AN EXISTENCE THEOREM FOR POSITIVE SOLUTIONS OF SECOND ORDER NEUTRAL DELAY DIFFERENCE EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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ABSTRACT. Positive solutions for a class of second order neutral difference equations with positive and negative coefficients are studied in this paper. A sufficient condition for existing positive solutions is obtained.

1 Introduction

Difference equations appear as natural descriptions of the observed evolution phenomena as well as in the study of equations and is rapidly broadening to various fields such as numerical analysis, control theory, finite mathematics, and computer science; in particular, the connection between the theory of difference equations and computer science has become more important in recent years, because of successful use of computers to solve difficult problems arising in practice.

Recently, some results [1–6] have been obtained in connection with the existence of nonoscillatory solutions for delay difference equations. However, most of the results are based on the constant positive or constant negative coefficient. In this paper, we employ some new techniques to study the existence of the positive solution for the neutral delay difference equations of second order with positive and negative coefficients

\[ \Delta^2(x_n + px_{n-\tau}) + q_n x_{n-\sigma} - r_n x_{n-\lambda} = 0, \quad n = 0, 1, \ldots, \]

where \( \tau \) is a positive integer; \( \sigma, \lambda \) are nonnegative integers; \( \{q_n\}, \{r_n\} \)

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are real positive sequences; $\Delta$ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$ and $\Delta^2 = \Delta(\Delta)$.

Let $\mu = \max\{\tau, \sigma, \lambda\}$. Then by a solution of (1), we mean a real sequence $\{x_n\}$, which is defined for $n \geq -\mu$, and satisfies equation (1) for $n = 0, 1, 2, \ldots$. A solution $\{x_n\}$ of (1) is said to be eventually positive if $x_n > 0$ for all large $n$.

2 Main Results

Theorem 1. Let $p \neq \pm 1$ and

\[
\sum_{n=n_0}^{\infty} nq_n < \infty, \quad \sum_{n=n_0}^{\infty} nr_n < \infty.
\]

If there exists a sufficiently large positive integer $n_1$ such that

\[
aq_n - r_n \geq 0 \text{ for every } n \geq n_1 \text{ and } a > 0,
\]

then equation (1) has a nonoscillatory solution.

Proof. Let $B_N$ denote the Banach space of all bounded real sequences $x = \{x_n\}_{n=\mu}^{\infty}$ with the sup norm $||x|| = \sup_{n \geq N-\mu} |x_n|$. Next we consider four cases.

Case 1. $0 \leq p < 1$. Choose a sufficiently large positive integer $N > n_0$ such that

\[
N \geq \max\{n_1, n_0 + \mu\},
\]

\[
\sum_{s=N}^{\infty} s(q_s + r_s) < 1 - p,
\]

\[
0 \leq \sum_{s=N}^{\infty} s(M_2q_s - M_1r_s) \leq p - 1 + M_2,
\]

\[
\sum_{s=N}^{\infty} s(M_1q_s - M_2r_s) \geq 0
\]

hold, where $M_1$ and $M_2$ are positive constants such that

\[
1 - M_2 < p \leq \frac{1 - M_1}{1 + M_2}
\]
holds.

Define the set
\[ \Omega = \{ x \in B^N : M_1 \leq x_n \leq M_2, \ n \geq N - \mu \} . \]

It is easy to see that \( \Omega \) is a bounded, closed, and convex subset of \( B^N \).

Define a mapping \( T : \Omega \to B^N \) as follows
\[
(Tx)_n = \begin{cases} 
1 - p - px_{n-\tau} + (n - 1) \sum_{s=n-1}^{\infty} (q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\
+ \sum_{s=N}^{n-2} s(q_s x_{s-\sigma} - r_s x_{s-\lambda}), & \text{for } n \geq N; \\
1 + p - px_{N-\tau}, & \text{for } N - \mu \leq n \leq N.
\end{cases}
\]

Clearly, \( T \) is a continuous mapping on \( \Omega \). For every \( x \in \Omega \) and \( n \geq N \), using (3) and (7) we get
\[
(Tx)_n = 1 - p - px_{n-\tau} + (n - 1) \sum_{s=n-1}^{\infty} (q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\
+ \sum_{s=N}^{n-2} s(q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\
\leq 1 - p + (n - 1) \sum_{s=n-1}^{\infty} (M_2 q_s - M_1 r_s) + \sum_{s=N}^{n-2} s(M_2 q_s - M_1 r_s) \\
\leq 1 - p + \sum_{s=N}^{\infty} s(M_2 q_s - M_1 r_s) + \sum_{s=N}^{n-2} s(M_2 q_s - M_1 r_s) \\
= 1 - p + \sum_{s=N}^{\infty} s(M_2 q_s - M_1 r_s) \leq M_2.
\]

Furthermore, in view of (3) and (7) we have
\[
(Tx)_n = 1 - p - px_{n-\tau} + (n - 1) \sum_{s=n-1}^{\infty} (q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\
+ \sum_{s=N}^{n-2} s(q_s x_{s-\sigma} - r_s x_{s-\lambda})
\]
\[
\begin{align*}
&\geq 1 - p - pM_2 + (n - 1) \sum_{s=n-1}^{\infty} (M_1 q_s - M_2 r_s) \\
&+ \sum_{s=N}^{n-2} s(M_1 q_s - M_2 r_s) \\
&\geq 1 - p - pM_2 \geq M_1.
\end{align*}
\]

Thus we proved that \( T \Omega \subset \Omega \). Next, we have to prove that \( T \) is a contraction mapping on \( \Omega \), in order to apply the contraction principle.

Now, for \( x^1, x^2 \in \Omega \) and \( n \geq N \) we have

\[
|T(x^1)_n - (Tx^2)_n| \leq p|x^1_{n-\tau} - x^2_{n-\tau}| + \\
+ (n - 1) \sum_{s=n-1}^{\infty} q_s|x^1_{s-\sigma} - x^2_{s-\sigma}| + (n - 1) \sum_{s=n-1}^{\infty} r_s|x^1_{s-\lambda} - x^2_{s-\lambda}| \\
+ \sum_{s=N}^{n-2} sq_s|x^1_{s-\sigma} - x^2_{s-\sigma}| + \sum_{s=N}^{n-2} sr_s|x^1_{s-\lambda} - x^2_{s-\lambda}| \\
\leq p\|x^1 - x^2\| + \|x^1 - x^2\| \left[ \sum_{s=n-1}^{\infty} s(q_s + r_s) + \sum_{s=N}^{n-2} s(q_s + r_s) \right] \\
= \|x^1 - x^2\| \left[ p + \sum_{s=N}^{\infty} s(q_s + r_s) \right] = \varepsilon_1 \|x^1 - x^2\|.
\]

This immediately implies that

\[
\|Tx^1 - Tx^2\| = \sup_{n \geq N} |(Tx^1)_n - (Tx^2)_n| = \sup_{n \geq N} \|(Tx^1)_n - (Tx^2)_n\|
\leq \varepsilon_1 \|x^1 - x^2\|,
\]

where in view of (5), \( \varepsilon_1 \leq 1 \), which proves that \( T \) is a contraction mapping. By the Banach contraction principle, consequently, \( T \) has the unique fixed point \( x \in \Omega \). It is easy to check that \( x = \{x_n\} \) is a positive solution of equation (1) for \( n \geq N \). This completes the proof of Case 1.

**Case 2.** \( p > 1 \). Choose a sufficiently large positive integer \( N > n_1 > n_0 \) such that

\[
N + \tau \geq n_0 + \max\{\sigma, \lambda\}
\]
\( \sum_{s=N}^{\infty} s(q_s + r_s) < p - 1, \)

\( 0 \leq \sum_{s=N}^{\infty} s(M_4q_s - M_3r_s) \leq 1 - p + pM_4, \)

and

\( \sum_{s=N}^{\infty} s(M_3q_s - M_4r_s) \geq 0, \)

where \( M_3 \) and \( M_4 \) are positive constants such that

\[ (1 - M_3)p \geq 1 + M_4 \quad \text{and} \quad p(1 - M_4) < 1. \]

Set \( \Omega = \{ x \in B_N : M_3 \leq x_n \leq M_4, \ n \geq N - \mu \} \). It is easy to see that \( \Omega \) is a bounded, closed, and convex subset of \( B_N \). Define a mapping \( T : \Omega \to B_N \) as follows:

\[
(Tx)_n = \begin{cases} 
1 - \frac{1}{p} x_{n+\tau} + \frac{n + \tau - 1}{p} \sum_{s=n+\tau-1}^{\infty} (q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\
\quad + \frac{1}{p} \sum_{s=N}^{n+\tau-2} s(q_s x_{s-\sigma} - r_s x_{s-\lambda}), & \text{for } n \geq N; \\
(Tx)_N, & \text{for } N - \mu \leq n \leq N.
\end{cases}
\]

Clearly, \( T \) is continuous. For every \( x \in \Omega \) and \( n \geq N \), using (3) and (11) we get

\[
(Tx)_n = 1 - \frac{1}{p} x_{n+\tau} + \frac{n + \tau - 1}{p} \sum_{s=n+\tau-1}^{\infty} (q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\
\quad + \frac{1}{p} \sum_{s=N}^{n+\tau-2} s(q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\
\leq 1 - \frac{1}{p} + \frac{n + \tau - 1}{p} \sum_{s=n+\tau-1}^{\infty} (M_4q_s - M_3r_s) \\
\quad + \frac{1}{p} \sum_{s=N}^{n+\tau-2} s(M_4q_s - M_3r_s)
\]
\begin{align*}
&\leq 1 - \frac{1}{p} + \frac{1}{p} \left[ \sum_{s=n+\tau-1}^{\infty} s(M_4q_s - M_3r_s) + \sum_{s=N}^{n+\tau-2} s(M_4q_s - M_3r_s) \right] \\
&= 1 - \frac{1}{p} + \frac{1}{p} \sum_{s=N}^{\infty} s(M_4q_s - M_3r_s) \leq M_4.
\end{align*}

Furthermore, in view of (11) we have
\begin{align*}
(Tx)_n &= 1 - \frac{1}{p} - \frac{1}{p} x_{n+\tau} + \frac{n + \tau - 1}{p} \sum_{s=n+\tau-1}^{\infty} (q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\
&\quad + \frac{1}{p} \sum_{s=N}^{n+\tau-2} s(q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\
&\geq 1 - \frac{1}{p} - \frac{M_4}{p} + \frac{n + \tau - 1}{p} \sum_{s=n+\tau-1}^{\infty} (M_3r_s - M_4r_s) \\
&\quad + \frac{1}{p} \sum_{s=N}^{n+\tau-2} s(M_3q_s - M_4r_s) \\
&\geq 1 - \frac{1}{p} - \frac{M_4}{p} \geq M_3.
\end{align*}

Thus we proved that $T\Omega \subset \Omega$.

Now for $x^1$, $x^2 \in \Omega$ and $n \geq N$ we have
\begin{align*}
|(Tx^1)_n - (Tx^2)_n| &\leq \frac{1}{p} |x_{n+\tau}^1 - x_{n+\tau}^2| \\
&\quad + \frac{n + \tau - 1}{p} \left[ \sum_{s=n+\tau-1}^{\infty} q_s |x_{s-\sigma}^1 - x_{s-\sigma}^2| + \sum_{s=n+\tau-1}^{\infty} r_s |x_{s-\lambda}^1 - x_{s-\lambda}^2| \right] \\
&\quad + \frac{1}{p} \left[ \sum_{s=N}^{n+\tau-2} s q_s |x_{s-\sigma}^1 - x_{s-\sigma}^2| + \sum_{s=n+\tau-1}^{\infty} s r_s |x_{s-\lambda}^1 - x_{s-\lambda}^2| \right] \\
&\leq \frac{1}{p} \|x^1 - x^2\| \left[ 1 + \sum_{s=N}^{\infty} s(q_s + r_s) \right] = \varepsilon_2 \|x^1 - x^2\|.
\end{align*}
This immediately implies that
\[ \|Tx^1 - Tx^2\| = \sup_{n \geq N-\mu} |(Tx^1)_n - (Tx^2)_n| = \sup_{n \geq N} |(Tx^1)_n - (Tx^2)_n| \leq \varepsilon_2 \|x^1 - x^2\|, \]
where in view of (9), \( \varepsilon_2 < 1 \), which proves that \( T \) is a contraction mapping. By the Banach contraction principle, consequently, \( T \) has the unique fixed point \( x^{\ast} \in \Omega \).

\( x^{\ast} = \{x_n\} \) is a positive solution of equation (1) for \( n \geq N \). This completes the proof of Case 2.

Case 3. \(-1 < p < 0\). Choose a sufficiently large positive integer \( N > n_1 > n_0 \) so that (4) and inequalities
\[
\sum_{s=N}^{\infty} s(q_s + r_s) < p + 1 \tag{12}
\]
\[
0 \leq \sum_{s=N}^{\infty} s(M_5q_s - M_5r_s) \leq (p + 1)(M_6 - 1) \tag{13}
\]
hold, where \( M_5 \) and \( M_6 \) satisfy \( 0 < M_5 \leq 1 < M_6 \).

Set \( \Omega = \{x \in B_N : M_5 \leq x_n \leq M_6, n \geq N - \mu\} \), which is obviously a bounded, closed, and convex subset of \( B_N \). Define a mapping \( T : \Omega \rightarrow B_N \) as follows
\[
(Tx)_n = \begin{cases} 
1 + p - px_{n-\tau} + (n - 1) \sum_{s=n-1}^{\infty} (q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\
+ \sum_{s=N}^{n-2} s(q_s x_{s-\sigma} - r_s x_{s-\lambda}), & \text{for } n \geq N; \\
(Tx)_N, & \text{for } N - \mu \leq n \leq N.
\end{cases}
\]
Clearly, \( T \) is continuous. For every \( x \in \Omega \) and \( n \geq N \), using (13) we get
\[
(Tx)_n = 1 + p - px_{n-\tau} + (n - 1) \sum_{s=n-1}^{\infty} (q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\
+ \sum_{s=N}^{n-2} s(q_s x_{s-\sigma} - r_s x_{s-\lambda})
\]
\[
\leq 1 + p - pM_6 + (n - 1) \sum_{s=n-1}^{\infty} (M_6q_s - M_5r_s) \\
+ \sum_{s=N}^{n-2} s(M_6q_s - M_5r_s) \\
\leq 1 + p - pM_6 + \sum_{s=N}^{\infty} s(M_6q_s - M_5r_s) \\
\leq 1 + p - pM_6 + (p+1)(M_6-1) = M_6.
\]

Furthermore, in view of (3) we have
\[
(Tx)_n = 1 + p - px_{n-\tau} + (n - 1) \sum_{s=n-1}^{\infty} (q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\
+ \sum_{s=N}^{n-2} s(q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\
\geq 1 + p - pM_5 + (n - 1) \sum_{s=n-1}^{\infty} (M_5q_s - M_6r_s) \\
+ \sum_{s=N}^{n-2} s(M_5q_s - M_6r_s) \\
\geq 1 + p - pM_5 \geq M_5.
\]

Thus, we proved that \( T\Omega \subset \Omega \).

Now for \( x^1, x^2 \in \Omega \) and \( n \geq N \) we have
\[
|(Tx^1)_n - (Tx^2)_n| \leq -p|x^1_n - x^2_n| + (n - 1) \sum_{s=n-1}^{\infty} q_s |x^1_{s-\sigma} - x^2_{s-\sigma}| \\
+ (n - 1) \sum_{s=n-1}^{\infty} r_s |x^1_{s-\lambda} - x^2_{s-\lambda}| + \sum_{s=N}^{n-2} sq_s |x^1_{s-\sigma} - x^2_{s-\sigma}| \\
+ \sum_{s=N}^{n-2} sr_s |x^1_{s-\lambda} - x^2_{s-\lambda}| \\
\leq -p\|x^1 - x^2\| + \|x^1 - x^2\| \cdot \sum_{s=N}^{\infty} s(q_s + r_s)
\]
This immediately implies that
\[ \| Tx^1 - T x^2 \| = \sup_{n \geq N - \mu} |(Tx^1)_n - (Tx^2)_n| = \sup_{n \geq N} |(Tx^1)_n - (Tx^2)_n| \]
\[ \leq \varepsilon_3 \| x^1 - x^2 \| , \]
where in view of (12), \( \varepsilon_3 < 1 \), which proves that \( T \) is a contraction mapping. By the Banach contraction principle, consequently, \( T \) has the unique fixed point \( x \in \Omega \).\( x = \{ x_n \} \) is a positive solution of equation (1) for \( n \geq N \). This completes the proof of Case 3.

Case 4. \( -\infty < p < -1 \). Choose a sufficiently large positive integer \( N > n_1 > n_0 \) such that (8) and inequalities
\[ \sum_{s = N}^{\infty} s(q_s + r_s) < -p - 1 \]
\[ 0 \leq \sum_{s = N}^{\infty} s(M_7 q_s - M_T r_s) \leq (p + 1)(M_T - 1) \]
hold, where \( M_T \) and \( M_7 \) satisfy \( 0 < M_T < 1 \leq M_7 \).

Set \( \Omega = \{ x \in B_N : M_7 \leq x_n \leq M_8, \ n \geq N - \mu \} \), which is obviously a bounded, closed, and convex subset of \( B_N \). Define a mapping \( T : \Omega \to B_N \) as follows:
\[
(Tx)_n = \begin{cases} 
1 + \frac{1}{p} x_{n+\tau} + \frac{n + \tau - 1}{p} \sum_{s=n+\tau-1}^{\infty} (q_s x_{s-\sigma} - r_s x_{s-\lambda}) + \frac{1}{p} \sum_{s=N}^{\infty} s(q_s x_{s-\sigma} - r_s x_{s-\lambda}), & \text{for } n \geq N; \\
(Tx)_N, & \text{for } N - \mu \leq n \leq N. 
\end{cases}
\]
Clearly, \( T \) is continuous. For every \( x \in \Omega \) and \( n \geq N \), using (3) we get
\[
(Tx)_n = 1 + \frac{1}{p} x_{n+\tau} + \frac{n + \tau - 1}{p} \sum_{s=n+\tau-1}^{\infty} (q_s x_{s-\sigma} - r_s x_{s-\lambda}) + \frac{1}{p} \sum_{s=N}^{\infty} s(q_s x_{s-\sigma} - r_s x_{s-\lambda})
\]
\[
\leq 1 + \frac{1}{p} - \frac{M_8}{p} + \frac{n + \tau - 1}{p} \sum_{s=n+\tau-1}^{\infty} (M_7 x_{s-\sigma} - M_8 x_{s-\lambda}) \\
+ \frac{1}{p} \sum_{s=N}^{n+\tau-2} s(M_7 q_s - M_8 r_s) \\
\leq 1 + \frac{1}{p} - \frac{M_8}{p} \leq M_8.
\]

Furthermore, in view of (15) we have
\[
(Tx)_n = 1 + \frac{1}{p} - \frac{1}{p} x_{n+\tau} + \frac{n + \tau - 1}{p} \sum_{s=n+\tau-1}^{\infty} (q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\
+ \frac{1}{p} \sum_{s=N}^{n+\tau-2} s(q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\
\geq 1 + \frac{1}{p} - \frac{M_7}{p} + \frac{n + \tau - 1}{p} \sum_{s=n+\tau-1}^{\infty} (M_7 q_s - M_7 r_s) \\
+ \frac{1}{p} \sum_{s=N}^{n+\tau-2} s(M_7 q_s - M_7 r_s) \\
\geq 1 + \frac{1}{p} - \frac{M_7}{p} + \frac{1}{p} \sum_{s=N}^{\infty} s(M_7 q_s - M_7 r_s) \\
\geq 1 + \frac{1}{p} - \frac{M_7}{p} + \frac{1}{p} (p+1)(M_7 - 1) = M_7.
\]

Thus, we proved that \(T\Omega \subset \Omega\).

Now for \(x^1, x^2 \in \Omega\) and \(n \geq N\) we have
\[
|(Tx^1)_n - (Tx^2)_n| \leq \frac{1}{p} |x^1_{n+\tau} - x^2_{n+\tau}| \\
- \frac{n + \tau - 1}{p} \left[ \sum_{s=n+\tau-1}^{\infty} q_s |x^1_{s-\sigma} - x^2_{s-\sigma}| + \sum_{s=n+\tau-1}^{\infty} r_s |x^1_{s-\lambda} - x^2_{s-\lambda}| \right] \\
- \frac{1}{p} \sum_{s=N}^{n+\tau-2} s q_s |x^1_{s-\sigma} - x^2_{s-\sigma}| + \sum_{s=N}^{n+\tau-2} s r_s |x^1_{s-\lambda} - x^2_{s-\lambda}| \right].
\]
\[ \geq -\frac{1}{p} \|x^1 - x^2\| - \frac{1}{p} \|x^1 - x^2\| \cdot \left[ \sum_{s=n+\tau-1}^{\infty} s(q_s + r_s) + \sum_{s=N}^{n+\tau-2} s(q_s + r_s) \right] \]
\[
= -\frac{1}{p} \left[ 1 + \sum_{s=N}^{\infty} s(q_s + r_s) \right] = \varepsilon_4 \|x^1 - x^2\|.
\]

This immediately implies that
\[
\|T x^1 - T x^2\| = \sup_{n \geq N-\mu} |(T x^1)_n - (T x^2)_n| = \sup_{n \geq N} |(T x^1)_n - (T x^2)_n| \leq \varepsilon_4 \|x^1 - x^2\|,
\]

where in view of (14), \( \varepsilon_4 < 1 \), which proves that \( T \) is a contraction mapping. By the Banach contraction principle, consequently \( T \) has the unique fixed point \( x \in \Omega \). \( x = \{x_n\} \) is a positive solution of equation (1) for \( n \geq N \). This completes the proof of Case 4.

The proof of the theorem is complete. \( \Box \)

**Corollary 1.** Consider the difference equation

\[ \Delta^2 (x_n + px_{n-\tau}) + q_n x_{n-\sigma} = 0, \quad n = 0, 1, 2, \ldots, \]

where \( p \in \mathbb{R}, \ p \neq 1, \ \tau \) is a positive integer; \( \sigma \) is a nonnegative integer; \( \{q_n\} \) is a real positive sequence and \( \sum_{n=n_0}^{\infty} n q_n < \infty \), then equation (16) has a nonoscillatory solution.

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**REFERENCES**


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