RELATIVE EQUILIBRIA IN THE CHARGED $n$-BODY PROBLEM


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ABSTRACT. The charged $n$-body problem studies the dynamic of $n$-point particles endowed with masses and electrostatic charges moving under the influence of the respective Newtonian and Coulombian forces. In this paper we study some important features for the global flow; pointing the differences with the classical $n$-body problem. In particular we prove the existence of spatial periodic orbits (relative equilibria) in a particular example.

1 Introduction. In the classical Newtonian $n$-body problem a central configuration is a particular position of the particles where the position and the acceleration vector of each particle are proportional, and the proportional constant is the same for the $n$ particles. Since the potential energy in this case only depends on the distances among the particles, the rotation of any central configuration around its center of mass becomes a periodic solution of the $n$-body problem. These are the unique explicity solutions known until now. Another important property of central configurations is that the total collision in the $n$-body problems is always asymptotic to central configurations, and therefore the knowledge of central configurations give us important insight into the dynamics near total collision. Also if we fix the total energy $h$ and the angular momentum $c$, then, the bifurcation points in the phase space occur at the level sets $(h,c)$ which contain central configurations [14].

The circular periodic orbits getting from central configurations become fixed points in rotating coordinates, hence the name of relative equilibria. Its study began in 1767 with a famous work of L. Euler where

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he found the 3 relative equilibria in the collinear three body problem, 5 years later, Lagrange rediscovered the Euler’s relative equilibria and found two new families of periodic orbits in the general three-body problem, the equilateral triangle solutions. (See [7] for a nice survey about relative equilibria).

In this paper we tackle the charged $n$-body problem, that is, we study the motion of $n$-point particles (endowed with a positive mass and an electrostatic charge), under the Newtonian and Coulombian forces at the same time. We generalize the concepts of central configuration and relative equilibrium for this new kind of potential. The goal of this work is to show the main differences between the charged and the classical problems, and how these differences can be used to obtain new surprising properties. A particular charged problem has been defined by I. Langmuir in 1920, in order to study the structure of the helium atom [8]. Daves et al. worked with charged problems to study periodic solutions of $n$-electrons and the $2n$-ion atom [5], among other authors that have been worked with charged problems we can cite [1], [2], [3], [4] and [11].

In section 2 we study the general aspects of the problem. We obtain the Lagrange-Jacobi identity, and the behaviour of the smallest and the largest distance among particles, which obviously it depends on the sign of the fix level of energy. Usually when the potential depends on different kind of forces it is possible to have spiraling solutions (see for example [6] and [15]), that is, we can get total collision with angular momentum different from zero, however in the charge problem we prove that total collision implies angular momentum zero.

Section 3 deals with relative equilibria; we give a necessary condition for the existence of relative equilibria and we prove that they must be on negative energy levels. The main results in this part are about the non-planar relative equilibria. In 1941, A. Wintner [16] shows that all relative equilibria in the Newtonian $n$-body problem in $\mathbb{R}^3$, must be on a fix plane in order to maintain the perfect balance between the gravitational and centrifugal forces. We prove that in the charged $n$-body problem it is possible to have non-planar relative equilibria; and even more, they are not generated by central configurations, a surprising result.

2 General aspects. The charged $n$-body problem corresponds to the analysis of the dynamic on $n$-point particles endowed with a positive mass $m_j$ and an electrostatic charge $q_i$ of any sign moving under the
Newtonian and Coulombian law of attraction. If we denote the position of the \(j\)-th particle as \(\vec{r}_j \in \mathbb{R}^3\) and we take as 1 the gravitational and electrostatic constants, the equations of motion are given by

\[
m_j \ddot{\vec{r}}_j = \sum_{i \neq j} \frac{m_i m_j - q_i q_j}{r_{ij}^3} (\vec{r}_i - \vec{r}_j) = \frac{\partial U}{\partial \vec{r}_j}, \quad j = 1, \ldots, n,
\]

where \(r_{ij} = |\vec{r}_i - \vec{r}_j|\), \(\lambda_{ij} = m_i m_j - q_i q_j\) and the potential \(U\) is

\[
U = \sum_{i<j} \frac{\lambda_{ij}}{r_{ij}}.
\]

We will suppose along this work that the center of mass is fixed at the origin, that is

\[
\sum_{j=1}^n m_j \vec{r}_j = 0.
\]

Let \(\Delta = \bigcup_{i<j} \Delta_{ij}\) be the set of collisions where

\[
\Delta_{ij} = \{ (\vec{r}_1, \ldots, \vec{r}_n) \in \mathbb{R}^{3n} \mid \vec{r}_i = \vec{r}_j \},
\]

and

\[
X = \left\{ (\vec{r}_1, \ldots, \vec{r}_n) \in \mathbb{R}^{3n} \mid \sum_{i=1}^n m_i \vec{r}_i = 0 \right\}.
\]

Then the configuration space is given by

\[
\Omega = X \setminus \Delta.
\]

Taking the mass matrix \(M = \text{diag}\{m_1, m_1, m_1, \ldots, m_n, m_n, m_n\}\) the position vector \(x = (\vec{r}_1, \ldots, \vec{r}_n) \in \Omega\) and the momenta vector \(\dot{p} = M \dot{x}\), the system (1) can be written in Hamiltonian form

\[
\dot{x} = \frac{\partial H}{\partial p},
\]

\[
\dot{p} = -\frac{\partial H}{\partial x},
\]

where \(H(x, p) = \frac{1}{2} p^t M^{-1} p - U(x)\), as usual the Hamiltonian \(H\) is a first integral. The momentum of inertia \(I\) is defined by

\[
I = \frac{1}{2} \sum_{j=1}^n m_j |\vec{r}_j|^2.
\]
Since the center of mass is fixed at the origin, \( I \) can be written as

\[
I = \frac{1}{\mu} \sum_{i \neq j} m_i m_j r_{ij}^2,
\]

where \( \mu = \sum_{i=1}^{n} m_i \). The angular momentum \( \tau \) is

\[
\tau = \sum_{j=1}^{n} m_j \vec{r}_j \times \vec{\dot{r}}_j.
\]

By straightforward computations using that \( U \) is a homogeneous function and the Euler’s theorem for homogeneous functions, we get the following identity.

**Proposition 2.1** (The Lagrange-Jacobi Identity).

\[
\bar{I} = U + 2h.
\]

As in the classical Newtonian case, we have the following result for charged problems.

**Proposition 2.2.** If \( H = h < 0 \) then the smallest distance among the particles remains bounded.

**Proof.** From the energy integral

\[
U = \frac{1}{2} p^t M^{-1} p - h \geq -h > 0,
\]

in particular this implies that not all \( \lambda_{ij} < 0 \). Let \( r = \min_{i \neq j} r_{ij} \), then

\[
U = \sum_{i < j} \frac{\lambda_{ij}}{r_{ij}} \leq \sum_{i < j} \frac{|\lambda_{ij}|}{r_{ij}} \leq \frac{1}{r} \sum_{i < j} |\lambda_{ij}|.
\]

Now by (8)

\[
0 < \frac{1}{U} \leq -\frac{1}{h},
\]

therefore, using (9) and (10)

\[
r \leq \frac{1}{U} \sum_{i < j} |\lambda_{ij}| \leq -\frac{1}{h} \sum_{i < j} |\lambda_{ij}|.
\]

What happens when we fix \( h \geq 0 \)? In this case we have:
Proposition 2.3. In the charged n-body problem, if \( H = h \geq 0 \), then the largest mutual distance among the particles can not remain bounded, except for the equilibrium solutions which exist for \( h = 0 \).

Proof. By Proposition 2.1 and the energy integral we have

\[
\dot{I} = \frac{1}{2} \rho^t M^{-1} p + h,
\]

so \( \dot{I} \geq h \). Then integrating twice from \( t_0 \) to \( t \) along a solution defined for all time, we obtain

\[
I(t) \geq \frac{1}{2} h(t - t_0)^2 + \dot{I}(t_0)(t - t_0) + I(t_0).
\]

For \( h > 0 \) the results follows trivially. For \( h = 0 \), if \( \dot{I}(t_0) \neq 0 \) the same holds taking \( t \to \infty \) or \( t \to -\infty \). If \( \dot{I}(t) \equiv 0 \), then \( I \) is constant; therefore by the Lagrange-Jacobi identity (Proposition 2.1), the potential \( \mathcal{U} \) is also constant, and the respective solution is an equilibrium of the vector field (4).

Another important inequality, for the global dynamics is the following:

Proposition 2.4. \( \ddot{I} - h - \frac{I^2}{4I} \geq \frac{2}{4I} \).

Proof. As usual, we will use the notation \( \dot{\vec{r}}_i^2 = |\dot{\vec{r}}_i|^2 \), taking the derivative with respect to time \( t \), we have \( \dot{\vec{r}}_i \cdot \ddot{\vec{r}}_i = |\dot{\vec{r}}_i||\dot{\vec{r}}_i| \), therefore the derivative of the momentum of inertia \( I \) defined in (5) can be written as

\[
\dot{I} = \sum_{i=1}^{n} m_i \dot{\vec{r}}_i \cdot \ddot{\vec{r}}_i = \sum_{i=1}^{n} m_i |\dot{\vec{r}}_i||\dot{\vec{r}}_i|,
\]

using the Schwartz inequality we obtain

\[
\dot{I}^2 \leq \left( \sum_{i=1}^{n} m_i \dot{\vec{r}}_i^2 \right) \left( \sum_{i=1}^{n} m_i |\dot{\vec{r}}_i|^2 \right),
\]

finally, since \( |\dot{\vec{r}}_i| = \frac{\dot{\vec{r}}_i \cdot \hat{\vec{r}}_i}{|\dot{\vec{r}}_i|} \) we get

\[
\dot{I}^2 \leq 2I \sum_{i=1}^{n} \frac{m_i (\dot{\vec{r}}_i \cdot \ddot{\vec{r}}_i)^2}{\dot{\vec{r}}_i^2}.
\]
From (7), the angular momentum is
\[ \vec{c} = \sum_{i=1}^{n} (m_i^{1/2} |\vec{r}_i|)(m_i^{1/2} \vec{r}_i \times \dot{\vec{r}}_i/|\vec{r}_i|), \]
here the Schwartz inequality implies
\[ c^2 \leq 2I \sum_{i=1}^{n} m_i (\vec{r}_i \times \dot{\vec{r}}_i)^2/\vec{r}_i^2, \]
where \( c = |\vec{c}| \). Adding (13) and (14), we have
\[ \dot{J}^2 + c^2 \leq 2I \sum_{i=1}^{n} m_i \left( (\vec{r}_i \cdot \dot{\vec{r}}_i)^2 + (\vec{r}_i \times \dot{\vec{r}}_i)^2/\vec{r}_i^2 \right). \]
Using the identity \((\vec{a} \cdot \vec{b})^2 + (\vec{a} \times \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2\) we obtain
\[ \dot{I}^2 + c^2 \leq 2I \sum_{i=1}^{n} m_i \vec{r}_i^2 = 2Ip^2M^{-1}p, \]
then considering the energy integral on a fix level \( h \), (16) takes the form
\[ \dot{I}^2 + c^2 \leq 4I(\mathcal{U} + h) = 4I(\bar{I} - h), \]
where in the last equality we have used the Lagrange-Jacobi identity (Proposition 2.1).

With the above inequality we can prove the main result in this section which connect the angular momentum with total collision.

**Theorem 2.5.** In the charged \( n \)-body problem, total collision implies angular momentum zero.

**Proof.** We define the function
\[ Q = I^{1/2} \left( -4h + \frac{\dot{I}^2 + c^2}{I} \right), \]
and take its derivative with respect to time \( t \),
\[ \dot{Q} = (\sqrt{I}) \left( 4\bar{I} - 4h - \left( \frac{\dot{I}^2 + c^2}{I} \right) \right). \]
By Proposition 2.4, the second factor in the right hand side of (19) is non-negative, therefore \( \dot{Q} \) and \((\sqrt{I})'\) do not have opposite signs, and therefore the same holds for \( Q \) and \( I \). Now, we apply this fact to analyze the behaviour of an orbit near total collision, which occurs at time \( t_0 \). That is, we consider that \( I \to 0 \) as \( t \to t_0, \ t < t_0 \), then \( Q \) is not negative and not increasing for \( t \sim t_0 \), therefore by (18), we have that at total collision

\[
\lim_{t\to t_0} \left( \frac{j^2 + c^2}{\sqrt{I}} \right)
\]

is finite and non-negative. It implies that \( c^2/\sqrt{I} \) is bounded as \( t \to t_0 \), then \( c = 0 \).

The above theorem shows that the spiraling collisions, very often when we study other potentials (for instance Manev [6] and Schwarzschild [15]) do not happen in charged problems, so:

**Corollary 2.6.** In the charged \( n \)-body problem any total collision is frontal, that is, it corresponds to a homothecie of a given fix configuration.

3 Central configurations and relative equilibria. Central configurations are special positions of the particles where the acceleration vector is proportional to the position vector, and the proportional constant is the same for all the particles, using the equations of motion (1) we can write directly.

**Definition 3.1.** We say that \( x \in \Omega \) is a central configuration if there exists \( \lambda \in \mathbb{R} \) such that

\[
M^{-1} \nabla \mathcal{U}(x) - \lambda x = 0.
\]

In the classical Newtonian \( n \)-body problem, one of the main properties of central configurations is that they generate the unique explicit solutions of the \( n \)-body problem. Given a central configuration, if we take the action of the group \( \text{SO}(3) \) on each particle we obtain a periodic solution, of course if we take the respective rotating frame the periodic orbits are transformed in equilibrium points, called relative equilibria.
In this paper we will analyze these concepts in the charged $n$-body problem. We will start defining the rotating matrix. Let $R(\theta)$ the $3n \times 3n$ block diagonal matrix, where each block is given by

$$
\begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

Given $x \in \Omega$, and $p = M \dot{x}$ we introduce the new rotating coordinates

$$
(22) \quad s = R(\omega t)x, \quad y = R(\omega t)p,
$$

where as is usual $\omega$ represents the constant angular velocity. The equations of motion (4) in the new coordinates take the form

$$
(23) \quad \dot{s} = Ks + M^{-1}y, \\
\dot{y} = \nabla U(s) + Ky,
$$

where $K = \dot{R}(\omega t)R^{-1}(\omega t)$. The equilibrium points must satisfy

$$
(24) \quad y = -MKs, \quad \nabla U(s) - MKs = 0.
$$

Since $KM = MK$ and $K^2 = -\omega^2E$, where $E$ is the $3n \times 3n$ block diagonal matrix, with blocks

$$
I_0 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix},
$$

the equations (24) take the form

$$
(25) \quad y = -MKs, \quad M^{-1}\nabla U(s) + \omega^2Es = 0.
$$

Given the importance of the above equation, we define:

**Definition 3.2.** A configuration $s \in \Omega$ is called a relative equilibrium, if it satisfies

$$
(26) \quad M^{-1}\nabla U(s) + \omega^2Es = 0.
$$

Physically, a relative equilibrium represents a particular solution of the charged $n$-body problem where the Newtonian and Coulombian forces are in perfect balance with the centrifugal ones. This fact is clear in the following result.
Proposition 3.3. A necessary condition in order to have a relative equilibrium $s$ in the charged $n$-body problem is that $U(s) > 0$.

Proof. Multiplying the equation (26) by $s^t M$, we obtain

$$(27) \quad s^t \nabla U(s) + \omega^2 s^t MEs = 0.$$ 

Now, using the fact that $U$ is a homogeneous function, with degree of homogeneity $-1$, and the Euler’s theorem for homogeneous functions, from (27) we get

$$(28) \quad \omega^2 = \frac{U(s)}{s^t MEs}.$$ 

The result follows trivially by (28). \hfill \Box

In other words, to obtain the perfect balance mentioned above we must have that the whole system must be attractive, that is $U(s) > 0$. If the Coulombian or repulsive forces predominate in the system, never it is possible to get relative equilibria. It is clear that we can have a central configuration $s$ even when $U(s) < 0$, then by Proposition 3.3, this central configuration does not generate a relative equilibrium or periodic orbit, this is a great difference with the classical Newtonian problem where any central configuration produces a periodic orbit. Another interesting question is about the levels of energy which contain relative equilibria. The answer is given in the following proposition.

Proposition 3.4. The periodic orbits obtained by relative equilibria in the charged $n$-body problem must be in negative energy levels.

Proof. By (24), $y = -MKs$ and since $K^2 = -\omega^2 E$, then the kinetic energy takes the form

$$(29) \quad \frac{1}{2} y^t M^{-1} y = \frac{1}{2} \omega^2 s^t MEs,$$

using (28) we have

$$(30) \quad \frac{1}{2} y^t M^{-1} y = \frac{1}{2} U(s).$$

Now, by the energy integral $\frac{1}{2} y^t M^{-1} y - U(s) = h$, we obtain

$$(31) \quad h = -\frac{1}{2} U(s).$$

The above equation and Proposition (3.3) give the result. \hfill \Box
3.1 Non-planar relative equilibria. It is well known that in the classical Newtonian \( n \)-body problem, when the particles are moving in \( \mathbb{R}^3 \), all relative equilibria are planar; that is, if the gravitational Newtonian forces are the unique forces among the particles, then in order to maintain the perfect balance with the centrifugal forces in the rotating system, all the particles must remain on the same plane. In this subsection we will show that this result fails in the charged \( n \)-body problem, where it is possible to obtain relative equilibria in \( \mathbb{R}^3 \). In this way we consider the following symmetrical charged 6-body problem. Let us consider the first 4 point particles located at the vertices of a square on the \( x \)-\( y \) plane, and the last two particles are symmetrically located on the orthogonal axis to the square at its center. Without loss of generality we can suppose that the above configuration \( s = (s_1, \ldots, s_6) \) is given by

\[
s_1 = -s_2 = (1, 0, 0), \quad s_3 = -s_4 = (0, 1, 0), \quad s_5 = -s_6 = (0, 0, l)
\]

where \( l \) is an arbitrary positive number. The masses and charges of the six particles are chosen symmetrically as follows:

\[
\begin{align*}
m_1 &= m_2 = 1, \quad m_3 = m_4 = m, \quad m_5 = m_6 = \mu, \\
n_1 &= n_2 = q, \quad n_3 = n_4 = m + 1 - q, \quad n_5 = n_6 = \mu,
\end{align*}
\]

where \( m \in \mathbb{R}^+, \mu \in \mathbb{R}^+ \) and \( q \in \mathbb{R} \). The parameters \( \lambda_{ij} = m_im_j - q_iq_j \) take the form:

\[
\begin{align*}
\lambda_{12} &= 1 - q^2, \quad \lambda_{34} = (1 - q)(q - 2m - 1), \quad \lambda_{56} = 0, \\
\lambda_{ik} &= \mu(1 - q), \quad \lambda_{ij} = (1 - q)(m - q), \quad \lambda_{jk} = -\mu(1 - q),
\end{align*}
\]

where \( i = 1, 2; j = 3 \) and \( k = 5, 6 \). The gradient \( \nabla U \) has as components

\[
\begin{align*}
\nabla_1 U &= -\nabla_2 U = -\left(\frac{\lambda_{12}}{4} + \frac{\lambda_{15}}{\sqrt{2}} + \frac{2\lambda_{15}}{(l^2 + 1)^{3/2}}\right)s_1, \\
\nabla_3 U &= -\nabla_4 U = -\left(\frac{\lambda_{34}}{4} + \frac{\lambda_{13}}{\sqrt{2}} + \frac{2\lambda_{35}}{(l^2 + 1)^{3/2}}\right)s_3, \\
\nabla_5 U &= \nabla_6 U = 0.
\end{align*}
\]

After some straightforward computations we can eliminate the parameter \( \omega^2 \) in (26), and obtain that the corresponding relative equilibrium must satisfy just one scalar equation given by

\[
(32) \quad \frac{(m - 1)(q + 1) + 2(m + 1)}{4} + \frac{(m - 1)(m - q)}{\sqrt{2}} + \frac{2\mu(m + 1)}{(l^2 + 1)^{3/2}} = 0.
\]
Of course, Proposition 3.3 gives us an additional constraint, which after some computations can be reduced to the following inequality

\begin{equation}
U(s) = (2\sqrt{2} - 1)(1 - q)(m - q) > 0. \tag{33}
\end{equation}

So in order to have a non-planar relative equilibrium, the parameters \(m, \mu, q\) and \(l\) must satisfy (32) and (33). Let us observe that the third term in the sum given in (32) is always positive, then in order to satisfy (32) the sum of the first two terms must be negative, in this way we have to analyze the relationship between just two parameters given by

\begin{equation}
(m - 1)q > \frac{\sqrt{8}m^2 + (3 - \sqrt{8})m + 1}{(\sqrt{8} - 1)}. \tag{34}
\end{equation}

Solving the inequality (34) we found that a necessary condition which guarantees the existence of the corresponding relative equilibrium is given by

\begin{equation}
0 < m < 1, \quad q < P(m) \quad \text{or} \quad m > 1, \quad q > P(m), \tag{35}
\end{equation}

where

\begin{equation}
P(m) = \frac{\sqrt{8}m^2 + (3 - \sqrt{8})m + 1}{(\sqrt{8} - 1)(m - 1)}. \tag{36}
\end{equation}

Then for \(m\) and \(q\) fix satisfying (35) we can choose \(\mu\) such that (32) holds, getting the non-planar relative equilibrium. We have proved the following result.

**Theorem 3.5.** In the charged \(n\)-body problem it is possible to have non-planar relative equilibria.

As an important consequence of the above theorem, we have the following result: Suppose that \(s\) is a non-planar relative equilibrium, then \(s\) must satisfy (26). It is easy to check that the third coordinate of the components of the vector \(Es\) is zero, that is, \(Es\) is a vector which has the form

\begin{equation}
(s_{11}, s_{12}, 0, \ldots, s_{n1}, s_{n2}, 0). \tag{37}
\end{equation}

Now, since \(M^{-1}\nabla U(s)\) and \(Es\) are parallel vectors, both must have the
same form (37). Therefore if \( s \) is a non-planar configuration, \( M^{-1}\nabla U(s) \) and \( s \) never can be parallel, in other words, the non-planar relative equilibrium cannot be generated by central configurations. This is a huge difference between the Newtonian on the charged problems. We have proved:

**Theorem 3.6.** The non-planar relative equilibria in the charged \( n \)-body problem are not generated by central configurations, in other words this special kind of configurations are invariant with respect to rotations, but not with respect to scale.

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