) can be restated as

\[\text{(2.1)} \quad \lim_{T \to 1} \frac{1}{T} \int_{t_0}^{T} (T - t)a(t) \, dt = +\infty,\]

which in turn implies that for any \(\alpha > 1\)

\[\text{(2.2)} \quad \lim_{T \to 1} \frac{1}{T^\alpha} \int_{t_0}^{T} (T - t)^\alpha a(t) \, dt = +\infty.\]

Clearly (2.2) implies \((A_4)\). To see (2.1) implies (2.2) for any \(\alpha > 1\) we consider the following identity upon integration by parts twice:

\[\int_{t_0}^{T} \left(1 - \frac{t}{T}\right)^\alpha a(t) \, dt = \frac{\alpha(\alpha - 1)}{T^2} \int_{t_0}^{T} \left(1 - \frac{t}{T}\right)^{\alpha - 2} tA(t) \, dt\]

where

\[A(t) = \frac{1}{\tau} \int_{t_0}^{t} \int_{t_0}^{\tau} a(\sigma) \, d\sigma \, d\tau.\]

For any \(M > 0\) we can by (2.1) choose \(t_1\) so that \(A(t) \geq M\) for \(t \geq t_1 \geq t_0\). Now using (2.2) and (2.3) we find

\[\int_{t_0}^{T} (T - t)^\alpha a(t) \, dt \geq \alpha(\alpha - 1)M \int_{t_1/T}^{1} (1 - u)^{\alpha - 2} u \, du.\]
Note that $\alpha(\alpha - 1) \int_0^1 (1 - u)^{\alpha - 2} u \, du = 1$. Since

$$\alpha(\alpha - 1) \int_0^1 (1 - u)^{\alpha - 2} u \, du = 1,$$

the right hand side of (2.4) can be made close to $M$ by choosing $T$ sufficiently large. On the other hand, since $M$ can be arbitrarily large, (2.4) proves that (A$_2$) implies (A$_1$).

Both (A$_3$) and (A$_4$) are weaker than (A$_2$). It is now a new challenge to prove that these two oscillation criteria remain valid for the Emden-Fowler equation (1.2). In the next section we shall describe that this is so by stating known results for the most general equation (1.1) subject to assumptions on $f(x)$ which are satisfied in the special Emden-Fowler case $f(x) = |x|^\gamma \text{sgn } x$, $\gamma > 0$.

3 In an earlier survey paper [66], we introduced the following generalized superlinear and sublinear assumptions for equation (1.1). The nonlinear function $f(x)$ is said to be superlinear if

$$0 < \int_y^\infty \frac{dx}{f(x)}, \quad \int_{-\infty}^{-y} \frac{dx}{f(x)} < \infty \quad \text{for all } y > 0,$$

and sublinear if

$$0 < \int_0^y \frac{dx}{f(x)}, \quad \int_{-y}^{0} \frac{dx}{f(x)} < \infty, \quad \text{for all } y > 0.$$

In addition, we introduced in [69, 70, 71] the additional assumptions which are called strictly superlinear and strictly sublinear following Naito [37, 38] as follows: $f(x)$ is strictly superlinear if there exists a positive constant $c(f)$ such that

$$f'(y) \int_y^\infty \frac{dx}{f(x)} \geq c(f) > 1 \quad \text{for all } y > 0,$$

and

$$f'(y) \int_{-\infty}^{-y} \frac{dx}{f(x)} \geq c(f) > 1 \quad \text{for all } y > 0.$$

Likewise, $f(x)$ is strictly sublinear if there exists a positive constant $d(f)$ such that

$$f'(y) \int_y^0 \frac{dx}{f(x)} \geq d(f) > 0 \quad \text{for all } y > 0.$$
and

\[ f'(y) \int_{-y}^{0} \frac{dx}{f(x)} \geq d(f) > 0 \quad \text{for all } y > 0. \]

Extension of Hartman’s oscillation criterion (A3) to equation (1.1) is given by:

**Theorem 1** (Wong [70]). Let \( f(x) \) satisfy either (3.1), (3.3) and (3.4), or (3.2), (3.5) and (3.6), i.e., \( f(x) \) is either strictly superlinear or strictly sublinear. Suppose that there exists \( \alpha \geq 1 \) such that \( a(t) \) satisfies

\[ -\infty < \lim_{T \to \infty} \inf \frac{1}{T^\alpha} \int_{t_0}^{T} (T - t)^\alpha a(t) \, dt \]

\[ < \lim_{T \to \infty} \sup \frac{1}{T^\alpha} \int_{t_0}^{T} (T - t)^\alpha a(T) \, dt \leq \infty. \]

Then equation (1.1) is oscillatory.

The proof of Theorem 1 was based upon the technique of integral averaging first used by Wintner [61] and further developed by Kamenev [21], Butler [7], Wong [68, 69, 70, 71], Philos [40, 41], [45]–[50], Philos and Purnaras [51, 52], Naito [37, 38] and others. We note that (3.7) and (A4) are not strictly comparable. In particular, (3.7) applies in situation where the integral average \( \frac{1}{T^\alpha} \int_{t_0}^{T} (T - t)^\alpha a(t) \, dt \) is finite for all \( T \geq t_0 \), for example, \( a(t) = \sin t \). On the other hand, condition (A4) allows the coefficient function \( a(t) \) to satisfy

\[ \lim_{T \to \infty} \inf \frac{1}{T^\alpha} \int_{t_0}^{T} (T - t)^\alpha a(t) \, dt = -\infty, \quad \alpha \geq 1, \]

a case which is excluded by (3.7). By improving upon the proofs given in Wong [70], Naito [38] succeeded in proving the following extension of Kamenev’s oscillation criterion:

**Theorem 2** (Naito [38]). Suppose that \( a(t) \) satisfies for some real number \( \alpha > 0 \)

\[ \lim_{T \to \infty} \sup \frac{1}{T^\alpha} \int_{t_0}^{T} (T - t)^\alpha a(t) \, dt = \infty. \]
(a) If $f(x)$ satisfies (3.1), (3.3) and (3.4) and $\alpha > c(f)(c(f) - 1)^{-1}$, then equation (1.1) is oscillatory;
(b) If $f(x)$ satisfies (3.2), (3.5) and (3.6) and $\alpha \geq 1$, then equation (1.1) is oscillatory.

When $f(x) = |x|^\gamma \text{sgn} \gamma$, then $c(f) = \gamma(\gamma - 1)^{-1}$ when $\gamma > 1$ and $d(f) = \gamma(1 - \gamma)^{-1}$ when $0 < \gamma < 1$. In the strictly superlinear case, the requirement in (3.8) becomes $\alpha > \gamma > 1$. This extends Kamenev’s criterion (A4) for linear case $\gamma = 1$ to equation (1.2) for all $\gamma \geq 1$. In the strictly sublinear case, (3.8) requires only $\alpha \geq 1$ which is weaker than (A4). Thus the task of proving that (A3) and (A4) remain valid for the oscillation of Emden-Fowler equation (1.2) for all $\gamma > 0$ is complete.

### 4 Integral averaging and general means

The technique of integral averaging was first used by Wintner [61] to prove an oscillation criterion (A2) for the linear equation (1.3). The effect of integral averaging is to cancel out certain negative parts of the coefficient function $a(t)$. Let us consider some simple examples. Suppose that $a(t) = 1 + 2 \sin t$ which takes on both positive and negative values for arbitrarily large values of $t$ and satisfies (A1), so the Emden-Fowler equation (1.2) is oscillatory. Now consider $a(t) = 1 + 2t \sin t$ which does not satisfy (A1) but satisfies (A2). Thus, integral averaging has the effect of smoothing out the wildly oscillatory functions. This idea had also been explored by Coles [9], Coles and Willett [10], Willett [60] for the linear equation (1.3) and further developed in Kwong and Zettl [31]. In the nonlinear case of equations (1.2) and (1.3), we refer to Kwong and Wong [28, 29].

Integral averaging of a function $a(t)$ can be considered as a special case of general means involving a non-negative kernel function $h(t,s)$, a concept first developed by Hartman [19] based on some results on averaging pairs by Macki and Wong [33]. Define the general mean of $b(t)$ by

\begin{equation}
G(b; t) = \frac{1}{h(t, t_0)} \int_{t_0}^{t} h(t, s)b(s)\, ds
\end{equation}

In case of (A2), $h(t,s) = (t-s)$ and for (A4), $h(t,s) = (t-s)^\alpha$, $\alpha > 1$. One can also consider more general kernel functions such as

\[ h(t,s) = \left( \int_s^t \frac{du}{g(u)} \right)^\alpha, \quad \alpha > 0, \]
where \( g(u) \) is a positive continuous function on \([t_0, \infty)\) such that

\[
\int_{t_0}^{\infty} \frac{du}{g(u)} = \infty.
\]

When \( g(u) = u \), then

\[
h(t, s) = \left( \log \frac{t}{s} \right)^\alpha.
\]

It is of interest to characterize the nature of the kernel function \( h(t, s) \) for the study of linear oscillation. In a recent paper [74], we have also extended Theorem 1 by using general means. In this section, we shall give a similar extension of Theorem 2 together with a simpler proof. Philos [47] had proved an extension of Kamenev’s criterion (A4) for the linear equation (1.3) by using general means.

Consider the class of non-negative kernel functions \( h(t, s) \) defined on \( D = \{(t, s) : t \geq s \geq t_0\} \) where \( h(t, s) \) is smooth in its two variables, say twice differentiable. We require the following assumptions on \( h(t, s) \):

\begin{align*}
(H_1) \quad & h(t, t) \equiv 0 \quad \text{for } t \geq t_0, \\
(H_2) \quad & \frac{\partial h}{\partial s}(t, s) \leq 0 \quad \text{for } t \geq s \geq t_0, \\
(H_3) \quad & \lim_{t \to \infty} \frac{h(t, s)}{h(t, t_0)} = 1 \quad \text{for all } t \geq s \geq t_0.
\end{align*}

Assumption (H3) ensures that the integral averaging process is compatible in the sense that if

\[
\lim_{T \to \infty} \int_{t_0}^{T} b(t) \, dt = k \leq \infty, \quad \text{then } \lim_{t \to \infty} G(b; t) = k \leq +\infty,
\]

i.e., the averaging preserves the limiting behaviour of the integral of the function. This is because \( h(t, s) \) is smooth so we can interchange the limit in (H3) with the integral in (4.1).

To give a generalization of Theorem 2, we need the following addi-
tional assumptions on \( h(t,s) \):

- (H_4) \[ \frac{\partial h}{\partial s}(t,s) \bigg|_{s=t} = 0 \text{ for } t \geq t_0, \]
- (H_5) \[ -h^{-1}(t,t_0) \frac{\partial h}{\partial s}(t,s) \bigg|_{s=t} = o(1) \]
  \text{ for all } t \geq t_0 \text{ as } t \to \infty, \]
- (H_6) \[ \left| \frac{\partial h}{\partial s}(t,s) \right|^2 \leq c_1 \frac{\partial^2 h}{\partial s^2}(t,s) h(t,s) \]
  \text{ for all } t \geq s \geq t_0, \text{ where } c_1 < c(f) \]

as defined by (3.3) and (3.4) in the strictly superlinear case. We are now ready to prove

**Theorem 3.** Let \( f(y) \) satisfy (3.1), (3.3) and (3.4) and \( h(t,s) \) satisfy (H_1)-(H_6). Suppose that \( a(t) \) satisfies

- \( \lim \sup \limits_{t \to \infty} \frac{1}{h(t,s)} \int_{t_0}^{t} h(t,s) a(s) ds = +\infty, \)

then equation (1.1) is oscillatory.

When \( h(t,s) = (t-s)^{\alpha}, \alpha > c(f)[c(f) - 1]^{-1}, \) Theorem 3 reduces to Theorem 2 of Naito [38]. Since

\[ c(f) = \frac{\gamma}{\gamma - 1} > c_1 = \frac{\alpha}{\alpha - 1}, \]

so \( \alpha > \gamma \) in (A_4) implies oscillation of equation (1.2) with \( \gamma > 1. \)

**Proof.** Define \( F(x) = \int_2^{\infty} dy/f(y) \). Let \( x(t) \) be a non-oscillatory solution of (1.1) and by (F_0), we can assume without loss of generality that \( x(t) > 0 \) for \( t \geq t_0 \). We shall show that (4.2) leads to a contradiction and hence it is an oscillation criterion for equation (1.1).

Denote \( w(t) = F(x(t)). \) Using (1.1), we note that \( w'(t) = u(t) \) satisfies the Riccati-like equation

- \[ u'(t) = a(t) + f'(x(t)) u^2(t). \]
Multiplying (4.3) by $h(t, s)$ and integrating by parts, we find by (H1)

$$h(t, s)u(s) = \int_{t_0}^t \frac{\partial h}{\partial s}(t, s)u(s) \, ds$$

Apply the Schwarz inequality to the first integral on the right hand side of (4.4) to obtain by (H6)

$$\left| \int_{t_0}^t \frac{\partial h}{\partial s}(t, s)u(s) \, ds \right|^2 \leq c_1 \left( \int_{t_0}^t h(t, s)f'(x(s))u^2(s) \, ds \right) \times \left( \int_{t_0}^t \frac{\partial^2 h}{\partial s^2}(t, s) \frac{ds}{f'(x(s))} \right).$$

Using (3.3), we can estimate the last integral in (4.5) as follows

$$\int_{t_0}^t \frac{\partial^2 h}{\partial s^2}(t, s) \frac{1}{f'(x(s))} \, ds \leq c(f)^{-1} \int_{t_0}^t \frac{\partial^2 h}{\partial s^2}(t, s)f(x(s)) \, ds.$$

Observe that

$$\int_{t_0}^t \frac{\partial^2 h}{\partial s^2}(t, s)f(x(s)) \, ds$$

$$= -\frac{\partial h}{\partial s}(t, s) \bigg|_{s=t_0} F(x(t_0)) - \int_{t_0}^t \frac{\partial h}{\partial s}(t, s)u(s) \, ds.$$
and

$$J(t) = \int_{t_0}^{t} h(t, s) f'(x(s)) u^2(s) \, ds.$$  

Denote

$$I(t) = \frac{\int_{t_0}^{t} \frac{\partial h}{\partial s}(t, s) u(s) \, ds}{J(t)}.$$  

By \(H_5\), for any \(\varepsilon > 0\) we can choose \(t_* > t_0\) so that \(N(t) \leq \varepsilon h(t, t_0)\).

We now rewrite (4.8) as

$$I(t) \leq \varepsilon h(t, t_0)(J(t))^{-1} + c_1 c(f)^{-1} \left( \frac{I(t)}{J(t)} \right).$$  

Since

$$\lim_{t \to \infty} \int_{t_0}^{t} f'(x(s)) u^2(s) \, ds = k \leq \infty$$  

where \(k\) is a positive number and can be infinite, by \(H_3\) we know that

$$\lim_{t \to \infty} \frac{1}{h(t, t_0)} \int_{t_0}^{t} h(t, s) f'(x(s)) u^2(s) \, ds = k \leq \infty.$$

We can therefore choose \(t_1 \geq t_*\) so that for all \(t \geq t_1\), \(h(t, t_0) J(t) \leq k_1\) where \(k_1 = \max(k^{-1}, 1)\). Now by assumption \(c_1 c(f)^{-1} < 1\), we can choose \(\varepsilon > 0\) sufficiently small such that

$$c_1 c(f)^{-1} + \frac{4k_1 \varepsilon}{c_1 c(f)^{-1}} < 1.$$  

Using this in (4.9) and writing \(K(t) = I(t)/J(t)\), we obtain from (4.9)

$$K(t)^2 \leq 2 \varepsilon k_1 + c_1 c(f)^{-1} K(t).$$  

We solve the quadratic inequality in \(K(t)\) above and obtain from (4.10) the following estimate

$$K(t) \leq c_1 c(f)^{-1} + 4\varepsilon k_1 c(f) c_1^{-1}.$$  

Substituting (4.11) into (4.4), we find by the choice of \(\varepsilon\) that

$$h(t, t_0) u(t_0) \geq J(t) \left( 1 - c_1 c(f)^{-1} + 4\varepsilon k_1 c(f) c_1^{-1} \right) + \int_{t_0}^{t} h(t, s) a(s) \, ds.$$
Note that the first term on the right of (4.12) is positive by the choice of \( \varepsilon \) above. Dividing (4.12) through by \( h(t, t_0) \) and taking limsup by (4.2), we obtained the desired contradiction from (4.12). This completes the proof of the theorem.

In the sublinear case, we can formulate a much stronger result (Cf. [74]; and [42, 43], [44, Theorem 1], and see also [69]).

**Theorem 4.** Let \( f(x) \) satisfy (3.2) and \( h(t, s) \) satisfy (H\(_1\)), (H\(_4\)), (H\(_5\)) and in addition

\[
(H_6)' \quad \frac{\partial^2 h}{\partial s^2}(t, s) \geq 0 \quad \text{for all } t \geq s \geq t_0.
\]

If \( a(t) \) satisfies (4.2), then equation (1.1) is oscillatory.

Note that Theorem 4 is valid for general sublinear equations not necessarily strictly sublinear. Also assumption \((H_6)'\) is considerably weaker than \((H_6)\), and \((H_2)\), \((H_3)\) are not required.

**Proof.** Let \( x(t) \) be a nonoscillatory solution of (1.1) which can be assumed to be positive for \( t \geq t_0 \). Define \( v(t) = x'(t)/f(x(t)) \). Then \( v(t) \) satisfies by (1.1) the Riccati-like equation

\[
(4.13) \quad v'(t) + f'(x(t))v^2(t) + a(t) = 0, \quad t \geq t_0.
\]

By \((F_0)\), \( f'(x(t)) \geq 0 \), so we obtain from (4.13) the simple differential inequality \( v'(t) + a(t) \leq 0 \). Multiply \( v' + a \leq 0 \) by \( h(t, s) \) to obtain

\[
(4.14) \quad \int_{t_0}^{t} h(t, s)v'(s)\,ds + \int_{t_0}^{t} h(t, s)a(s)\,ds \leq 0.
\]

If we now integrate the first integral in (4.14) by parts, we find by \((H_1)\), \((H_4)\) that

\[
(4.15) \quad \int_{t_0}^{t} h(t, s)v'(s)\,ds = -h(t, t_0)v(t_0) + \frac{\partial h}{\partial s}(t, s)\bigg|_{s=t_0}^{s=x(t_0)} \int_{0}^{x(t_0)} \frac{dy}{f(y)}
\]

\[
+ \int_{t_0}^{t} \frac{\partial^2 h}{\partial s^2}(t, s)\int_{0}^{x(s)} \frac{dy}{f(y)}\,ds.
\]

Dividing (4.14) through by \( h(t, t_0) \) and using \((H_5)\) and \((H_6)'\) and (3.2) in (4.15), we deduce the desired contradiction because of (4.2). This completes the proof.
Corollary. Let \( f(y) \) satisfy (3.2). If \( a(t) \) satisfies for some \( \alpha \geq 1 \),
\[
\lim_{t \to \infty} \sup \left\{ (\log t)^{-\alpha} \int_{t_0}^t \left( \log \frac{t}{s} \right)^{\alpha} a(s) \, ds \right\} = +\infty,
\]
then equation (1.1) is oscillatory.

Similar results in the sublinear case can be found in Philos [50], Wong and Yeh [75] when \( h(t, s) = (t - s)^{\alpha}, \alpha \geq 1 \).

5 In the previous sections, we showed how the program of extending the validity of the three important oscillation criteria \((A_2), (A_3)\) and \((A_4)\) for the linear equation (1.3) to that of the Euler-Fowler equation (1.2) was initiated by Waltman’s result [38] for the case of Fite-Wintner’s oscillation criterion \((A_1)\). In this final section, we discuss various other extensions and related work generated from the interest of Waltman’s result. We also pose some open problems to interested readers for further research.

(a) Waltman’s theorem was extended to the second order nonlinear equation of a form more general than that of equation (1.1). Wong [63] proved in 1966 that \((A_1)\) is an oscillation criterion for the following equation:
\[
(x'' + a(t)f(x)g(x') = 0,
\]
where \( f(x) \) satisfies \((F_0)\) and \( g(u) \) is a positive continuous function in \( u \in \mathbb{R} \). The proof was based upon modification of Waltman’s original arguments and substantively different from another extension of Waltman’s theorem by Bhatia [4] also in 1966 for equation (1.1). Bhatia’s proof is more elegant and it provided the impetus for many of the results subsequently developed by others and reported in this paper.

Waltman’s argument was proved useful by Coles and Willett [10] in another approach to the integral averaging method developed by Coles [9], Willett [60]. This formed the basis of Butler’s work on the nonlinear equations (1.1) and (1.2) as reported in [7].

It is however not known whether Butler’s extension of Wintner’s criterion \((A_2)\) can be extended to equations of the form (5.1). If so, what additional assumptions are required for \( f(x) \) and \( g(x')? \)

(b) Waltman’s theorem has also been extended to second order systems of the form
\[
\begin{cases}
x'(t) = a_1(t)f_1(y(t)) \\
y'(t) = -a_2(t)f_2(x(t))
\end{cases}
\]
by Kwong and Wong [30] in 1988. Oscillation of system (5.2) was first studied by Mirzov [34, 35, 36] under the additional assumption that both $a_1(t)$ and $a_2(t)$ are nonnegative. To extend Waltman’s result to system (5.2), one needs to allow $a_2(t)$ to take on positive and negative values for all large values of $t$. For further refinements of Kwong and Wong’s extension, we refer to Kordonis and Philos [27]. Oscillation of difference equations with alternating coefficients was studied by Thandapani, Gyori and Lalli [57], and difference systems by Graef and Thandapani [16]. Again, the subject of extending (A_2) remains at large for differential and difference systems.

(c) In 1976, Butler [5] studied the nonlinear analogue of Hill’s equation in a special form of equation (1.2), namely

$$x'' + (m + q(t))|x|^{\gamma} \text{sgn} x = 0, \quad \gamma > 0,$$

where $m$ is a real number and $q(t)$ is periodic with mean value zero. Clearly if $m > 0$, then (A_1) is satisfied, so by Waltman’s theorem equation (5.3) is oscillatory. Butler [5] showed that even if $m = 0$, equation (5.3) is also oscillatory. Butler’s proof is rather complicated, but this conclusion now follows readily from Theorem 1. For $m < 0$, Nazr [39] used techniques developed by El-Sayed [13] and Wong [72] for the linear equation and proved results on oscillation of (5.3) when $\gamma > 1$, but the results in this direction are rather scanty.

(d) Waltman’s theorem has been extended via the Olech-Opial-Wazewski oscillation criterion by Butler and Erbe [8] and Erbe and Kong [14]. Kiguradze [24] in 1967 proved an oscillation result generalizing both Waltman’s theorem and that of Atkinson [27] for the case when $a(t)$ is non-negative, (see also Kiguradze and Chanturia [26]). Waltman’s theorem had been studied by Johnson and Yan [20] Wong [65], Wong and Yeh [62, 75], and Elabbasy [12]. Also Butler [6] and Wong [73] studied extensions of Waltman’s theorem to equations with damping.

The above demonstrates the significance of Waltman’s contribution to the subject of second order nonlinear oscillation. We pay tribute to Paul Waltman for his vision and pioneering work on this topic.
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City University of Hong Kong, 83 Tat Chee Avenue, Kowloon Tong, Kowloon, Hong Kong, SAR