A PERTURBATION METHOD FOR ASYMMETRIC PROBLEMS IN EDDY CURRENT TESTING

M. Ya. ANTIMIROV, A.A. KOLYSHKIN AND RÉMI VAILLANCOURT

ABSTRACT. A perturbation method is used to solve eddy-current testing problems for a flaw of arbitrary shape situated in a conducting half-space when the electric conductivities of the flaw and the surrounding medium are nearly the same. A general formula for the change in impedance is obtained. Examples of cylindrical flaws of finite length and spherical flaws are given. Detailed numerical results are given in the case of a flaw in the shape of a circular cylinder of finite length whose axis is shifted sideways with respect to the axis of the testing coil.

1. Introduction. Nondestructive eddy current testing is often used to ascertain the presence of a flaw within a conducting material by measuring the impedance change in a coil which is excited by an alternating current. As in any inverse problem, it is difficult, in general, to determine the shape, or even the presence, of a flaw from readings of a change in impedance in the probe because it is not known how this change depends upon: (a) the parameters of the flaw, (b) the properties of the conducting medium, and (c) the relative position of the eddy current probe with respect to the flaw. In fact, inverse problems are often ill posed because their solutions, if they exist, may be many and may depend in a discontinuous way upon the input data. In such cases, one often studies the direct problem in an attempt to tabulate solution patterns. For example, one minimizes (in some norm) the difference between experimental and theoretical (or numerical) data, and the unknown parameters of the flaw may be found by an iterative procedure applied to the minimizing function. In the present paper, we deal exclusively with the solution to the direct problem.

From a mathematical point of view, a three-dimensional problem of electrodynamics is to be solved with complicated boundary conditions

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at the interface between the conducting media. Therefore, analytical solutions to problems which model eddy current testing processes have been found only for simple domains and, as a rule, in axisymmetric cases (see [2, 8, 9]).

In practice, however, the determination of a flaw is an asymmetric problem because the search is carried out by the movement of an eddy current probe along the exterior surface of the material to be inspected.

Much attention has been given in recent years to numerical modeling of three-dimensional problems in eddy current testing [5, 7, 10]. But, on the whole, considerations are restricted mainly to axisymmetric problems because of large computational difficulties in three-dimensional asymmetric cases. In [4, 11] the Green function technique is used to estimate the impedance change in a coil due to a flaw.

Perturbation methods are powerful tools for the investigation of natural and technical processes (see, for example, [6]) because they often simplify considerably the solution to a given problem, especially if a small parameter is naturally or artificially introduced in the problem. Fortunately, such a small parameter arises naturally in the eddy current testing of spot welds. In fact, spot welding produces a cast core whose electrical conductivity, \( \sigma_2 \), is close to the conductivity, \( \sigma_1 \), of the base material, with a difference of at most 10 percent. If the cast core is considered as a flaw, then it is natural to introduce the small parameter, \( \epsilon = 1 - \sigma_2/\sigma_1 \), and seek a solution in the form of a power series in \( \epsilon \).

This approach is used in [3] for a flaw in the form of a circular cylinder of finite length, whose axis coincides with the coil axis. In this axisymmetric case, the only movement of the coil allowed is vertical, and hence it is not very sensitive to detect flaws.

This perturbation method is also used in [1] to compute the change in impedance in a double conductor line situated above a uniform conducting medium with a flaw in the form of an infinitely long cylinder of rectangular cross-section. This asymmetric case, in which the coil may move sideways, is still an idealization because the flaw is infinitely long with a rectangular cross-section.

In the present paper, the perturbation method is used in a very general form to compute the impedance change in a coil situated above a conducting half-space which contains a flaw of arbitrary shape, provided the electrical conductivity of the flaw is close to that of the
surrounding material. This asymmetric case corresponds rather closely to a real situation, as compared to [3] and [1]. A relatively simple general formula for the impedance change in the coil is obtained. Some particular cases are also considered.

2. Formulation of the problem. The equation for the vector potential \( A \) in a conducting nonferromagnetic medium is

\[
\Delta A = \mu_0 \sigma \frac{\partial A}{\partial t} - \mu_s i^c,
\]

where \( \Delta = \nabla^2 \) is, in general, the three-dimensional Laplacian, \( \sigma \) is the electrical conductivity of the medium, \( \mu_0 \) is the magnetic constant, and \( i^c \) is the density of the external current.

We consider a coil of radius \( R \) situated at height \( h \) above a uniform conducting half-space whose electrical conductivity is \( \sigma_1 \). The coil is excited by an alternating current \( i = I e^{j\omega t} \), where \( j = \sqrt{-1} \). A convex region \( V \) bounded by a smooth closed surface \( S \) is situated in the conducting half-space. It is assumed that the conductivity, \( \sigma_2 \), of the region \( V \) is close to that of the surrounding medium, so that

\[
\varepsilon = 1 - \sigma_2 / \sigma_1
\]

is a small parameter. The geometry of the problem is shown in Figure 1.
3. Mathematical analysis. We introduce a system of cylindrical polar coordinates \((r, \varphi, z)\) with the origin at the point \(0\).

The magnetic flux \(\psi\) crossing the coil can be found as the circulation of the induced vector potential along the contour \(C\) of the coil:

\[
(2) \quad \psi = \int_C \mathbf{A} \cdot d\mathbf{l}.
\]

The electromotive force induced in the coil is

\[
(3) \quad E = -\partial \psi / \partial t;
\]

hence the induced change in impedance, \(Z_{\text{ind}}\), is

\[
(4) \quad Z_{\text{ind}} = -E / I.
\]

Note that, in cylindrical polar coordinates,

\[
(5) \quad \mathbf{A} \cdot d\mathbf{l} = A_r dr + A_\varphi r d\varphi + A_z dz,
\]

where \(\mathbf{A} = (A_r, A_\varphi, A_z)\).

By integrating (5) along \(C\), with \(z = h\) and \(r = R\), we have \(dr = 0\) and \(dz = 0\) so that, from (2) and (5), we obtain

\[
(6) \quad \psi = \int_C A_\varphi r d\varphi.
\]

In other words, according to (3), (4) and (6), the impedance change in the coil is determined only by the \(A_\varphi\) component of the vector potential.

In general one has to solve the vector equation (1) in each of the three regions \(R_0\), \(R_1\) and \(R_2\), and take into account the continuity of the vector potential and its normal derivative on the interface between the media. Since there are no external currents in the \(z\)-direction and the boundary conditions between the regions \(R_1\) and \(R_2\) are not used in the perturbation method, we obtain a homogeneous system of differential equations for the third component, \(A_z\), of the vector potential. At \(z = 0\), the equation and boundary conditions for \(A_z\) are uncoupled from those of the first and second components, \(A_r\) and \(A_\varphi\). Therefore,
we may take $A_z = 0$ in the three regions. Using this fact, we obtain the following system of equations for $A_r$ and $A_\phi$:

\begin{equation}
\Delta A_r^{(0)} - \frac{A_r^{(0)}}{r^2} - \frac{2}{r^2} \frac{\partial A_\phi^{(0)}}{\partial \varphi} = 0,
\end{equation}

\begin{equation}
\Delta A_\phi^{(0)} - \frac{A_\phi^{(0)}}{r^2} + \frac{2}{r^2} \frac{\partial A_r^{(0)}}{\partial \varphi} = -\mu_0 I \delta(r - R) \delta(z - h),
\end{equation}

\begin{equation}
\Delta A_r^{(1)} - \frac{A_r^{(1)}}{r^2} - \frac{2}{r^2} \frac{\partial A_\phi^{(1)}}{\partial \varphi} + k_1^2 A_r^{(1)} = 0,
\end{equation}

\begin{equation}
\Delta A_\phi^{(1)} - \frac{A_\phi^{(1)}}{r^2} + \frac{2}{r^2} \frac{\partial A_r^{(1)}}{\partial \varphi} + k_1^2 A_\phi^{(1)} = 0,
\end{equation}

\begin{equation}
\Delta A_r^{(2)} - \frac{A_r^{(2)}}{r^2} - \frac{2}{r^2} \frac{\partial A_\phi^{(2)}}{\partial \varphi} + k_2^2 A_r^{(2)} = 0,
\end{equation}

\begin{equation}
\Delta A_\phi^{(2)} - \frac{A_\phi^{(2)}}{r^2} + \frac{2}{r^2} \frac{\partial A_r^{(2)}}{\partial \varphi} + k_2^2 A_\phi^{(2)} = 0,
\end{equation}

with the boundary conditions

\begin{equation}
A^{(0)}|_{z=0} = A^{(1)}|_{z=0}, \quad \frac{\partial A^{(0)}}{\partial z} \bigg|_{z=0} = \frac{\partial A^{(1)}}{\partial z} \bigg|_{z=0},
\end{equation}

\begin{equation}
A^{(1)}|_s = A^{(2)}|_s, \quad \frac{\partial A^{(1)}}{\partial n} \bigg|_s = \frac{\partial A^{(2)}}{\partial n} \bigg|_s,
\end{equation}

where, for $m = 0, 1, 2$, and $l = 1, 2$, 

\begin{equation}
A^{(m)} = (A_r^{(m)}(r, \varphi, z), A_\phi^{(m)}(r, \varphi, z), 0), \quad k_i^2 = -j \omega \sigma l \mu_0,
\end{equation}
$S$ is the boundary of $V$, $\mathbf{n}$ is the outer unit normal to $S$, $\delta(x)$ is the Dirac delta function, and
\[
\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.
\]

We denote by $z = z_2(r, \varphi)$ and $z = z_1(r, \varphi)$, $z_2 \geq z_1$, the equations describing the upper and lower parts of $S$, respectively. The projection of $V$ on the plane $z = 0$ is a convex region $D$ with a smooth boundary, as shown in Figure 2.

We suppose that the two parts $ACB$ and $AEB$ of the boundary of $D$ are described in polar coordinates, respectively, by the equations
\[
r = r_1(\varphi), \quad r = r_2(\varphi),
\]
and the rays $OA$ and $OB$ form angles $\varphi_2$ and $\varphi_1$, respectively, with the positive $x$-direction.

If we introduce the small parameter
\[
(15) \quad \varepsilon = 1 - \sigma_2/\sigma_1,
\]
then we have
\[
(16) \quad k_2^2 = k_1^2 (1 - \varepsilon).
\]
We now seek the solution to (7)--(14) in the form
\[
(17) \quad A^{(m)}_r = A^{(m)}_{r_0} + \varepsilon A^{(m)}_{r_1} + \cdots,
\]
\[
A^{(m)}_\varphi = A^{(m)}_{\varphi_0} + \varepsilon A^{(m)}_{\varphi_1} + \cdots,
\]
where $m = 0, 1, 2$.

By substituting (16) and (17) into (7)–(14) and comparing the coefficients of each power of $\varepsilon$, we obtain a system of equations for each such power, and if the conducting medium is uniform ($\varepsilon = 0$), we have the following problem:

\begin{equation}
\Delta A_{\phi_0}^{(0)} - \frac{A_{\phi_0}^{(0)}}{r^2} = -\mu_0 I \delta(r - R) \delta(z - h),
\end{equation}

\begin{equation}
\Delta A_{\phi_0}^{(1)} - \frac{A_{\phi_0}^{(1)}}{r^2} + k_1^2 A_{\phi_0}^{(1)} = 0,
\end{equation}

\begin{equation}
A_{\phi_0}^{(0)}|_{z=0} = A_{\phi_0}^{(1)}|_{z=0}, \quad \frac{\partial A_{\phi_0}^{(0)}}{\partial z}|_{z=0} = \frac{\partial A_{\phi_0}^{(1)}}{\partial z}|_{z=0}.
\end{equation}

The solution to problem (18)–(20) is given in [8], and, in general, it is the sum of two terms, the first corresponding to the vector potential of a solitary coil in unbounded space, and the second representing the vector potential due to the conducting half-space. However, when speaking about the solution to this problem, we shall have in mind only the second term. In particular, the function $A_{\phi_0}^{(1)}$ has the form

\begin{equation}
A_{\phi_0}^{(1)}(r, z) = \mu_0 IR \int_0^\infty J_1(uR)J_1(u) \frac{u}{u + q} e^{u_z - u h} du,
\end{equation}

where $q = \sqrt{u^2 - k_1^2}$.

If we substitute (16) and (17) into (7)–(14) and compare the coefficients of $\varepsilon$ to the first power, we obtain the following system:

\begin{equation}
\Delta A_{r_1}^{(0)} - \frac{A_{r_1}^{(0)}}{r^2} - 2 \frac{\partial A_{\phi_1}^{(0)}}{\partial \phi} = 0,
\end{equation}

\begin{equation}
\Delta A_{\phi_1}^{(0)} - \frac{A_{\phi_1}^{(0)}}{r^2} + 2 \frac{\partial A_{r_1}^{(0)}}{\partial \phi} = 0,
\end{equation}
\begin{align*}
(24) \quad \Delta A_{r_1}^{(1)} - \frac{A_{r_1}^{(1)}}{r^2} - \frac{2}{r^2} \frac{\partial A_{\varphi_1}^{(1)}}{\partial \varphi} + k_1^2 A_{r_1}^{(1)} &= 0, \\
(25) \quad \Delta A_{\varphi_1}^{(1)} - \frac{A_{\varphi_1}^{(1)}}{r^2} + \frac{2}{r^2} \frac{\partial A_{r_1}^{(1)}}{\partial \varphi} + k_1^2 A_{\varphi_1}^{(1)} &= \Phi(r, \varphi, z),
\end{align*}

with boundary conditions

\begin{align*}
(26) \quad A_{r_1}^{(0)}|_{z=0} = A_{r_1}^{(1)}|_{z=0}, & \quad A_{\varphi_1}^{(0)}|_{z=0} = A_{\varphi_1}^{(1)}|_{z=0}, \\
(27) \quad \left. \frac{\partial A_{r_1}^{(0)}}{\partial z} \right|_{z=0} = \left. \frac{\partial A_{r_1}^{(1)}}{\partial z} \right|_{z=0}, & \quad \left. \frac{\partial A_{\varphi_1}^{(0)}}{\partial z} \right|_{z=0} = \left. \frac{\partial A_{\varphi_1}^{(1)}}{\partial z} \right|_{z=0},
\end{align*}

where

\[ \Phi(r, \varphi, z) = \begin{cases} 
  k_1^2 A_{\varphi_0}^{(1)} & \text{if } M(r, \varphi, z) \in V, \\
  0 & \text{if } M(r, \varphi, z) \notin V.
\end{cases} \]

To derive (22)–(27) we have combined the equations for the regions \( R_1 \) and \( R_2 \) into one equation and used the same upper index (1) for the functions in the whole lower half-space \( z < 0 \). It can easily be shown that the boundary conditions (14) are satisfied automatically. This fact simplifies considerably the solution to the given problem.

We seek the solution to (22)–(27) in the form of Fourier series:

\begin{align*}
(28) \quad A_{r_1}^{(m)}(r, z) &= a_0^{(m)}(r, z) \\
&\quad + \sum_{n=1}^{\infty} \left[ a_{n_1}^{(m)}(r, z) \cos n\varphi + a_{n_2}^{(m)}(r, z) \sin n\varphi \right], \\
A_{\varphi_1}^{(m)}(r, z) &= b_0^{(m)}(r, z) \\
&\quad + \sum_{n=1}^{\infty} \left[ b_{n_1}^{(m)}(r, z) \cos n\varphi + b_{n_2}^{(m)}(r, z) \sin n\varphi \right].
\end{align*}

Substitution of (28) and (29) into (22)–(27) yields systems of equations for the coefficients \( a_0^{(m)}(r, z) \), \( a_{nk}^{(m)}(r, z) \), \( b_0^{(m)}(r, z) \) and \( b_{nk}^{(m)}(r, z) \),
\( m = 0, 1, \ k = 1, 2 \). Since we are only interested in the impedance change in the coil, it follows from (2)–(4), (6), (28) and (29) that we need only know the function \( b_0^{(0)}(r, z) \) to compute this change. Comparing the coefficients which do not depend upon \( \varphi \) leads to the conclusion that \( a_0^{(m)}(r, z) \equiv 0 \).

Hence the determination of the functions \( b_0^{(m)}(r, z) \) reduces to the following boundary value problem

(30) \[ Lb_0^{(0)} - \frac{b_0^{(0)}}{r^2} = 0, \]

(31) \[ Lb_0^{(1)} - \frac{b_0^{(1)}}{r^2} + k_1^2 b_0^{(1)} = F(r, z), \]

(32) \[ b_0^{(0)} \big|_{z=0} = b_0^{(1)} \big|_{z=0}, \quad \frac{\partial b_0^{(0)}}{\partial z} \bigg|_{z=0} = \frac{\partial b_0^{(1)}}{\partial z} \bigg|_{z=0}, \]

where

(33) \[ F(r, z) = \frac{1}{2\pi} \int_0^{2\pi} \Phi(r, \varphi, z) \, d\varphi \]

and

(34) \[ L := \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \]

Now, by applying the Hankel transform

(35) \[ \tilde{b}_0^{(m)}(\lambda, z) = \int_0^{\infty} b_0^{(m)}(r, z) r J_1(\lambda r) \, dr, \]

to problem (30)–(32), we have

(36) \[ d^2b_0^{(0)}/dz^2 - \lambda^2\tilde{b}_0^{(0)} = 0, \]

(37) \[ d^2\tilde{b}_0^{(1)}/dz^2 - (\lambda^2 - k_1^2)\tilde{b}_0^{(1)} = f(\lambda, z), \]
(38) \[ \tilde{b}^{(0)}_0 |_{z=0} = \tilde{b}^{(1)}_0 |_{z=0}, \quad \frac{d\tilde{b}^{(0)}_0}{dz} |_{z=0} = \frac{d\tilde{b}^{(1)}_0}{dz} |_{z=0}, \]

where

(39) \[ f(\lambda, z) = \int_0^{\infty} F(r, z) r J_1(\lambda r) \, dr. \]

By solving problem (36)-(38), applying the inverse Hankel transform

\[ b^{(m)}_0(r, z) = \int_0^{\infty} \tilde{b}^{(m)}_0(\lambda, z) \lambda J_1(\lambda r) \, d\lambda, \]

to the given solution and using (33) and (39), we obtain the following expression for \( b^{(0)}_0(r, z) \):

\[ b^{(0)}_0(r, z) = \frac{k_1^2 \mu_0 IR}{2\pi} \int_0^{\infty} \frac{\lambda J_1(\lambda r) e^{-\lambda z}}{\lambda + \sqrt{\lambda^2 - k_1^2}} \, d\lambda \int_{\varphi_1}^{\varphi_2} \, d\varphi \]

\[ \cdot \int_{r_1(\varphi)}^{r_2(\varphi)} \rho J_1(\lambda \rho) \, d\rho \]

\[ \cdot \int_0^\infty J_1(uR) J_1(u\rho) \]

\[ \cdot \frac{ue^{-uh}}{(u + \sqrt{u^2 - k_1^2})(\sqrt{\lambda^2 - k_1^2} + \sqrt{u^2 - k_1^2})} \]

\[ \cdot \exp[(\sqrt{\lambda^2 - k_1^2} + \sqrt{u^2 - k_1^2})\xi] \bigg|_{\xi = z_2(\rho, \varphi)}^{\xi = z_1(\rho, \varphi)} \]

\[ du. \]

Now it is easy to compute the impedance change in the coil by using formulae (3), (4), (6) and (40):

\[ Z_{\text{ind}} = k_1^2 j \mu_0 \omega R^2 \int_0^{\infty} \frac{\lambda J_1(\lambda R) e^{-\lambda h}}{\lambda + \sqrt{\lambda^2 - k_1^2}} \, d\lambda \int_{\varphi_1}^{\varphi_2} \, d\varphi \]

\[ \cdot \int_{r_1(\varphi)}^{r_2(\varphi)} \rho J_1(\lambda \rho) \, d\rho \]

\[ \cdot \int_0^\infty J_1(uR) J_1(u\rho) \]

\[ \cdot \frac{ue^{-uh}}{(u + \sqrt{u^2 - k_1^2})(\sqrt{\lambda^2 - k_1^2} + \sqrt{u^2 - k_1^2})} \]

\[ \cdot \exp[(\sqrt{\lambda^2 - k_1^2} + \sqrt{u^2 - k_1^2})\xi] \bigg|_{\xi = z_2(\rho, \varphi)}^{\xi = z_1(\rho, \varphi)} \]

\[ du. \]
It is important to stress again that expression (41) multiplied by ε corresponds to the impedance change in a coil due to a flaw in the conducting medium (if we restrict ourselves to terms of order ε in (17)); this is precisely the quantity which can be found from experimental data.

We consider some particular cases where formula (41) can be simplified.

(a) **Circular flaw.** A flaw in the form of a circular cylinder of finite length whose axis coincides with the axis of the coil.

In this case the cylinder is described by the following inequalities:

\[
0 \leq \varphi \leq 2\pi, \quad 0 \leq r \leq \rho_0, \quad -b \leq z \leq -a,
\]

where \(\rho_0\) is the radius of the cylinder. Hence,

\[
\varphi_1 = 0, \quad \varphi_2 = 2\pi, \quad \tau_1(\varphi) = 0, \quad \tau_2(\varphi) = \rho_0,
\]

and

\[
z_2(\rho, \varphi) = -a, \quad z_1(\rho, \varphi) = -b,
\]

so that the integrals with respect to \(\rho\) and \(\varphi\) in (41) may be computed. Therefore, the change in impedance has the form

\[
Z_{\text{ind}} = 2\pi k_1^2 j \mu_0 \omega R^2 \rho_0 \int_0^\infty \frac{\lambda J_1(\lambda R)e^{-\lambda h}}{\lambda + \sqrt{\lambda^2 - k_1^2}} d\lambda
\]

\[
\cdot \int_0^\infty \frac{u J_1(\lambda \rho_0) J_0(u \rho_0) - \lambda J_0(\lambda \rho_0) J_1(u \rho_0)}{\lambda^2 - u^2} \frac{J_1(uR)ue^{-uh}}{(\sqrt{\lambda^2 - k_1^2} + \sqrt{u^2 - k_1^2})(u + \sqrt{u^2 - k_1^2})}
\]

\[
\cdot [e^{-a(\sqrt{\lambda^2 - k_1^2} + \sqrt{u^2 - k_1^2})} - e^{-b(\sqrt{\lambda^2 - k_1^2} + \sqrt{u^2 - k_1^2})}] du.
\]

Formula (42) is obtained in [3].

(b) **Spherical flaw.** A spherical flaw centered on the z-axis.
We consider the sphere

\[ r^2 + (z + c)^2 = \rho_0^2, \]

of radius \( \rho_0 \) and center \( z_0 = -c \) shifted along the \( z \)-axis with respect to the origin (we suppose that \( c > \rho_0 \), i.e., flaws on the surface \( z = 0 \) are not considered). Then

\[ \varphi_1 = 0, \quad \varphi_2 = 2\pi, \quad r_1 = 0, \quad r_2 = \rho_0, \]

and

\[ z_1 = -c - \sqrt{\rho_0^2 - r^2}, \quad z_2 = -c + \sqrt{\rho_0^2 - r^2}, \]

and (41) becomes

\[
Z_{\text{ind}} = 2\pi k_1^2 j\mu_0 \omega R^2 \int_0^\infty \frac{\lambda J_1(\lambda R)e^{-\lambda h}}{\lambda + \sqrt{\lambda^2 - k_1^2}} \int_0^\rho_0 \rho J_1(\lambda \rho) d\rho \\
\cdot \int_0^\infty J_1(uR)J_1(u\rho) \frac{ue^{-uh}}{(u + \sqrt{u^2 - k_1^2})(\sqrt{\lambda^2 - k_1^2} + \sqrt{u^2 - k_1^2})} \\
\cdot \exp[(\sqrt{\lambda^2 - k_1^2} + \sqrt{u^2 - k_1^2})\xi] \bigg|_{\xi = -c - \sqrt{\rho_0^2 - r^2}}^{\xi = -c + \sqrt{\rho_0^2 - r^2}} du
\]

(c) Shifted circular flaws. A flaw in the form of a circular cylinder of finite length whose center is shifted by a distance \( c \) along the \( x \)-axis.

We consider the cylinder, of radius \( \rho_0 \),

\[ \varphi_1 = -\arctan(\rho_0/c), \quad \varphi_2 = \arctan(\rho_0/c), \]

\[ r_1 = c\cos \varphi - \sqrt{\rho_0^2 - c^2}\sin^2 \varphi, \quad r_2 = c\cos \varphi + \sqrt{\rho_0^2 - c^2}\sin^2 \varphi, \]

\[ z_1 = -b, \quad z_2 = -a. \]
In this case formula (41) becomes

\[
Z_{\text{ind}} = k_1^2 j \mu_0 \omega R^2 \int_0^\infty \frac{\lambda J_1(\lambda R) e^{-\lambda h}}{\lambda + \sqrt{\lambda^2 - k_1^2}} d\lambda \int_{\varphi_1}^{\varphi_2} d\varphi \\
\cdot \int_0^\infty J_1(uR) \frac{u e^{-uh}}{(u + \sqrt{u^2 - k_1^2})(\sqrt{\lambda^2 - k_1^2} + \sqrt{u^2 - k_1^2})} \\
\cdot \{\exp[-a(\sqrt{\lambda^2 - k_1^2} + \sqrt{u^2 - k_1^2})] - \exp[-b(\sqrt{\lambda^2 - k_1^2} + \sqrt{u^2 - k_1^2})]\} \\
\cdot \frac{[u \rho J_1(\lambda \rho) J_0(u \rho) - \lambda \rho J_0(\lambda \rho) J_1(u \rho)]}{\lambda^2 - u^2} \bigg|_{\rho = r_2(\varphi)}^{\rho = r_1(\varphi)} du.
\]

We introduce the following dimensionless variables:

\[
\alpha = h/R, \quad \beta = R/\omega \sigma_1 \mu_0, \quad \delta = a/R,
\gamma = b/R, \quad r_0 = \rho_0/R, \quad x_0 = c/R.
\]

Then formula (4) can be written in the form (43)

\[
\tilde{Z}_{\text{ind}} = \beta^2 \int_{\varphi_1}^{\varphi_2} d\varphi \int_0^\infty \frac{x J_1(\beta x) e^{-\alpha \beta x}}{x + \sqrt{x^2 + j}} dx \\
\cdot \int_0^\infty \frac{y J_1(\beta y) e^{-\alpha \beta y}}{y + \sqrt{y^2 + j}} \\
\cdot e^{-\delta \beta (\sqrt{x^2 + j} + \sqrt{y^2 + j}) - \gamma \beta (\sqrt{x^2 + j} + \sqrt{y^2 + j})} \\
\cdot \frac{[y \rho J_1(\beta \rho x) J_0(\beta \rho y) - x \rho J_0(\beta \rho x) J_1(\beta \rho y)]}{\rho = r_2(\varphi)}^{\rho = r_1(\varphi)} dy,
\]

where \(\tilde{Z}_{\text{ind}} = Z_{\text{ind}}/(\omega \mu_0 R)\).

A modification of the IMSL subroutine DMLIN to admit a complex-valued function was used to compute integral (43) with a relative error of 1 percent or less.

This computation calls for some explanation. First, l'Hôpital's rule is used to determine the value of the integrand when \(x = y\), since the
numerator and denominator vanish there. Second, care should be taken with the integration limits \( \varphi_1 \) and \( \varphi_2 \), and with the functions \( r_1 \) and \( r_2 \) as the parameters \( r_0 \) and \( x_0 \) vary, that is,

(a) if \( x_0 > r_0 \), then

\[
\varphi_1 = -\arctan(r_0/x_0), \quad \varphi_2 = \arctan(r_0/x_0),
\]
\[
\rho_1(\varphi) = x_0 \cos \varphi - \sqrt{r_0^2 - x_0^2 \sin^2 \varphi},
\]
\[
\rho_2(\varphi) = x_0 \cos \varphi + \sqrt{r_0^2 - x_0^2 \sin^2 \varphi};
\]

(b) if \( x_0 = r_0 \), then

\[
\varphi_1 = -\pi/2, \quad \varphi_2 = \pi/2,
\]
\[
\rho_1(\varphi) = 0, \quad \rho_2(\varphi) = 2x_0 \cos \varphi;
\]

(c) if \( 0 < x_0 < r_0 \), then

\[
\varphi_1 = 0, \quad \varphi_2 = 2\pi,
\]
\[
\rho_1(\varphi) = 0, \quad \rho_2(\varphi) = x_0 \cos \varphi + \sqrt{r_0^2 - x_0^2 \sin^2 \varphi}.
\]

Finally, in the symmetric case (i.e., if \( x_0 = 0 \)),

\[
\rho_1(\varphi) = 0, \quad \rho_2(\varphi) = r_0, \quad \varphi_1 = 0, \quad \varphi_2 = 2\pi,
\]

and the integral with respect to \( \varphi \) may be easily computed. Thus, formula (43) becomes

\[
\tilde{Z}_{\text{ind}} = 2\pi \beta^2 \int_0^\infty \frac{x J_1(\beta x)e^{-\alpha \beta x}}{x + \sqrt{x^2 + j}} \, dx \int_0^\infty \frac{y J_1(\beta y)e^{-\alpha \beta y}}{y + \sqrt{y^2 + j}} \, dy \cdot \left[ yr_0 J_1(\beta yr_0)J_0(\beta yr_0) - yr_0 J_0(\beta xr_0)J_1(\beta yr_0) \right] dy.
\]
Formula (44) was also obtained in [3].

In the computations presented below, the values of the parameters $\alpha = 0.1$, $\delta = 0.1$ and $\gamma = 0.5$ are kept constant.

In Figure 3, the real and imaginary parts of $\tilde{Z}_{\text{ind}} = R_{\text{ind}} + jX_{\text{ind}}$ are shown in the case $r_0 = 0.5$. It is seen that the abscissa, $x_0$, of the maximum of the absolute value of any curve remains in the interval $[0.9, 1.0]$ as $\beta$ varies in the interval $[0.5, 2.0]$. This means that the maximum change in impedance occurs when the projection of the coil on the plane $z = -a$ intersects the axis of the flaw. It is also seen
FIGURE 4. Dependence of real and imaginary parts of $\tilde{Z}_{\text{ind}}$ upon $x_0$ and $\beta$, at $r_0 = 1$.

that the best identification of the flaw occurs when $\beta = 2$ because the maximum is sharper in this case. Computation done with $\beta = 3$ has shown that $|X_{\text{ind}}|$ increases and $R_{\text{ind}}$ decreases as compared with the case $\beta = 2$.

In Figure 4, the functions $R_{\text{ind}}$ and $X_{\text{ind}}$ exhibit a completely different behavior in the case $r_0 = 1$.

For each value of $\beta$ shown, the maxima occur in the symmetric case ($x_0 = 0$). These functions decrease monotonically as the coil axis is moved away from the flaw axis.
FIGURE 5. Dependence of impedance change in the \((R_{\text{ind}}, X_{\text{ind}})\)-plane upon \(x_0\) and \(\beta\), at \(r_0 = 1\).

In Figure 5, the dependence of the impedance change upon the parameters \(x_0\) and \(\beta\) is shown in the case \(r_0 = 1\).

The dashed lines show that the modulus of the change in impedance decreases as \(x_0\) increases. Moreover, an analysis of the hodographs shows that the range \(1 \leq \beta \leq 3\) is best for eddy current testing in this case.

Finally, we underline the importance of the parameter \(r_0 = \rho_0 / R\) in our model. For the chosen values of \(r_0, \alpha, \delta\) and \(\gamma\) in Figures 1 and 2, computation shows that a flaw is better identified by an eddy current
probe if the radius, $R$, of the coil is greater than the radius, $\rho_0$, of the flaw.

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DEPARTMENT OF APPLIED MATHEMATICS, RIGA TECHNICAL UNIVERSITY, RIGA, LATVIA, 226010

DEPARTMENT OF APPLIED MATHEMATICS, RIGA TECHNICAL UNIVERSITY, RIGA, LATVIA, 2226010

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF OTTAWA, OTTAWA, ON, CANADA K1N 6N5