Math 201 FAQ

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1. Power Series Solutions

1.1. Why is a power series sometimes referred to as "formal"?.

Ans.

A power series is a combination of numbers a_0, a_1, ..., x_0 and a symbol x in the following particular manner:

$$a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots$$

or equivalently $\sum_{n=0}^{\infty} a_n (x - x_0)^n$.

Such an infinite sum is often called "formal" because

1. At this stage, without assigning any value to x, it is just a particular way of writing the numbers $a_0, a_1, ..., x_0$ and the symbol x together, nothing more. In particular it does not represent anything such as a "function of x".
2. Even if we assign some value to x, say let $x = 1$, the resulting infinite sum of numbers

   $$a_0 + a_1 (1 - x_0) + a_2 (1 - x_0)^2 + \cdots$$

   still may or may not converge. When it converges, it represents a number; When it does not, it represents nothing.

In summary, a power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ represents a function only inside some interval $|x| < \rho$. Outside, the meaning of $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is not clear and is thus purely formal (Many researchers have been trying to give meaning to the sum for x outside the convergence disk. Such research results in many "named sums": Cesaro sum, Borel sum, etc.)

1.2. Why is the "radius of convergence" the distance from x_0 to its closest *complex* singular point?.

Ans.

First the distance from x_0 to its closest complex singular point is just a lower bound of the radius of convergence of the power series solution – meaning the radius is at least as large.

A full understanding of the "why" involves tedious calculation which can fill a couple of pages. But quick "pseudo-understanding" may be achieved through the following.

Consider a power series $y = \sum a_n (x - x_0)^n$ which solves a linear differential equation whose singular points are $z_1, ..., z_k$,.

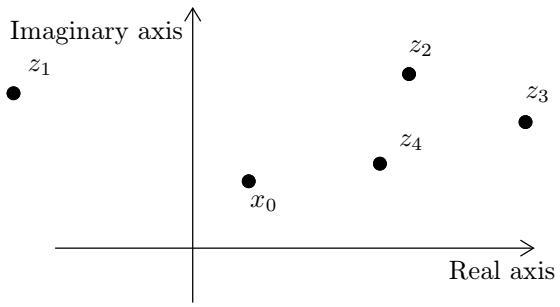

Figure 1. (Hypothetical) distribution of singular points

First, it is crucial to think of the x in a power series not as only a real number, but a complex number. Thus for any complex number c, we can set $x = c$ in the formal power series and obtain an infinite sum of numbers:

$$\sum a_n (c - x_0)^n.$$

Next, the fact is that, for each value c, this infinite sum either converges or does not converge. It turns out that, for all c such that $|c-x_0|<\rho$, the sum converges, while for all c such that $|c-x_0|>\rho$, the sum diverges. As a consequence, all the c's in the complex plane with the sum converges form a disk centered at x_0 with radius ρ – hence the name "radius of convergence" (If we only consider the real line, "distance of convergence" would be a better name!).

It turns out that, this converging disk can be at least so large that no singular point is inside it. Therefore the best we can do is to "expand" this disk until its boundary "touches" a singular point:

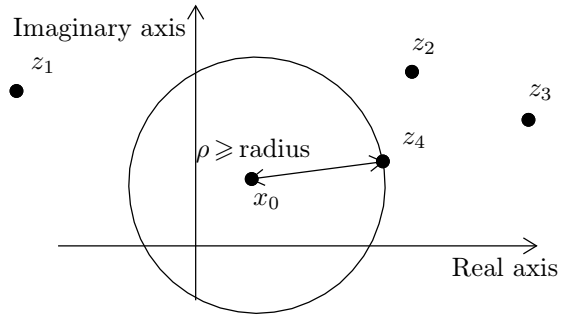

Figure 2. Largest disk we can have without including any singular point.

1.3. Is the radius of convergence useful in practice?.

Ans.

Indeed it is. For example, suppose after some great effort we have found out the first 4 terms of a power series solution

$$y(x)=1+3\,x+5\,x^2+7\,x^3+\cdots$$

and have concluded that it is not possible to write down a general formula for the generic coefficient a_n.

Thus we have to get some idea of $y(x)$ using the first 4 terms. Naïvely we expect $y(x)$ is close to $1+3\,x+5\,x^2+7\,x^3$. But how confident are we? Do we have any idea for which x this is true and for which x this is not? We have no idea.

Things change when we have the extra information of radius of convergence. Say we found out that $\rho=2$. Now we can conclude:

1. For $|x|>2$, it's not a good idea to use $1+3\,x+5\,x^2+7\,x^3$ to approximate $y(x)$;
2. For $|x|<2$, such approximation is possible. Furthermore the smaller $|x|$ is, the better the approximation.

2. Fourier Series and Related

2.1. What is an eigenvalue problem? What is an eigenvalue?.

Ans.

An "eigenvalue problem" is a linear, homogeneous boundary value problem involving one unknown number. For example

$$X''+3\,x\,X'-(5+\lambda)\,X=0;\qquad X(0)=0;\quad X'(1)+3\,X(1)=0$$

is an eigenvalue problem. So an "eigenvalue problem" is in fact a collection of **infinitely many** boundary value problems.

If we assign a number to λ, the eigenvalue problem "collapses" to a usual boundary value problem. For example, if we set $\lambda=1$, the above problem "collapses" to

$$X''+3\,x\,X'-6\,X=0;\qquad X(0)=0;\quad X'(1)+3\,X(1)=0.$$

As an "eigenvalue problem" is linear and homogeneous, $X=0$ is always a solution, no matter what number λ is assigned. On the other hand, there usually exist a bunch of special numbers such that, when assigned to λ, the resulting boundary value problem has (besides $X=0$) non-zero solutions. These "special numbers" are called "eigenvalues".

For example, consider the eigenvalue problem:

$$X'' - \lambda X = 0; \qquad X(0) = X(1) = 0.$$

We try:

- $\lambda = 0$. The problem becomes $X'' = 0$; $X(0) = X(1) = 0$. Solving it, we see that the only solution is $X = 0$. So 0 is not an eigenvalue for the problem.
- $\lambda = -\pi^2$. The problem becomes $X'' + \pi^2 X = 0, X(0) = X(1) = 0$ which has, besides $X = 0$, at least one non-zero solution $\sin(\pi x)$ (as well as $2\sin(\pi x)$, $5\sin(\pi x)$ and in fact $C\sin(\pi x)$ for arbitrary number C). Therefore $-\pi^2$ is an eigenvalue for the problem.

A word of caution here: An eigenvalue problem consists of three parts: an equation (involving λ) and two boundary conditions. *Slight change to any one part of the three may lead to big change in the looks of eigenvalues/eigenfunctions as well as the range of* n!

2.2. Why are eigenvalues that are larger than zero insignificant?.
Ans.

They are **not** insignificant. All eigenvalues are significant, larger than zero or not. Following N. Trefethen, we can say the set of all eigenvalues is the "signature" of the differential equation. We only see non-positive eigenvalues in class because we have only solved a couple of the simplest eigenvalue problems. It's purely accidental that there is no positive eigenvalue for these problems. If we have chance to see more sophsticated ones, there will be eigenvalues of both signs.

It should be emphasized that **the question itself is not correct**. For the eigenvalue problems we dealt in class, any $\lambda > 0$ cannot be an eigenvalue. So it's not that "eigenvalues ... larger than zero insignificant", but "no eigenvalues larger than zero at all".

2.3. In the case $\lambda = 0$, how would you ever get an eigenvalue? Your general solution does not even have a λ!.
Ans.

Recall that an eigenvalue is just a number such that, if λ is set to this number, the resulting boundary value problem has non-zero solutions. So the whole discussion of the case $\lambda = 0$ is just checking whether 0 is an eigenvalue or not: If we set $\lambda = 0$ in the problem, does the resulting boundary value problem have any non-zero solution? If the answer is yes, then 0 is an eigenvalue; If the answer is no, then 0 is not an eigenvalue.

For example, consider the eigenvalue problem

$$X'' - \lambda X = 0; \qquad X(0) = X(1) = 0.$$

If we set $\lambda = 0$, the problem becomes $X'' = 0, X(0) = X(1) = 0$ which gives $X = 0$ as the only solution. So 0 is not an eigenvalue for this problem.

On the other hand, if we consider a different eigenvalue problem

$$X'' - \lambda X = 0; \qquad X'(0) = X'(1) = 0.$$

Setting $\lambda = 0$ gives $X'' = 0, X'(0) = X'(1) = 0$ which indeed has non-zero solutions, for example $X(x) = 1$. So 0 is an eigenvalue for this problem.

2.4. Why do we discuss three cases $\lambda > 0, = 0, < 0$?.
Ans.

First it should be emphasized that this only happens when the equation in our problems if $X'' - \lambda X = 0$. If we change the equation, the cases will be different.

To understand why, we track how we find eigenvalues.

- The basic idea of finding eigenvalues is to try every number: Assign it to λ and see whether any non-zero solution exists;
- Naïve implementation of this strategy clearly would not work. Instead, we observe that, for the problem with equation $X'' - \lambda X = 0$, all possible values of λ can be categorized into three cases: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$. In each case, we can write down a formula for the solution X which is true for all λ in that case. For example, no matter what number we assign to λ, as long as it's positive, X must take the form $C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$.

2.5. Why is there generally still a constant in the eigenfunction? Will this always be the case?.

Ans.

Yes this will always be the case. There will always be arbitrary constants in the formulas for eigenfunctions, and the number of such constants can be any positive number: one, two, three...

To understand why, we take a look at the eigenvalue problems we have solved:

$$X''-\lambda X=0; X(0)=X(L)=0; \Longrightarrow X_n=A_n\sin\Big(\frac{n\,\pi\,x}{L}\Big);$$

$$X''-\lambda X=0; X'(0)=X'(L)=0; \Longrightarrow X_n=A_n\cos\Big(\frac{n\,\pi\,x}{L}\Big);$$

$$X''-\lambda X=0; X(0)=X(L); X'(0)=X'(L) \Longrightarrow X_n=A_n\cos\Big(\frac{n\,\pi\,x}{L}\Big)+B_n\sin\Big(\frac{n\,\pi\,x}{L}\Big);$$

$$X''-\lambda X=0; X(0)=0; X(\pi)+X'(\pi)=0 \Longrightarrow X_n=A_n\sin(p_n\,x);$$

in the last one p_n are the roots of the transcendental equation $-p=\tan(p\,\pi)$. Also notice that in the 3rd problem two arbitrary constants are involved in the formula of the eigenfunction.

All these problems are *linear and homogeneous*, which means if X_1, X_2 solves the problem, so is $C_1\,X_1+C_2\,X_2$, where C_1, C_2 are arbitrary constants. This property is enjoyed by all eigenvalue problems. As a consequence, if there is any nonzero solution to the problem, then automatically its constant multiples are also solutions.

2.6. When solving the eigenvalue problems, why do we sometimes write $\sqrt{-\lambda}$ instead of $\sqrt{\lambda}$?.

Ans.

The reason is we would like to write every complex number in its standard form $a+b\,i$ where a, b are real. For example, if we have $\sqrt{-1}$, we usually write it as $\sqrt{-(-1)}\,i=i$ instead of just $\sqrt{-1}$.

So when $\lambda>0$, we just write $\sqrt{\lambda}$ as this is a real number; But when $\lambda<0$, we prefer writing $\sqrt{-\lambda}\,i$ over $\sqrt{\lambda}$ because the latter is not in "standard form".

2.7. Why is the factor $2/L$ for Fourier cosine and Fourier sine, but $1/L$ for Fourier?.

Ans. The reason lies in that all three are special cases of orthogonal systems. $\Big\{1, \cos\frac{\pi\,x}{L}, \cos\frac{2\,\pi\,x}{L}, ...\Big\}$ and $\Big\{\sin\frac{\pi\,x}{L}, \sin\frac{2\,\pi\,x}{L}, ...\Big\}$ are orthogonal systems with weight $w(x)=1$ over the interval $0<x<L$, while $\Big\{1, \cos\frac{\pi\,x}{L}, \sin\frac{\pi\,x}{L}, \cos\frac{2\,\pi\,x}{L}, \sin\frac{2\,\pi\,x}{L}, ...\Big\}$ is an orthogonal system with weight $w(x)=1$ over the interval $-L<x<L$ (note the interval is different!).

If $\{X_n\}$ is an orthogonal system over $a<x<b$ with weight w, then the coefficients of the expansion

$$f\sim\sum c_n\,X_n$$

can be found through

$$c_n=\frac{\int_a^b f\,X_n\,w\,\mathrm{d}x}{\int_a^b X_n^2\,w\,\mathrm{d}x}.$$

Now we have

$$\frac{1}{\int_0^L\big[\cos\big(\frac{n\,\pi\,x}{L}\big)\big]^2\,\mathrm{d}x}=\frac{1}{\int_0^L\big[\sin\big(\frac{n\,\pi\,x}{L}\big)\big]^2\,\mathrm{d}x}=\frac{2}{L};\quad \frac{1}{\int_{-L}^L\big[\cos\big(\frac{n\,\pi\,x}{L}\big)\big]^2\,\mathrm{d}x}=\frac{1}{\int_{-L}^L\big[\sin\big(\frac{n\,\pi\,x}{L}\big)\big]^2\,\mathrm{d}x}=\frac{1}{L}.$$

which explains the different factors.

2.8. Why is the constant term written as $\frac{a_0}{2}$ in Fourier and Fourier cosine series?.

Ans.

Consider the Fourier series, where any f is expanded with respect to the orthogonal system $\big\{1, \cos\frac{\pi\,x}{L}, \sin\frac{\pi\,x}{L}, ...\big\}$ (which is orthogonal over $-L<x<L$ with weight $w=1$). From the theory of orthogonal systems, we know that if we write

$$f\sim a_0\cdot 1+a_1\cos\frac{\pi\,x}{L}+b_1\sin\frac{\pi\,x}{L}+\cdots$$

then the coefficients are given by

$$a_0 = \frac{\int_{-L}^{L} f \cdot 1 \, \mathrm{d}x}{\int_{-L}^{L} 1^2 \, \mathrm{d}x} = \frac{1}{2L} \int_{-L}^{L} f \, \mathrm{d}x;$$

$$a_1 = \frac{\int_{-L}^{L} f \cos\frac{\pi x}{L} \, \mathrm{d}x}{\int_{-L}^{L} \left(\cos\frac{\pi x}{L}\right)^2 \mathrm{d}x} = \frac{1}{L} \int_{-L}^{L} f \cos\frac{\pi x}{L} \, \mathrm{d}x;$$

$$b_1 = \frac{\int_{-L}^{L} f \sin\frac{\pi x}{L} \, \mathrm{d}x}{\int_{-L}^{L} \left(\sin\frac{\pi x}{L}\right)^2 \mathrm{d}x} = \frac{1}{L} \int_{-L}^{L} f \sin\frac{\pi x}{L} \, \mathrm{d}x;$$

and so on. We see that the formulas for all a_n except a_0 can be written as

$$a_n = \frac{1}{L} \int_{-L}^{L} f \cos\frac{n\pi x}{L} \, \mathrm{d}x;$$

This is not beautiful. To make things look better, instead of $f \sim a_0 \cdot 1 + a_1 \cos\frac{\pi x}{L} + b_1 \sin\frac{\pi x}{L} + \cdots$ we write $f \sim \frac{a_0}{2} \cdot 1 + a_1 \cos\frac{\pi x}{L} + b_1 \sin\frac{\pi x}{L} + \cdots$ so that the new a_0 is the same as two times the old one, and can be computed through

$$a_0 = \frac{1}{L} \int_{-L}^{L} f \cos\left(\frac{0\pi x}{L}\right) \mathrm{d}x.$$

2.9. Are Fourier cosine and Fourier sine series special cases of Fourier series?.
Ans. No.

- On the theoretical side, all three (Fourier cosine, Fourier sine and Fourier) are special cases of orthogonal systems arising from solving eigenvalue problems.
  - $X'' - \lambda X = 0, X(-L) = X(L), X'(-L) = X'(L)$ gives Fourier;
  - $X'' - \lambda X = 0, X(0) = X(L) = 0$ gives Fourier sine;
  - $X'' - \lambda X = 0, X'(0) = X'(L) = 0$ gives Fourier cosine.

  No one is at a higher level or "more general" than another.
- On the other hand, on the practical side, one can obtain the coefficients in a Fourier cosine or Fourier sine expansion of a certain function f by computing the coefficients of the Fourier expansion of **another** function which is related to f through:
  - If the Fourier cosine expansion of f is

    $$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

    then the a_n's turn out to be the same as those a_n's in the Fourier expansion of the even extension of f,

    $$f_e = \begin{cases} f(x) & 0 < x < L \\ f(-x) & -L < x < 0 \end{cases}.$$

  - If the Fourier sine expansion of f is

    $$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

    then the b_n's turn out to be the same as those b_n's in the Fourier expansion of the odd extension of f.

    $$f_o = \begin{cases} f(x) & 0 < x < L \\ -f(-x) & -L < x < 0 \end{cases}.$$

Such properties can be used to analyze the convergence properties of the Fourier cosine and Fourier sine expansions (although such detour becomes obsolete once one learns the full Sturm-Liouville theory).

3. Separation of Variables

3.1. How do I know when it's Fourier Cosine and when it's Fourier Sine?.
Ans.

The short answer is, you know automatically from solving the eigenvalue problem. If after solving the eigenvalue problem, you get $\cos\big(\frac{n\,\pi\,x}{L}\big)$ for certain L (**usually** given in the problem in the form $0<x<L$ – it can be given in other forms), then the initial value should be expanded into Fourier cosine series; If you get $\sin\big(\frac{n\,\pi\,x}{L}\big)$, Fourier sine.

The above guarantees quick reaction in exams. But a quiet mind and fearless heart can be reached through understanding the reason behind this mess. The fundamental reason is the following:

- The eigenfunctions X_n form an orthogonal system with certain weight $w(x)$, which means $\int_0^L X_n\,X_m\,w\,\mathrm{d}x=0$ whenever $n\neq m$. Consequently the expansion of any function f into these eigenfunctions
$$f\sim\sum c_n\,X_n$$
can be computed through
$$c_n=\frac{\int_0^L f\,X_n\,w\,\mathrm{d}x}{\int_0^L X_n^2\,\mathrm{d}x}.$$
When the eigenvalue problem is $X''-\lambda\,X=0$ + boundary conditions, the weight function w is always the constant function 1.
- Therefore, when the eigenfunctions are $\cos\big(\frac{n\,\pi\,x}{L}\big)$, the coefficients are given by
$$a_n=\frac{\int_0^L f(x)\cos\big(\frac{n\,\pi\,x}{L}\big)\,\mathrm{d}x}{\int_0^L\cos\big(\frac{n\,\pi\,x}{L}\big)^2\,\mathrm{d}x}=\frac{2}{L}\int_0^L f\cos\Big(\frac{n\,\pi\,x}{L}\Big)\,\mathrm{d}x.$$