MATHEMATICS 118 Review

1. Sketch the curves whose equations in polar coordinates are

   (a) $r = \theta$, $-2\pi \le \theta \le 2\pi$,

   (b) $r = \theta^2$, $-2\pi \le \theta \le 2\pi$,

   (c) $r = a\cos 2\theta$, $0 \le \theta \le 2\pi$, $(a > 0)$.

2. (a) Show that the area enclosed by one loop of the curve in #1(c) is $\pi a^2/8$.

   (b) Find the centroid of the loop in (a) as a quotient of 2 integrals. It is not necessary to evaluate the integrals.

3. Show that each of the series is convergent if $p > 1$ and divergent if $p \le 1$. **Verify completely** that the conditons for any test that you use are satisfied.

$$\text{(i)} \sum_{k=1}^{\infty} \frac{1}{k^p} \qquad\qquad \text{(ii)} \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$$

4. For what values of $p \in \mathbb{R}$ are the following series absolutely convergent, conditionally convergent, divergent? **Verify completely** that the conditions for any test that you use are satisfied.

$$\text{(i)} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^p} \qquad\qquad \text{(ii)} \sum_{k=2}^{\infty} \frac{(-1)^k}{k(\ln k)^p}$$

5. Express $F(x) = \frac{1}{(1-x)(1+x^2)}$ as a power series in x. ANS: $F(x) = \sum_{n=0}^{\infty}(x^{4n} + x^{4n+1})$, $|x| < 1$. Try to "discover" the power series rather than work back from the given answer.

6. Find the sum of each of the series:

   (a) $x + 2x^2 + 3x^3 + 4x^4 + \cdots$,

   (b) $4 + 5x + 6x^2 + 7x^3 + 8x^4 + \cdots$,

   (c) $1 + 4x + 9x^2 + 16x^3 + 25x^4 + \cdots$,

   (d) $x - \frac{x^3}{3^2} + \frac{x^5}{5^2} - \frac{x^7}{7^2} + \cdots$,

   (e) $x - \frac{x^9}{9} + \frac{x^{17}}{17} - \frac{x^{25}}{25} + \cdots$.

   ANS: (a) $\frac{x}{(1-x)^2}$, (b) $\frac{4-3x}{1-x^2}$, $|x| < 1$, (c) $\log(1-x^4)^{-1/4}$, $|x| < 1$, (d) $\int_0^x \frac{\arctan t}{t} dt$, $|x| < 1$, (e) $\int_0^x \frac{1}{1+t^8} dt$.

   Again, try to "discover" the answer.

7. Show that

$$\left| \int_0^1 \frac{1}{1+t^8} dt - 1 + \frac{1}{9} - \frac{1}{17} + \frac{1}{25} \right| < \frac{1}{33}.$$

8. If $0 \le a_k$ and $\sum_{k=0}^{\infty} a_k$ is convergent, show that $\sum_{k=0}^{\infty} (a_k)^2$ is also convergent.

9. For the series

$$\sum_{k=2}^{\infty} \frac{(-1)^k}{k^p (\log k)^q},$$

determine all values of $p, q \in \mathbb{R}$ for which it is absolutely convergent, conditionally convergent, divergent.

ANS: (i) Abs C $p > 1$ all q, (ii) Cond C $0 < p < 1$ all q, (iii) D $p < 0$ all q, (iv) Cond C $p = 0$, $q > 0$, (v) D $p = 0$, $q \leq 0$.

10. Show $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^p} \log\left(1 + \frac{1}{k}\right)$ is absolutely convergent if and only if $p > 0$. For what value(s) of p is it conditionally convergent?

11. Show that

$$\sum_{k=1}^{\infty} (-1)^k \left[\frac{1.3.5 \cdots (2k-1)}{2.4.6 \cdots 2k}\right]^p$$

is absolutely convergent if $p > 2$, conditionally convergent if $0 > p \geq 2$ and divergent if $p \geq 0$.

12. If b is not a negative integer, show that

$$\sum_{k=1}^{\infty} (-1)^k \frac{(a+1)(a+2)\cdots(a+k)}{(b+1)(b+2)\cdots(b+k)}$$

is absolutely convergent if $b - a > 1$, conditionally convergent if $0 < b - a \leq 1$ and divergent if $b - a \leq 0$.

13. Show that

$$\sum_{k=1}^{\infty} (-1)^k \frac{k^k}{(k+1)^k}$$

is conditionally convergent.

14. Show in TWO WAYS that

$$\int x^2 \arctan x dx = \frac{x^3}{3} \arctan x - \frac{1}{6}x^2 + \frac{1}{5}\log\left(1 + x^2\right) + C$$

(a) By direct integration.

(b) By expressing both sides of the equation as power series in x.

15. How many terms should be taken in the series below if the error in the approximation is not to exceed 0.01

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{5} + \frac{1}{9} - \cdots.$$

It may be accepted that the sum is as stated but the proof of the error estimate should be given in detail.

16. Estimate the error in the approximation

$$\int_0^1 \frac{1}{(1+x^{20})^{1/3}} dx \simeq \int_0^1 \left(1 - \frac{1}{3}x^{20}\right) dx.$$

17. Consider the integral $\int_1^3 \sin x dx$. Estimate the errors in the trapezoidal and Simpson's approximations if the interval is partitioned into (a) 10, (b) 20 subintervals.

18. Let $\sinh^{-1}$ be the inverse function of the hyperbolic function $\sinh$.

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

(a) What are the domain and range of $\sinh^{-1}$?

(b) Show that $D \sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$.

(c) Show that $\sinh^{-1} x = \log\left(x + \sqrt{1+x^2}\right)$.

19. Which is larger e^π or π^e? Prove.

20. For what value of the constant a does

$$\lim_{x \to 0} \left(x^{-3} \sin x + a x^{-2}\right)$$

exist? What is the value of the limit in that case? Explain.

21. Let $f(x) = \log(1+x) - x$, $x > -1$.

(a) Find the Taylor polynomial $p_4(x)$ of degree 4 in powers of x for $f(x)$ (i.e. $a = 0$).

(b) Give explicit upper and lower bounds for the remainder

$$r_4(x) = f(x) - p_4(x), \text{ if } 0 \leq x \leq 1.$$

Your *final* form for the bounds should not involve an undetermined quantity c.

22. (a) Show that the equation $x + \log x = 0$ has one root.

(b) PLAN the location of x_0 and the number of Newton iterates for the solution of the equation so that the error is less than 10^{-5}.

23. (a) Show that the equation $x + \log 4x = 0$ has two roots.

(b) PLAN the location of x_0 and the number of Newton iterates for the computation of the smaller of the two roots so that the error is less than 10^{-5}.

24. A solid has a circular base of radius a in the $(x, y)-$plane. If each section of the solid by a plane perpendicular tothe $x-$axis is an equilateral triangle, show that the volume of the solid is $\frac{4}{\sqrt{3}}a^3$.

25. The area bounded by the curve $y^2 = 4ax$ and the line $x = a$ is rotated rigidly about the line $x = 2a$ $(a > 0)$. Find the volume generated. ANS: $112\pi a^3/13$.

26. Two circles have a common diameter and lie in perpendicular planes. A square moves in such a way that its plane is perpendicular to this diameter and its diagonals are chords of the circles. Find the volume of the solid generated. ANS: $\frac{8}{3}r^3$, where r is the radius of the circles.