PARAMETRIZATION OF TWO-DIMENSIONAL SOLUTIONS FOR THE NAVIER STOKES EQUATIONS

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ABSTRACT. The present paper discusses a complex variable method by which Navier Stokes solutions of general type can be parametrized in terms of the flow quantities.

1. Introduction. It is sometimes possible to exhibit solutions of nonlinear differential equations in parametric form, that is, the dependent and independent variables can be expressed as functions of an auxiliary parameter. The parameters in the case of an ordinary differential equation are usually the derivative(s) of the dependent variables with respect to the independent variable. An elementary example illustrates the procedure. The differential equation \( x = f(p), p = \frac{dy}{dx} \) is not readily invertible for \( p \) as a function of \( x \). However, differentiating with respect to \( y \), gives \( \frac{1}{p} = f'(p) \frac{dp}{dy} \) and inversion of \( \frac{dp}{dy} = \frac{1}{x} \frac{dx}{dp} \) followed by integration with respect to \( p \) gives \( y = \int p f'(p) dp + c \). The general solution is then determined in parametric form with \( p = \frac{dy}{dx} \) as parameter and \( c \) is an arbitrary constant.

The present paper discusses the extension of this basic solution method of parametrization to a class of partial differential equations and in particular the simplest, non degenerate case of the Navier Stokes equations, namely, the equations describing the steady motion of a two-dimensional viscous, incompressible liquid.

The starting point for the analysis is a concise complex variable formulation of the steady two-dimensional Navier Stokes equations. This representation was first given in [1] in connection with boundary layer theory, and subsequently rediscovered by others. A derivation of this equation is given in [2]. The complex variable formulation has the advantages of being quasi linear, autonomous and containing only \( z = x - iy \) as independent variable.

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A main feature of the present method of approach is to extend the domain of stream functions from real to complex valued. Even though the complex valued functions are used as an analytical artifice and have no obvious physical meaning, they form an essential ingredient of the method. In [2] one complex stream function $F(z, \bar{z})$ is utilized to start the integration process, but here it is necessary to use a second complex function $G(z, \bar{z})$ with $G \neq F$ in order to achieve greater generality. In fact, the analysis demonstrates that the flow quantities can be parametrized in terms of two arbitrary complex functions or equivalently four arbitrary real functions.

In concluding this introduction, it is remarked that, although existence, regularity and uniqueness have been established for the two-dimensional Navier Stokes equations, the methods for constructing solutions has been reserved for, "state of the art," numerical methods. The techniques for analytical solutions which arise from classical differential geometry are mainly only suitable for partial differential equations of second order and their generalization to equations of higher order is not straightforward if not intractable. As a consequence, it is not possible to determine a simple display of viscosity dependent general solutions for the steady two-dimensional Navier Stokes equation at the present time.

2. The equations of motion. The descriptive equations governing the steady motion of a viscous, incompressible fluid can be written in the form

\[(1) \quad -[q \times \text{curl} \ q] = -\text{grad} \ H + \nu \nabla^2 q,\]

\[(2) \quad \text{div} \ q = 0, \quad H = p/\rho + \frac{1}{2} |q|^2,\]

where $q$ is the fluid velocity, $p$ the pressure, $\rho$ the density, $\nu$ the kinematic viscosity, and $H$ the Bernoulli function or total head of pressure. In this representation of the flow equations part of the usual convective term $(q \cdot \nabla)q$ has been absorbed into the Bernoulli function. For two-dimensional flow there is a concise complex variable
formulation of (1) and (2) which is described in [1, 2] and can be expressed by

\[ \phi_{zz} + i\psi_{zz} + \frac{1}{2\nu} \psi_z^2 = 0, \tag{3} \]

\[ H = -\nu \nabla^2 \phi = -4\nu \phi_{zz} = \frac{p}{\rho} + 2\psi_z \psi_z, \tag{4} \]

where the complex velocity is \( q = 2i\psi_z \) and \( \psi \) is the stream function. The real function \( \phi \) plays an auxiliary role in defining the equations. The vorticity of the fluid is \( \omega = \nabla^2 \psi = 4\psi_{zz} \) where \( 2 \frac{\partial}{\partial z} \equiv \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \).

The complex conjugate of (3) is

\[ \phi_{zz} - i\psi_{zz} + \frac{1}{2\nu} \psi_z^2 = 0, \tag{5} \]

and elimination of \( \phi \) from (3) and (5) gives the usual vorticity equation

\[ \nu \nabla^4 \psi = \frac{\partial(\psi, \nabla^2 \psi)}{\partial(x, y)}. \tag{6} \]

There are a few cases in which the equations of flow can be integrated or partially integrated. First, in the formal limit \( \nu \to 0 \), equation (6) is equivalent to

\[ \nabla^2 \psi = B(\psi), \tag{7} \]

where \( B \) is an arbitrary function of \( \psi \) and lines of constant vorticity coincide with the streamlines. Also in the limit \( \nu \to \infty \), equation (3) reduces to Stokes flow given by

\[ \phi_{zz} + i\psi_{zz} = 0, \tag{8} \]

for which the solution is readily shown to be

\[ \phi + i\psi = \bar{z}D_1(z) + E_1(z), \tag{9} \]

where \( D_1, E_1 \) are analytic in an appropriate region. The only known general solution of either (3) or (6) for arbitrary \( \nu \) is given by
\( \psi = k(x^2 + y^2) + n(x, y), \quad \nabla^2 n = 0, \)

where \( k \) is a constant and the flow is viscosity independent.

**3. Method of solution.** It is appropriate to define the following operators

\[
L \equiv \phi_{zz} + i\psi_{zz} + \frac{1}{2\nu} \psi_z^2, \\
L_1 \equiv \phi_{zz} + iF_{zz} + \frac{1}{2\nu} F_z^2, \\
L_2 \equiv \phi_{zz} + iG_{zz} + \frac{1}{2\nu} G_z^2,
\]

where \( \psi, \phi \) are real functions of \( x, y \) and \( F, G \) complex functions of \( z = x + iy, \bar{z} = x - iy \) with \( G \neq F \). The functions \( F, G \) may be interpreted as complex stream functions which have no clear physical significance and in essence are used as artifices in constructing the real physical stream function \( \psi \). The equation

\[
L_1 - L \equiv ie^{- \frac{G + \psi}{2\nu\imath}} \frac{\partial}{\partial \bar{z}} \left\{ e^{\frac{G + \psi}{2\nu\imath}} (F_z - \psi_z) \right\} = 0,
\]

is of Riccati type and implies

\[
e^{\frac{G + \psi}{2\nu\imath}} (F_z - \psi_z) = f(z),
\]

where \( f(z) \) is arbitrary and analytic in a suitable region of the fluid domain. In a similar way the equation

\[
L_2 - L \equiv ie^{- \frac{G + \psi}{2\nu\imath}} \frac{\partial}{\partial \bar{z}} \left\{ e^{\frac{G + \psi}{2\nu\imath}} (G_z - \psi_z) \right\} = 0,
\]

implies

\[
e^{\frac{G + \psi}{2\nu\imath}} (G_z - \psi_z) = g(z),
\]

where again \( g(z) \) is arbitrary and analytic in a suitable region. Now from (15) and (17)

\[
f(z)e^{\frac{G + \psi}{2\nu\imath}} (G_z - \psi_z) = g(z)e^{\frac{F - \psi}{2\nu\imath}} (F_z - \psi_z),
\]
and integration produces the complex equation

\begin{equation}
(19) \quad f(z)e^{\frac{G}{2\nu i}} - g(z)e^{\frac{F}{2\nu i}} = h(z)e^{\frac{\psi}{2\nu i}},
\end{equation}

with \(h(z)\) analytic in an appropriate region of the fluid. Also from (15) and (17)

\begin{align}
G_z - F_z &= g(z)e^{-\frac{(G+\psi)}{2\nu i}} - f(z)e^{-\frac{(F+\psi)}{2\nu i}}, \\
&= e^{-\frac{(F+G)}{2\nu i}} \left[ g(z)e^{\frac{F}{2\nu i}} - f(z)e^{\frac{G}{2\nu i}} \right] \\
&= -h(z)e^{-\frac{(F+G)}{2\nu i}},
\end{align}

or equivalently

\begin{equation}
(20) \quad e^{\frac{F+G}{2\nu i}} (G_z - F_z) = -h(z).
\end{equation}

Equation (21) can also be determined by observing that

\begin{equation}
(22) \quad L_2 - L_1 \equiv ie^{-\frac{(F+G)}{2\nu i}} \frac{\partial}{\partial z} \left\{ e^{\frac{F+G}{2\nu i}} (G_z - F_z) \right\} = 0,
\end{equation}

which implies (21). Now from (12) and (13) it follows both \(F, G\) are invariant under the transformations \(F \rightarrow F + e(z), G \rightarrow G + m(z)\) where \(e(z), m(z)\) are arbitrary, and in view of integrability conditions to be considered at a later stage in the analysis it is expedient to set \(f(z) = 1, h(z) = -1\). Since \(G \neq F\) the function \(g(z)\) will remain arbitrary and equation (19) can now be written as

\begin{equation}
(23) \quad e^{\frac{G}{2\nu i}} - g(z)e^{\frac{F}{2\nu i}} + e^{\frac{\psi}{2\nu i}} = 0.
\end{equation}

Since \(\psi\) is real, the equation (23) is equivalent to

\begin{equation}
(24) \quad [g(z)e^{\frac{F}{2\nu i}} - e^{\frac{G}{2\nu i}}][\bar{g}(\bar{z})e^{-\frac{F}{2\nu i}} - e^{-\frac{G}{2\nu i}}] = 1,
\end{equation}

and

\begin{equation}
(25) \quad \psi = \nu i \log \left\{ \frac{g(z)e^{\frac{F}{2\nu i}} - e^{\frac{G}{2\nu i}}}{\bar{g}(\bar{z})e^{-\frac{F}{2\nu i}} - e^{-\frac{G}{2\nu i}}} \right\}.
\end{equation}
Also (21) can now be written as

\begin{equation}
\exp^{\frac{G\phi F}{2\nu t}} (G_{\bar{z}} - F_{\bar{z}}) = 1,
\end{equation}

and it is observed that (24), (25), together with (26) imply

\begin{equation}
\exp^{\frac{G\phi}{2\nu t}} (L_2 - L) - g(z) \exp^{\frac{F\phi}{2\nu t}} (L_1 - L) = 0, \quad L_2 - L_1 = 0,
\end{equation}

which in turn yield

\begin{equation}
L_2 = L_1 = L,
\end{equation}

producing equations (15) and (17).

To preserve the generality of the analysis it is profitable to proceed in the following way. First, if $G$ is regarded as a function of $\phi, F$, then equation (26) can be written as

\begin{equation}
\exp^{\frac{G\phi F}{2\nu t}} [G\phi_{\bar{z}} + (G_F - 1)F_{\bar{z}}] = 1
\end{equation}

and secondly, consider the equation

\begin{equation}
AL_1 + BL_2 = \frac{\partial}{\partial z} (C\phi_{\bar{z}} + DF_{\bar{z}}) + (K\phi_{\bar{z}} + QF_{\bar{z}} + M)(C\phi_{\bar{z}} + DF_{\bar{z}} - 1)
+ (E\phi_{\bar{z}} + G_0F_{\bar{z}} + J) \left[ \exp^{\frac{G\phi F}{2\nu t}} (G\phi_{\bar{z}} + (G_F - 1)F_{\bar{z}}) - 1 \right],
\end{equation}

where $A, B, C, D, E, G_0, J, K, M, Q$ are functions of $\phi, F$. Written explicitly in terms of the derivatives with respect to $\phi$ and $F$ equation (30) is of the form

\begin{equation}
A_1\phi_{\bar{z}\bar{z}} + A_2F_{\bar{z}\bar{z}} + A_3\phi_{\bar{z}}^2 + A_4F_{\bar{z}}^2 + A_5\phi_{\bar{z}}F_{\bar{z}} + A_6\phi_{\bar{z}} + A_7F_{\bar{z}} + A_8 = 0
\end{equation}

and is an identity provided that $A_j = 0, j = 1, 2\ldots, 8$. This yields the
following set of eight equations relating the unknown functions of \( \phi, F \):

\[
\begin{align*}
(32) \quad \phi_{xx} : C &= A + B(1 + iG_{\phi}), \\
(33) \quad F_{xx} : D &= iA + iBG_F, \\
(34) \quad \phi_{x}^2 : C_{\phi} + EG_{\phi}e^{\frac{G+G}{2\nu t^t}} + CK &= B(iG_{\phi} + \frac{1}{2\nu}G_{\phi}^2), \\
(35) \quad \phi_x F_x : C_F + D_{\phi} + E(G_F - 1)e^{\frac{G+G}{2\nu t}} + G_o e^{\frac{G+G}{2\nu t^t}} G_{\phi} + KD + CQ &= 2B(iG_{\phi} + \frac{1}{2\nu}G_{\phi}G_F), \\
(36) \quad F_x^2 : D_F + G_0 e^{\frac{G+G}{2\nu t}} (G_F - 1) &= \frac{A}{2\nu} + B(iG_{FF} + \frac{1}{2\nu}G_{F}^2), \\
(37) \quad \phi_x : -E + Je^{\frac{G+G}{2\nu t}} G_{\phi} + MC - K &= 0, \\
(38) \quad F_x : -G_0 + Je^{\frac{G+G}{2\nu t}} (G_F - 1) + MD - Q &= 0, \\
(39) \quad 1 : J + M &= 0.
\end{align*}
\]

If the derivatives of \( C, D \) are eliminated then the second derivatives of \( G \) also disappear through symmetry and the resulting equations are:

\[
\begin{align*}
(40) \quad A_{\phi} + B_{\phi}(1 + iG_{\phi}) + Ee^{\frac{G+G}{2\nu t}} G_{\phi} + CK &= \frac{BG_{\phi}^2}{2\nu}, \\
(41) \quad A_F + iA_{\phi} + B_{F}(1 + iG_{\phi}) + iB_{\phi}G_F + Ee^{\frac{G+G}{2\nu t^t}} (G_F - 1) \\
+ G_0 e^{\frac{G+G}{2\nu t}} G_{\phi} + KD + CQ &= \frac{B}{\nu} G_{\phi}G_F, \\
(42) \quad iA_F + iB_{F}g_F + G_0 e^{\frac{G+G}{2\nu t}} (G_F - 1) + QD &= \frac{A}{2\nu} + \frac{B}{2\nu}G_{F}^2
\end{align*}
\]

In addition elimination of \( M \) from (37), (38), (39) gives

\[
\begin{align*}
(43) \quad E &= J(e^{\frac{G+G}{2\nu t}} G_{\phi} - C) - K, \\
(44) \quad G_0 &= J(e^{\frac{G+G}{2\nu t}} (G_F - 1) - D) - Q.
\end{align*}
\]

Addition of equations (40), (41), (42) results in one complex equation
expressed by

\[(1 + i)(A_\phi + A_F) + (B_\phi + B_F)[1 + i(G_\phi + G_F)]
+ (E + G_0)e^{G_\phi + G_F} (G_\phi + G_F - 1) + (C + D)(Q + K)
= \frac{A}{2\nu} + \frac{B}{2\nu}(G_\phi + G_F)^2.\]

(45)

At this point the most satisfactory manner to proceed is to set

\[Q + K = -\Gamma_\phi - \Gamma_F, \quad G_\phi + G_F - 1 = 0,\]

(46)

where \(\Gamma\) is an arbitrary function of \(F\) and \(\phi\). The latter equation (46) can be integrated directly to give

\[G = F + N(\phi - F),\]

(47)

with \(N\) an arbitrary complex function of its argument. Equation (45) with the aid of (47) simplifies to

\[(1 + i)\{(A + B)\phi + (A + B)_F\} = \frac{A + B}{2\nu} + (1 + i)(A + B)(\Gamma_\phi + \Gamma_F),\]

(48)

for which the solution is

\[A + B = a(\phi - F)e^{\Gamma + \pi(\frac{F}{i+1})},\]

(49)

where \(a(\phi - F) \neq 0\) is an arbitrary complex function of \(\phi - F\). From (26), (32), (33), (49) it follows that if

\[C\phi_z + DF_z = A(\phi_z + iF_z) + B(\phi_z + iG_z)
= a(\phi - F)e^{\Gamma + \pi(\frac{F}{i+1})}(\phi_z + iF_z) + iBe^{-\frac{F+z}{2\nu}} = 1,\]

(50)

then from (30), \(AL_1 + BL_2 = 0\), which in turn from (26) implies

\[L = L_1 = L_2 = 0,\]

(51)
since \( A + B \neq 0 \). It is necessary to consider the remaining equations in the system represented by (40) and (42). From (47) these can be written as

\[
(52) \quad ES + CK = \frac{BG^2_\phi}{2\nu} - A_\phi - B_\phi(1 + iG_\phi),
\]

\[
(53) \quad QD - G_0S = \frac{A}{2\nu} + \frac{B}{2\nu}G_F^2 - iA_F - iB_FG_F,
\]

with \( S = e^{\frac{G+F}{2\nu}}N'(\phi - F) \). Elimination of \( E, G_0 \) from (43) and (44) results in the equations

\[
(54) \quad S(S - C)J + (C - S)K = \frac{BG^2_\phi}{2\nu} - A_\phi - B_\phi(1 + iG_\phi),
\]

\[
(55) \quad S(S + D)J + (S + D)Q = \frac{A}{2\nu} + \frac{BG^2_F}{2\nu} - iA_F - iB_FG_F.
\]

Further elimination of \( Q \) from (53) using (446) produces the equation

\[
(56) \quad (SJ - K)(D + S) = \frac{A}{2\nu} + \frac{B}{2\nu}G_F^2 - iA_F - iB_FG_F + (D + S)(\Gamma_\phi + \Gamma_F),
\]

and finally elimination of \( SJ - K \) from (54), (56) gives the equation

\[
(57) \quad (S + D)\left[\frac{BG^2_\phi}{2\nu} - A_\phi - B_\phi(1 + iG_\phi)\right]
\]

\[
= (S - C)\left[\frac{A}{2\nu} + \frac{BG^2_F}{2\nu} - iA_F - iB_FG_F + (S + D)(\Gamma_\phi + \Gamma_F)\right].
\]

This equation written explicitly in terms of \( A, B, \Gamma \) and \( N \) is expressed by

\[
\left[ e^{\frac{G+F}{2\nu}}N' + i(A + B - BN') \right] \left[ \frac{BN'^2}{2\nu} - A_\phi - B_\phi(1 + iN') \right]
\]

\[
= \left[ e^{\frac{G+F}{2\nu}}N' - A - B - iBN' \right] \left\{ \frac{A}{2\nu} + \frac{B}{2\nu}(1 - N')^2 - iA_F - iB_F(1 - N') \right\}
\]

\[
+ (\Gamma_\phi + \Gamma_F)\left[ e^{\frac{G+F}{2\nu}}N' + i(A + B - BN') \right].
\]
It follows from (26), (47) that if
\begin{equation}
(59) \quad e^{\frac{N(\phi - F) + 2F}{2\nu i}} N'(\phi - F) (\phi_\tilde{z} - F_\tilde{z}) = 1,
\end{equation}
then \( \phi, F \) satisfy \( L_1 = 0 \) since \( A, B, \Gamma \) are essentially determined from (49), (50) and (58). It is evident from (49) and (58) the function \( a(\phi - F) \) can be absorbed into the complex function \( \Gamma \), so without loss of generality set \( a(\phi - F) = 1 \). From (46) it follows \( J \) or \( K \) may be set equal to unity. It is now necessary to consider the application of integrability conditions which is the subject of the next section.

4. Integrability conditions. It follows from equations (26), (47) that
\begin{equation}
(60) \quad e^{\frac{N(\phi - F) + 2F}{2\nu i}} N'(\phi - F) (\phi_\tilde{z} - F_\tilde{z}) = 1,
\end{equation}
with \( N(\phi - F) \) arbitrary yields solutions which are consistent with \( L = L_1 = 0 \), since \( A, B, \Gamma \) are determined from (49), (50), (59) and the integrability conditions \( \phi_{z\tilde{z}} = \phi_{\tilde{z}z}, \psi_{z\tilde{z}} = \psi_{\tilde{z}z}, F_{z\tilde{z}} = F_{\tilde{z}z} \). The basic equations in which integrability conditions are to be applied can be written as
\begin{equation}
(61) \quad L_1 = 0, \quad L_3 \equiv e^{\frac{N(\phi - F) + 2F}{2\nu i}} N'(\phi - F) (\phi_\tilde{z} - F_\tilde{z}) - 1 = 0,
\end{equation}
\begin{equation}
(62) \quad L_4 \equiv e^{\frac{F + \psi}{2\nu i}} (F_\tilde{z} - \psi_\tilde{z}) - 1 = 0 \quad L_5 \equiv \phi_\tilde{z} + iF_\tilde{z} - k = 0,
\end{equation}
where \( k = [1 - iBe^{\frac{F - \phi}{2\nu i}}] \times e^{-\frac{\Gamma - \psi z (F_\tilde{z} + i)}{2\nu i}} \) is for present essentially arbitrary. In symbolic form the integrability conditions can be expressed as
\begin{equation}
(63) \quad L_1 = \frac{\partial L_1}{\partial z} = \frac{\partial L_1}{\partial \tilde{z}} = L_j = \frac{\partial L_j}{\partial z} = \frac{\partial L_j}{\partial \tilde{z}} = \frac{\partial^2 L_j}{\partial z^2} = \frac{\partial^2 L_j}{\partial \tilde{z}^2} = 0,
\end{equation}
for \( j = 3, 4, 5 \). In total there are 42 independent equations containing the 40 independent variables \( \frac{\partial^m F}{\partial z^m \partial \tilde{z}^m}, \frac{\partial^m \phi}{\partial z^m \partial \tilde{z}^m}, \frac{\partial^m \psi}{\partial z^m \partial \tilde{z}^m}, \frac{\partial^m \phi}{\partial z^m \partial \tilde{z}^m}, \frac{\partial^m \psi}{\partial z^m \partial \tilde{z}^m}, m = 0, 1, \ldots n, n = 0, 1, 2, 3 \). It follows there are two equations from this system which can be utilized to determine \( k \) and \( \bar{k} \), and without solving
explicitly for $k, \tilde{k}$ it can be inferred the integrability conditions can be satisfied by equations (63). Now equation (60) can be written as

$$\phi_z - F_z = e^{i\phi/\nu} \alpha(\phi - F), \quad \alpha(\phi - F) = \frac{1}{N'(\phi - F)} e^{\frac{2(\phi - F) - N(\phi - F)}{2\nu^4}}.$$ 

Now the equation

$$F_z = e^{i\phi/\nu} \beta(\phi - F),$$

is consistent with $L_1 = 0$, provided the complex function $\beta$ satisfies

$$\alpha(\alpha' + \beta') + \frac{i}{\nu} (\alpha + \beta)^2 + i[\beta' \alpha + \beta (\alpha + \beta)] + \frac{1}{2\nu} \beta^2 = 0,$$

where $\alpha \equiv \alpha(\phi - F), \beta \equiv \beta(\phi - F)$. Also from equations (64), (65)

$$\phi_z - F_z = \frac{\alpha}{\beta} F_z,$$

which can be integrated to give

$$\gamma(\phi - F) - F = j(z), \quad \gamma(\phi - F) = \int_0^{\phi - F} \frac{\beta(\omega)}{\alpha(\omega)} d\omega.$$ 

Equations (64), (65) which give rise to (68) provide a considerable simplification in satisfying the integrability conditions. In fact it is sufficient to apply the integrability conditions to the set of equations

$$F_z - \psi_z = e^{-\frac{\psi + F}{2\nu^2}}, \quad \phi_z - F_z = \alpha e^{i\phi/\nu}, \quad \gamma - F = j(z).$$

Differentiation of each of equations (69) with respect to $z$ produces the equations

$$F_{zz} - \psi_{zz} + \frac{1}{2\nu^2} (\psi_z + F_z) e^{-\frac{\psi + F}{2\nu^2}} = 0,$$

$$\phi_{zz} - F_{zz} = e^{i\phi/\nu} \left[ \frac{i\alpha}{\nu} \phi_z + (\phi_z - F_z) \alpha' \right],$$

$$\gamma' - F_z = j'(z).$$
Also differentiation of \( \gamma - F = j(z) \) with respect to \( \tilde{z} \) gives

\begin{equation} \tag{73} \label{eq73}
(\phi \z - F \z)\gamma' - F \z = 0,
\end{equation}

and differentiation of (72) with respect to \( \z \) produces the equation

\begin{equation} \tag{74} \label{eq74}
(\phi \z \z - F \z \z)\gamma' + (\phi \z - F \z)(\phi \z - F \z)\gamma'' - F \z \z = 0.
\end{equation}

Elimination of \( \psi \z \z, \psi \z, \psi \z \) from (70), (69) gives the single equation

\begin{equation} \tag{75} \label{eq75}
F \z \z - \bar{F} \z \z + \frac{1}{2\nu_i}(F \z + \bar{F} \z)e^{-(\psi + F) \nu_i} + \frac{1}{2\nu_i}e^{\frac{\psi + F}{2\nu_i}} - \frac{1}{\nu_i}e^{\frac{\bar{F} - F}{2\nu_i}} = 0.
\end{equation}

Again elimination of \( F \z \z, \bar{F} \z \z, \phi \z \z \) from the system results in the equations

\begin{equation} \tag{76} \label{eq76}
-\frac{i}{\nu}e^{i\phi \z \nu} + (\phi \z - F \z)\alpha' + e^{-i\phi \nu} \left[ (\bar{\alpha}' - \frac{i\bar{\alpha}}{\nu})\phi \z - \bar{\alpha}' \bar{F} \z \right]
\end{equation}

\begin{equation} \tag{77} \label{eq77}
+ \frac{1}{2\nu_i}(F \z + \bar{F} \z)e^{-(\psi + F) \nu_i} + \frac{1}{2\nu_i}(F \z + \bar{F} \z)e^{\frac{\psi + F}{2\nu_i}} - \frac{1}{\nu_i}e^{\frac{\bar{F} - F}{2\nu_i}} = 0,
\end{equation}

and

\begin{equation} \tag{78} \label{eq78}
(1 + \gamma')e^{i\phi \nu} \left[ (\alpha' + \frac{i\alpha}{\nu})\phi \z - \alpha' F \z \right] + \gamma''(\phi \z - F \z)(\phi \z - F \z)
\end{equation}

\begin{equation} \tag{79} \label{eq79}
= (1 + \bar{\gamma}')e^{-i\bar{\phi} \nu} \left[ (\bar{\alpha}' - \frac{i\bar{\alpha}}{\nu})\phi \z - \bar{\alpha}' \bar{F} \z \right] + \bar{\gamma}''(\phi \z - \bar{F} \z)(\phi \z - \bar{F} \z).
\end{equation}

Again from equations (69)

\begin{equation} \tag{80} \label{eq80}
\phi \z = \alpha(1 + \gamma')e^{i\phi \nu}, \quad F \z = \alpha \gamma'e^{i\phi \nu}, \quad F \z = \frac{\bar{\alpha} \gamma'(1 + \bar{\gamma}')e^{-i\bar{\phi} \nu} - j'(z)}{(1 + \gamma')}.
\end{equation}

and elimination of the first derivatives from (76), (77), (78) gives the equations

\begin{equation} \tag{81} \label{eq81}
\frac{\alpha'}{1 + \gamma'} \left[ \gamma' \bar{\alpha}'(1 + \bar{\gamma}') - j'(z)e^{\frac{i\gamma'}{\nu}} \right] - \bar{\alpha}(1 + \bar{\gamma}')(\alpha' + \frac{i\alpha}{\nu})
\end{equation}

\begin{equation} \tag{82} \label{eq82}
+ \frac{1}{2\nu_i}e^{-\frac{(\psi + F) \nu_i}{2\nu_i}} \left\{ [\bar{\gamma}''(1 + \bar{\gamma}')e^{-\frac{i\gamma'}{\nu}} - j'(z)] + \bar{\alpha}'e^{-\frac{i\gamma'}{\nu}} \right\}
\end{equation}

\begin{equation} \tag{83} \label{eq83}
+ \alpha(1 + \gamma')(\bar{\alpha}' - \frac{i\bar{\alpha}}{\nu}) - \frac{\bar{\alpha}'}{1 + \bar{\gamma}'} \left[ \bar{\gamma}'(1 + \gamma')\alpha - \bar{\gamma}''(z)e^{-\frac{i\gamma'}{\nu}} \right]
\end{equation}

\begin{equation} \tag{84} \label{eq84}
+ \frac{1}{2\nu_i}e^{\frac{\psi + F}{2\nu_i}} \left\{ \alpha \left[ \frac{(\gamma + 1 + \gamma')e^{\frac{i\gamma'}{\nu}} - j'(\bar{z})}{(1 + \gamma')} \right] + \alpha \gamma'e^{\frac{i\gamma'}{\nu}} \right\} - \frac{1}{\nu_i}e^{\frac{\bar{F} - F}{2\nu_i}} = 0,
\end{equation}
and

\[
(1 + \gamma') \left\{ \left( \alpha' + \frac{i\alpha}{\nu} \right) \bar{\alpha}(1 + \bar{\gamma}') - \frac{\alpha'}{1 + \gamma'} [\gamma'(1 + \gamma') \bar{\alpha} - j'(z)e^{i\phi}] \right\} \\
+ \alpha \gamma'' \left\{ \bar{\alpha}(1 + \bar{\gamma}') - \frac{[\gamma'(1 + \gamma') - j'(z)e^{i\phi}]}{(1 + \gamma')} \right\}
\]

(80) \[
= (1 + \bar{\gamma}')(\bar{\alpha}' - \frac{i\bar{\alpha}}{\nu})(1 + \gamma')\alpha - \frac{\bar{\alpha}'}{1 + \bar{\gamma}'} [\bar{\gamma}'(1 + \gamma')\alpha \\
- \bar{j}'(\bar{z})e^{-i\phi}] \\
+ \bar{\alpha} \gamma'' \left\{ \alpha(1 + \gamma') - \frac{[\alpha \gamma'(1 + \gamma') - e^{-i\phi} \bar{j}'(\bar{z})]}{1 + \bar{\gamma}'} \right\}
\]

Together with the equations

(81) \[
\gamma - F = j(z),
\]

where \( \gamma' = \frac{\bar{\beta}}{\alpha} \) and

(82) \[
\alpha(\alpha' + \beta') + \frac{1}{\nu}(\alpha + \beta)^2 + i[\alpha \beta' + \beta(\alpha + \beta)] + \frac{1}{2\nu} \beta^2 = 0,
\]
equations (79), (80), (81) define a general solution of the two dimensional flow equations in implicit form and containing the two arbitrary complex functions \( \alpha(\phi - F) \) and \( j(z) \). The complex functions \( \beta \) and \( \gamma \) are defined in terms of \( \alpha \).

Finally, the method of solution can be checked by recovering a known potential flow solution. The simplest nondegenerate form for the function \( \phi - F \) is expressed by

(83) \[
N(\phi - F) = \phi - F + i\pi,
\]

and the equations

(84) \[
e^{-\frac{N(\phi - F) + 2F}{2\nu t}} N'(\phi - F)(\phi - F) + \frac{1}{2\nu t} \beta^2 = 1,
\]

(85) \[
e^{-\frac{\phi - F}{2\nu t}} (F - \psi) = 1,
\]

are satisfied by

(86) \[
\phi = \psi = f(z) + \bar{f}(\bar{z}), \quad F = -f(z) + J_0(\bar{z}),
\]
where the function \( J_\nu(\bar{z}) \) is defined by

\[
(87) \quad e^{\frac{J_0(z)+f(z)}{2\nu i}} [J_0'(\bar{z}) - \bar{F}'(\bar{z})] = 1.
\]

The Navier Stokes equation is satisfied providing

\[
(88) \quad (1 + i)\bar{F}''(\bar{z}) + \frac{1}{2\nu} [\bar{F}'(\bar{z})]^2 = 0,
\]

for which the solution is

\[
\phi = \psi = 2\nu (1 - i) \log(z + a_0) + 2\nu (1 + i) \log(\bar{z} + \bar{a}_0),
\]

representing the flow produced by a linear combination of a line source and a line vortex filament.

REFERENCES
