DIFFERENCE EQUATIONS FROM DISCRETIZATION OF A CONTINUOUS EPIDEMIC MODEL WITH IMMIGRATION OF INFECTIVES

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ABSTRACT. A continuous-time epidemic model with immigration of infectives is introduced. Systems of difference equations obtained from the continuous-time model by using nonstandard discretization technique are presented. Comparisons between the continuous-time model and its discrete counterpart are made.

1 Introduction Although the continuous-time logistic equation has only equilibrium dynamics, its discrete counterpart, the well known discrete logistic equation, exhibits period doubling bifurcation cascade to chaos [2, 5, 6]. This discrete logistic equation can be obtained via a simple forward Euler approximation with step size \( \Delta t = 1 \). Since the step size \( \Delta t = 1 \) is large, the discrepancy between the ordinary differential equation and its difference approximation is inevitable.

Another attempt is to consider the well-studied Lotka-Volterra ordinary differential equations of two populations. Ushiki [15] presented a forward Euler approximation with step size \( h \). It was demonstrated again that the discrete model possesses period-doubling bifurcation route to chaos. Several other researchers used piecewise constant arguments to obtain a discrete analogue of the Lotka-Volterra equation. Jiang and Rogers [7] studied the competitive case and Krawcewicz and Rogers [8] discussed the cooperative case. Both studies showed dynamical inconsistency between the continuous and discrete models.

Liu and Elaydi [9], Al-Kahby, Dennan and Elaydi [3] on the other hand used Mickens nonstandard discretization method [10] to derive a discrete version of the two dimensional continuous-time Lotka-Volterra model very successfully. They proved dynamical consistency between the continuous and discrete-time models in these two pioneering papers.
Since then researchers such as Roeger and Allen [11], Roeger [12, 13, 14] have applied the method to continuous-time three dimensional May-Leonard competitive systems and showed similar dynamics between the discrete and continuous-time models.

The purpose of this paper is to use Mickens nonstandard discretization technique to study the discrete analogue of a continuous SIS model investigated by Brauer and van den Driessche [4]. This work is motivated by the recent surge of research interest in the area. See Jiang and Rogers [7], Krawcewicz and Rogers [8], Al-Kabhy et al. [3], Liu and Elaydi [9], Roeger and Allen [11], Roeger [12, 13, 14], and references cited therein for discussion between continuous-time and discrete-time models by using nonstandard discretization technique.

It is shown here that the existence criteria of the steady states between the continuous-time and discrete-time models are the same. Moreover, both continuous and discrete time models also have the same equilibria. However, unlike the continuous SIS model for which global asymptotical stability of the steady state can be easily established by using well known theory of Dulac criterion and Poincaré-Bendixson Theorem, the local stability of the interior steady state for the discrete model is not trivial. For the steady state on the boundary, global asymptotic stability of the discrete model can be obtained by using simple comparison arguments. For the interior equilibrium, we are only able to prove its local stability. It is demonstrated that the discrete models derived from Mickens nonstandard discretization method possess similar dynamics as the continuous model.

2 Model derivation  Let $S(t)$ and $I(t)$ denote the number of susceptible and infective of a population at time $t$. It is assumed that there is a constant flow of $A$ new members into the population, of which a fraction $p$ ($0 \leq p \leq 1$) is infective. Let $d > 0$ be the per capita natural death rate of the population. The disease related death rate is denoted by $\alpha \geq 0$. In this model, a simple mass action $\beta SI$ is used to model disease transmission, where $\beta$ is a positive constant, and a fraction $\gamma \geq 0$ of these infectives recovers. We refer the reader to Brauer and van den Driessche [4] for more biological discussion about the continuous-time model presented below. Epidemic models with different transmission assumptions were also discussed in their work.

The continuous SIS model studied by Brauer and van den Driessche [4] can be written as the following two-dimensional ordinary differential
The dynamics of model (2.1) are well understood. All solutions of (2.1) converge to the steady state when the system has only a single equilibrium. When there are two equilibria, one is on the boundary and the other is in the interior, solutions with positive initial condition always asymptotically approach to the interior steady state [4].

We now describe our discretization procedure. Let \( \phi_i(h) = h + O(h^2) \), \( 0 < \phi_i(h) < 1 \) for \( i = 1, 2 \). We replace \( S \) by \( (S_{n+1} - S_n)/\phi_1(h) \), \( I \) by \( (I_{n+1} - I_n)/\phi_2(h) \), \( SI \) by \( S_{n+1}I_n \) in \( S \) and by \( S_nI_n \) in \( I \). Notice that there are several different ways to discretize \( SI \). We choose \( S_{n+1}I_n \) in \( S \) and \( S_nI_n \) in \( I \) so that solutions of the resulting difference equations will remain nonnegative as seen below. Substituting these into equation (2.1) yields

\[
\frac{S_{n+1} - S_n}{\phi_1(h)} = (1 - p)A - \beta S_{n+1}I_n - dS_{n+1} + \gamma I_n
\]

\[
\frac{I_{n+1} - I_n}{\phi_2(h)} = pA + \beta S_nI_n - (d + \gamma + \alpha)I_{n+1}.
\]

For simplicity, we write \( \phi_i \) instead of \( \phi_i(h) \) for \( i = 1, 2 \). Then the system of difference equations is obtained as follows.

\[
S_{n+1} = \frac{S_n + (1 - p)\phi_1A + \gamma \phi_1 I_n}{1 + \beta \phi_1 I_n + d \phi_1}
\]

(2.2)

\[
I_{n+1} = \frac{I_n + \phi_2 pA + \beta \phi_2 S_nI_n}{1 + (d + \gamma + \alpha)\phi_2}
\]

\( S_0, I_0 \geq 0 \).

When \( \beta = 0 \) there is no disease transmission between individuals in the population. In this case the only new infectives are coming from
immigration, and system (2.2) becomes

\[ S_{n+1} = \frac{S_n + (1 - p) \phi_1 A + \gamma \phi_1 I_n}{1 + d \phi_1} \]

(2.3)

\[ I_{n+1} = \frac{I_n + \phi_2 p A}{1 + (d + \gamma + \alpha) \phi_2} \]

System (2.3) is a special case of system (2.2) when \( \beta = 0 \). We remark that biological meanings of the original ordinary differential equations in general are lost after discretization. For discrete epidemic models we refer the readers to [1] and references cited therein.

System (2.3) always has a unique steady state \( E_0 = (S_0^*, I_0^*) \), where

\[ S_0^* = \frac{(1 - p) A}{d} + \frac{\gamma}{d} I_0^* \quad \text{and} \quad I_0^* = \frac{p A}{d + \gamma + \alpha}. \]

As in the continuous-time model, there is a relationship between steady states of systems (2.3) and (2.2). A brief discussion after introducing the notation for steady state of (2.2) is given before Proposition 2.5. Since the equation for \( I_n \) can be decoupled from \( S_n \), the global dynamics of (2.3) can be easily understood.

**Theorem 2.1.** If \( \beta = 0 \), then every solution of (2.2) converges to \( E_0 \).

**Proof.** Let

\[ f(I) = \frac{I + \phi_2 p A}{1 + (d + \gamma + \alpha) \phi_2}. \]

Then \( f'(I) > 0 \) for \( I \geq 0 \), \( 0 < f(0) < I_0^* \), and \( f(I) > I \) if and only if \( I < I_0^* \). Therefore if \( 0 \leq I_0 < I_0^* \) is given, then \( I_1 = f(I_0) > I_0 \) and \( I_2 = f(I_1) < f(I_0) = I_0^* \), i.e., \( I_0 < I_1 < I_2 < I_0^* \). Similarly, \( I_0 < I_1 < I_2 = f(I_1) < I_0^* \) and \( \{I_n\} \) is an increasing sequence of real numbers which is bounded above by \( I_0^* \). It follows that \( \lim_{n \to \infty} I_n \) exists, and is a fixed point of \( f \) by the continuity of \( f \). Thus \( \lim_{n \to \infty} I_n = I_0^* \) if \( 0 \leq I_0 \leq I_0^* \). A similar argument can be applied to the case when \( I_0 > I_0^* \). We conclude that \( \lim_{n \to \infty} I_n = I_0^* \).

The proof of \( \lim_{n \to \infty} S_n = S_0^* \) can be shown by a simple comparison argument as follows. For any \( \epsilon > 0 \), there exists \( n_0 > 0 \) such that \( I_0^* - \epsilon < I_n < I_0^* + \epsilon \) for \( n \geq n_0 \). Hence for \( n \geq n_0 \)

\[ S_{n+1} \leq \frac{S_n + (1 - p) \phi_1 A + \gamma \phi_1 (I_0^* + \epsilon)}{1 + d \phi_1}. \]
Consider

(2.4) \[ x_{n+1} = \frac{x_n + (1-p)\phi_1 A + \gamma \phi_1 (I_0^* + \epsilon)}{1 + d\phi_1} \]

with \(x_0 = S_{n_0}\). Equation (2.4) has a unique steady state

\[ x^* = \frac{(1-p)A}{d} + \frac{\gamma (I_0^* + \epsilon)}{d}. \]

By the same argument as we did for the equation \(I_n\), we can prove that \(\lim_{n \to \infty} x_n = x^*\). Hence

\[ \limsup_{n \to \infty} S_n \leq S_0^* \]

Letting \(\epsilon \to 0^+\), we have

\[ \limsup_{n \to \infty} S_n \leq S_0^*. \]

On the other hand since

\[ S_{n+1} \geq \frac{S_n + (1-p)\phi_1 A + \gamma \phi_1 (I_0^* - \epsilon)}{1 + d\phi_1} \quad \text{for } n \geq n_0, \]

it can be shown that \(\liminf_{n \to \infty} S_n \geq S_0^*\). Therefore \(\lim_{n \to \infty} S_n = S_0^*\) and the proof is complete.

Theorem 2.1 illustrates that the discrete model derived from Mickens discretization method \([10]\) has the same asymptotic dynamics as the original continuous-time model when \(\beta = 0\). Suppose now \(\beta > 0\) and \(p = 0\), i.e., all the immigrants are susceptible but there is disease transmission within the population. System (2.2) takes the following form.

(2.5) \[ I_{n+1} = \frac{I_n + \beta \phi_2 S_n I_n}{1 + (d + \gamma + \alpha)\phi_2} \]

\[ S_0, I_0 \geq 0. \]
System (2.5) may be rewritten as $(S_{n+1}, I_{n+1}) = H(S_n, I_n)$, where $S_{n+1} = F(S_n, I_n)$, $I_{n+1} = G(S_n, I_n)$ and $H = (F, G)$. A steady state $(S, I)$ of (2.5) must satisfy

$$(d + \gamma + \alpha)I = \beta I \frac{A + \gamma I}{\beta I + d}$$

One solution is $I = 0$ and another solution is

$$(2.6) \quad I = \frac{\beta A - d(d + \gamma + \alpha)}{(d + \alpha)\beta}.$$ 

As in the continuous model, we let

$$\sigma = \beta A - d(d + \gamma + \alpha).$$

Then $(A/d, 0)$ is the only feasible steady state of (2.5) if $\sigma < 0$, and in addition a nontrivial steady state $(S^*, I^*)$ exists if $\sigma > 0$, where $I^*$ is given by (2.5) and

$$S^* = \frac{A + \gamma I^*}{\beta I^* + d} = \frac{d + \gamma + \alpha}{\beta}.$$

The following lemma is trivial but it can be used to study system (2.5).

**Lemma 2.2.** Let $a = \max\{\gamma/\beta, A/d\}$. Then

$$\frac{\phi_1 A + \gamma \phi_1 x}{1 + \beta \phi_1 x + d\phi_1} \leq a \quad \text{for all} \ x \geq 0.$$

If $\sigma < 0$, then (2.5) has only boundary steady state $(A/d, 0)$, which can be shown to be globally asymptotically stable.

**Theorem 2.3.** If $\beta > 0$, $p = 0$ and $\sigma < 0$, then every solution of (2.2) converges to $(A/d, 0)$.

**Proof.** If $I_0 = 0$, then $I_n = 0$ for $n \geq 0$ and thus $\lim_{n \to \infty} S_n = A/d$. We may assume $I_0 > 0$, then $I_n > 0$ for $n \geq 0$. If there exists $k = 0, 1, \ldots$ such that

$$S_k \leq \frac{d + \gamma + \alpha}{\beta},$$
then
\[ S_{k+1} \leq \frac{d + \gamma + \alpha}{\beta} + \frac{\phi_1 A + \gamma \phi_1 I_k}{1 + \beta \phi_1 I_k + d \phi_1} < \frac{d + \gamma + \alpha}{\beta} \quad \text{as} \quad \sigma < 0. \]

Therefore \( S_n < (d + \gamma + \alpha)/\beta \) for \( n > k \). As a result,
\[ I_{n+1} = \frac{(1 + \beta \phi_2 S_n) I_n}{1 + (d + \gamma + \alpha) \phi_2} < I_n \]
for \( n > k \), which implies \( \lim_{n \to \infty} I = \bar{I} \geq 0 \) exits. Notice that if \( \bar{I} = 0 \), then for any \( \epsilon > 0 \) there exists \( n' > 0 \) such that \(-\epsilon < I_n < \epsilon \) for \( n \geq n' \). Thus
\[ S_{n+1} \geq \frac{S_n + \phi_1 A}{1 + \beta \phi_1 \epsilon + d \phi_1} \quad \text{for} \quad n \geq n', \]
and we have \( \liminf_{n \to \infty} S_n \geq A/d \). Similarly it can be proven that \( \limsup_{n \to \infty} S_n \leq A/d \). Hence \( \lim_{n \to \infty} S_n = A/d \) and the assertion is shown. Suppose now \( \bar{I} > 0 \). Then
\[ 1 = \lim_{n \to \infty} \frac{I_{n+1}}{I_n} = \lim_{n \to \infty} \frac{1 + \beta \phi_2 S_n}{1 + (d + \gamma + \alpha) \phi_2} \]
implies \( \lim_{n \to \infty} S_n = (d + \gamma + \alpha)/\beta \). Consequently, solutions of (2.5) converge to \(( (d + \gamma + \alpha)/\beta, \bar{I}) \), a fixed point of \( H \). Since \(( A/d, 0) \) is the only fixed point of \( H \), we obtain a contradiction and the result follows.

Suppose on the other hand that
\[ S_n > \frac{d + \gamma + \alpha}{\beta} \quad \text{for} \quad n = 0, 1, \ldots. \]
Then \( I_{n+1} > I_n \) for \( n = 0, 1, \ldots \) and thus \( \lim_{n \to \infty} I_n > 0 \) exists (maybe \( \infty \)). Notice that
\[ S_{n+1} \leq \frac{S_n}{1 + d \phi_1} + a, \quad \text{for} \quad n \geq 0 \]
by Lemma 2.2, and hence
\[ \limsup_{n \to \infty} S_n \leq \frac{a(1 + d \phi_1)}{d \phi_1}. \]
Consequently, if \( \lim_{n \to \infty} I_n = \infty \), then from the first equation of (2.5), we have
\[ \lim_{n \to \infty} S_{n+1} = \frac{\gamma}{\beta} < \frac{d + \gamma + \alpha}{\beta} \leq \liminf_{n \to \infty} S_n \]
and obtain a contradiction. Therefore \( \lim_{n \to \infty} I_n = \bar{I} \) is a positive real number. As a consequence,

\[
\lim_{n \to \infty} S_n = \frac{d + \gamma + \alpha}{\beta}.
\]

But since \( \sigma < 0 \),

\[
\frac{d + \gamma + \alpha}{\beta} = \lim_{n \to \infty} S_{n+1} = \frac{d + \gamma + \alpha}{\beta} + \phi_1 A + \gamma \phi_1 \bar{I}
\]

\[
\leq \frac{(d + \gamma + \alpha)(1 + d \phi_1 + \beta \phi_1 I)}{\beta(1 + \beta \phi_1 I + d \phi_1)} = \frac{d + \gamma + \alpha}{\beta}.
\]

We thus arrive at a contradiction. Therefore, there must exist \( k_0 \geq 0 \) such that \( S_{k_0} \leq (d + \gamma + \alpha)/\beta \) and hence \((A/d, 0)\) is globally asymptotically stable.

The result of Theorem 2.3 demonstrates that the system of difference equations and its corresponding system of ordinary differential equations have the same qualitative behavior when \( \beta > 0, p = 0 \) and \( \sigma < 0 \). If \( \beta > 0, p = 0 \) and \( \sigma > 0 \), then system (2.2) has two steady states \((A/d, 0)\) and \((S^*, I^*)\). Their stability are summarized below. It shows again that the discrete model and its original continuous-time model have the same number of steady states and the same local stability.

**Theorem 2.4.** If \( \beta > 0, p = 0 \) and \( \sigma > 0 \), then \((A/d, 0)\) is unstable and \((S^*, I^*)\) is locally asymptotically stable. Moreover,

\[
\frac{\gamma}{\beta} \leq \liminf_{n \to \infty} S_n \leq \limsup_{n \to \infty} S_n \leq \frac{A}{d}
\]

for any solution \((S_n, I_n)\) of (2.2) with \( S_0, I_0 > 0 \).

**Proof.** It is straightforward to verify that \((S^*, I^*)\) is locally asymptotically stable. Indeed, the linearization of system (2.5) about the steady state yields the following Jacobian matrix

\[
J = \begin{pmatrix}
1 & \phi_1(\gamma + d \phi_1 - \beta S^* - A \beta \phi_1) \\
1 + \beta \phi_1 I^* + d \phi_1 & \frac{\beta \phi_1 I^*}{(1 + \beta \phi_1 I^* + d \phi_1)^2}
\end{pmatrix}.
\]
Note that

\[(2.7) \quad \gamma + d\gamma \phi_1 - \beta S^* - A\beta \phi_1 < 0\]

implies

\[\det J = \frac{1}{1 + \beta \phi_1 I^* + d\phi_1} \frac{(\beta \phi_1 \phi_2 I^*)(\gamma + d\gamma \phi_1 - S^* \beta - A\beta \phi_1)}{[1 + (d + \gamma + \alpha)\phi_2][1 + \beta \phi_1 I^* + d\phi_1]^2} > 0,\]

and also

\[\text{tr} J = 1 + \frac{1}{1 + \beta \phi_1 I^* + d\phi_1} > 0.\]

Applying the Jury conditions [2], we have eigenvalues \(\lambda\) of \(J\) satisfying \(|\lambda| < 1\) if and only if \(\det J < 1\) and \(\text{tr} J < 1 + \det J\). Clearly \(\text{tr} J < 1 + \det J\) by (2.7), and \(\det J < 1\) if and only if

\[-\beta \phi_2 I^*(\gamma + d\gamma \phi_1 - \beta S^* - A\beta \phi_1) < \beta I^* + d.\]

Since \(A\beta = d^2 + d\gamma + d\alpha + (\alpha + d)\beta I^*\), a straightforward calculation shows that the latter inequality is true. Hence the steady state \((S^*, I^*)\) is locally asymptotically stable.

Furthermore, the linearization of system (2.5) about steady state \((A/d, 0)\) has the following Jacobian matrix

\[J_0 = \begin{pmatrix}
\frac{1}{1 + d\phi_1} & \frac{\phi_1(\gamma + \phi_1 d - \frac{A}{d} \beta - A\beta \phi_1)}{(1 + d\phi_1)^2} \\
0 & \frac{1 + \phi_2 \beta \frac{A}{d}}{1 + (d + \gamma + \alpha)\phi_2}
\end{pmatrix}.\]

Clearly the eigenvalues of \(J_0\) are

\[\lambda_1 = \frac{1}{1 + d\phi_1} \quad \text{and} \quad \lambda_2 = \frac{1 + \beta \phi_2}{1 + (d + \gamma + \alpha)\phi_2}.\]

Since \(\alpha > 0\), we have \(\lambda_2 > 1\) and \((A/d, 0)\) is unstable.

We proceed to prove the rest of the assertions. Observe that

\[\frac{\gamma}{\beta} < \frac{d + \gamma + \alpha}{\beta} < \frac{A}{d}.\]
If $S_n < \gamma/\beta$ for $n = 0, 1, \ldots$, then $I_{n+1} < I_n$ for $n \geq 0$ and thus $\lim_{n \to \infty} I_n = \bar{I} \geq 0$ exists. If $\bar{I} = 0$, then we have $\liminf_{n \to \infty} S_n \geq A/d$ and obtain a contradiction. If $\bar{I} > 0$, then by using a similar argument as in the proof of the previous theorem, we have $\lim_{n \to \infty} S_n = (d+\gamma+\alpha)/\beta$. But this again is impossible as $S_n < \gamma/\beta$ for $n \geq 0$. We therefore conclude that $S_n \geq \gamma/\beta$ for some $n^* \geq 0$. It is then straightforward to show that $S_{n^*+1} > \gamma/\beta$. As a consequence $S_n \geq \gamma/\beta$ for all $n \geq 0$ and $\lim_{n \to \infty} I_n = I > 0$ exists. By using similar arguments as in the proof of Theorem 2.3, we have $\lim_{n \to \infty} S_{n+1} = \gamma/\beta$ if $\bar{I} = \infty$ and $\lim_{n \to \infty} S_n = (d+\gamma+\alpha)/\beta$ if $\bar{I}$ is a real number. In any case we obtain a contradiction. Hence $S_{k'} \leq A/d$ for some $k' \geq 0$ and $S_{k'+1} \leq \frac{A + dA\phi_1 + \gamma d\phi_1 I_{k'}}{d(1 + \beta\phi_1 I_{k'} + d\phi_1)} = \frac{A}{d}$. Therefore $S_n \leq A/d$ for all $n \geq 0$ and $\limsup_{n \to \infty} S_n \leq A/d$ is shown.

Suppose now $\beta > 0$ and $p > 0$. Then steady state $(S, I)$ of (2.2) must satisfy

$$\beta(d+\alpha)I^2 - \sigma I - dpA = 0,$$

where $\sigma = \beta A - d(d+\gamma+\alpha)$. One root is negative and the other root is

$$\bar{I} = \frac{\sigma + \sqrt{\sigma^2 + 4\beta dpA(d+\alpha)}}{2\beta(d+\alpha)} > 0.$$

Consequently,

$$\bar{S} = \frac{A + \gamma \bar{I}}{\beta \bar{I} + d}$$

and $(\bar{S}, \bar{I})$ is the only feasible steady state of system (2.2). Similar to the continuous model, it can be shown that $\bar{I} > I_0^*$, where $I_0^*$ is the $I$-component of the steady state of (2.2) when $\beta = 0$. The linearization of (2.2) at $(\bar{S}, \bar{I})$ yields the following Jacobian matrix

$$J = \begin{pmatrix}
1 & \frac{\phi_1(\gamma + d\gamma\phi_1 - \beta \bar{S} - (1-p)A\beta\phi_1)}{(1 + \beta\phi_1 I + d\phi_1)^2} \\
\frac{1}{\beta\phi_2 I} & \frac{1 + \beta\phi_2 \bar{S}}{1 + (d+\gamma+\alpha)\phi_2}
\end{pmatrix}.$$
Unlike the continuous model for which global asymptotic stability of the positive steady state can be easily proved by using the Dulac criterion and the Poincaré-Bendixson Theorem, the local stability of the steady state for the discrete model is not trivial and requires a lot of computations. However, these computations are straightforward. We will use the Jury condition to show \((S, I)\) is locally asymptotically stable for (2.2) if \(\phi_1 < (d + \alpha)/(d\gamma - A\beta)\) when \(d\gamma > A\beta\) is true. When \(d\gamma \leq A\beta\), then the inequality imposed on \(\phi_1\) is unnecessary.

**Proposition 2.5.** If \(d\gamma \leq A\beta\), then \((S, I)\) is locally asymptotically stable for model (2.2). If \(d\gamma > A\beta\), then \((S, I)\) is locally asymptotically stable if \(\phi_1 < (d + \alpha)/(d\gamma - A\beta)\).

**Proof.** Since \(\text{tr} \ J > 0\), the Jury condition states that \((S, I)\) is locally asymptotically stable if and only if the Jacobian matrix \(J\) satisfying \(\text{tr} \ J < 1 + \det \ J < 2\). We first claim that \(\det \ J < 1\), where \(\det \ J\) is

\[
\frac{1 + \beta\phi_2\bar{S}}{(1 + \beta\phi_1 I + d\phi_1)[1 + (d + \gamma + \alpha)\phi_2]} - \frac{\beta\phi_1\phi_2\bar{I}[\gamma + d\gamma\phi_1 - \beta\bar{S} - (1 - p)A\beta\phi_1]}{[1 + (d + \gamma + \alpha)\phi_2][1 + \beta\phi_1 I + d\phi_1]^2}.
\]

Thus \(\det \ J < 1\) if and only if the following inequality is true

\[
2\beta \phi_2 \bar{S} + 2\beta^2 \phi_1 \phi_2 \bar{S} \bar{I}
+ \alpha \beta \phi_1 \phi_2 \bar{S} - \beta \gamma \phi_1 \phi_2 \bar{I} - \beta d \gamma \phi_1^2 \phi_2 \bar{I} + (1 - p) A \beta^2 \phi_1^2 \phi_2 \bar{I}
< (d + \gamma + \alpha)\phi_2 \beta \phi_1 \phi_2 \bar{I} + 2\beta(d + \gamma + \alpha)\phi_1 \phi_2 \bar{I} + \beta^2 \phi_1^2 \bar{I}^2
+ d \phi_1 + 2d(d + \gamma + \alpha)\phi_1 \phi_2
+ \beta^2(d + \gamma + \alpha)\phi_1^2 \phi_2 \bar{I}^2 + 2d\beta \phi_1^2 \bar{I}
+ 2d\beta(d + \gamma + \alpha)\phi_1^2 \phi_2 \bar{I} + d^2 \phi_1^2 + d^2(d + \gamma + \alpha)\phi_1^2 \phi_2.
\]

Note by (2.2) we have

\[
\beta \bar{S} \bar{I} = (d + \gamma + \alpha) \bar{I} - pA < (d + \gamma + \alpha) \bar{I}.
\]
By using (2.9) and after some simplifications, inequality (2.8) becomes

\[
2 \beta \phi_1 \phi_2 p A - \beta \gamma \phi_1 \phi_2 I - \beta d \gamma \phi_1^2 \phi_2 I + (1 - p) A \beta^2 \phi_1^2 \phi_2 I
\]

\[
< \beta \phi_1 I + \beta^2 \phi_1^2 I^2 + \beta^2(d + \gamma + \alpha) \phi_1^2 \phi_2 I^2
\]

\[
+ 2d \beta \phi_1^2 I + 2d \beta(d + \gamma + \alpha) \phi_1^2 \phi_2 I
\]

\[
+ d^2 \phi_1^2 + d^2(d + \gamma + \alpha) \phi_1^2 \phi_2.
\]

The only positive term which is left on the left hand side of (2.10) is

\[
(1 - p) A = \beta \bar{S} I + d \bar{S} - \gamma I,
\]

we can conclude from (2.10) that \( \det J < 1 \).

To show \( \text{tr} J < 1 + \det J \), it is equivalent to show the inequality

\[
\beta^2 \phi_1 \phi_2 \bar{S} I + \beta^3 \phi_1^2 \phi_2 \bar{S} I^2
\]

\[
+ d \beta \phi_1 \phi_2 \bar{S} + 2d \beta^2 \phi_1^2 \phi_2 \bar{S} I + \beta d^2 \phi_1^2 \phi_2 \bar{S}
\]

\[
< \beta(d + \gamma + \alpha) \phi_1 \phi_2 I + \beta^2(d + \gamma + \alpha) \phi_1^2 \phi_2 I^2
\]

\[
+ d(d + \gamma + \alpha) \phi_1 \phi_2 + 2d \beta(d + \gamma + \alpha) \phi_1^2 \phi_2 I
\]

\[
+ d^2(d + \gamma + \alpha) \phi_1^2 \phi_2 - \beta \gamma \phi_1 \phi_2 I
\]

\[
- d \beta \gamma \phi_1^2 \phi_2 - (1 - p) A \beta^2 \phi_1^2 \phi_2 I.
\]

Applying (2.9), (2.11) becomes

\[
\beta d \gamma \phi_1^2 \phi_2 I - 2d \beta p A \phi_1^2 \phi_2 < \beta(d + \alpha) \phi_1 \phi_2 I + A \beta^2 \phi_1^2 \phi_2 I.
\]

If \( d \gamma \leq A \beta \), then the inequality is trivially true. If \( d \gamma > A \beta \), then a sufficient condition for inequality (2.12) to be true is by requiring

\[
\phi_1 < (d + \alpha)/(d \gamma - A \beta).
\]

Numerical simulations of system (2.2) and (2.5) showed that solutions converge to the interior steady state. Moreover, global asymptotic stability of the interior equilibrium was also proved for the continuous-time model (2.1). Therefore, it is strongly suspected that the interior steady state is globally asymptotically stable for the discrete models. However, since the system is neither competitive nor cooperative, one may construct a Liapunov function to show the conjecture.
REFERENCES


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