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Transmission and Reflection of Three-Dimensional Internal Gravity Wave Packets in Nonuniform Retrograde Shear Flow

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AMI Seminar, March 18, 2022

Outline of today's talk

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- What is a stratified fluid?
- Waves in a stratified fluid
- Wave-induced mean flow
- Waves in a stratified fluid with background shear: Linear & nonlinear theory
- 3-D wave packets in retrograde background shear flow
  - Numerics
  - Results

Stably stratified fluids

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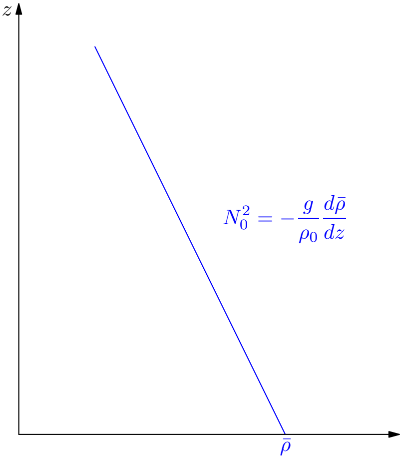
- A fluid in which "effective density" decreases with height



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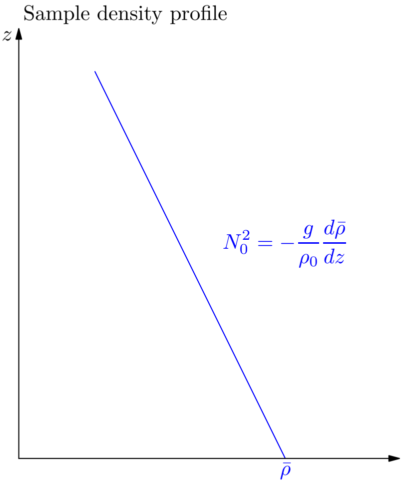
- If the change is continuous, the fluid is "continuously stratified"



Stably stratified fluids

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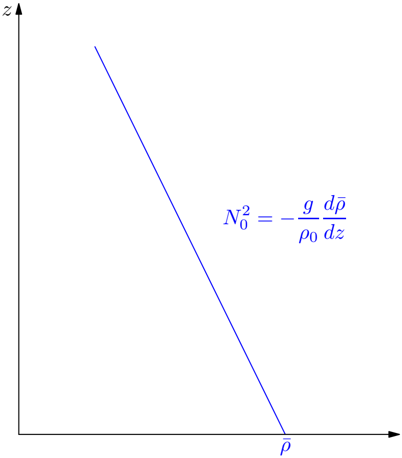
- A fluid in which "effective density" decreases with height
- If the change is continuous, the fluid is "continuously stratified"
- Example: Idealized mass density profile and squared buoyancy frequency, N_0^2

Sample density profile



Waves in stratified fluids

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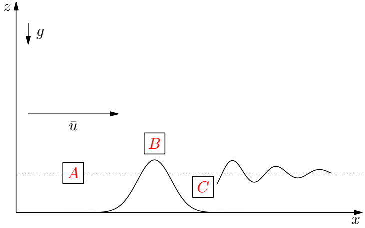
- A: Fluid is advected by $\bar{u}$ at its equilibrium level

Orographic wave generation



Waves in stratified fluids

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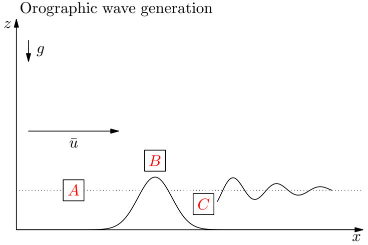
- A: Fluid is advected by $\bar{u}$ at its equilibrium level
- B: The parcel of fluid encounters topography, and is forced to rise



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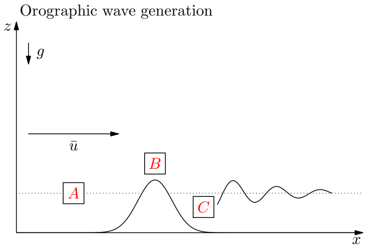
- C: Gravity forces the (relatively dense) fluid to descend below its equilibrium level



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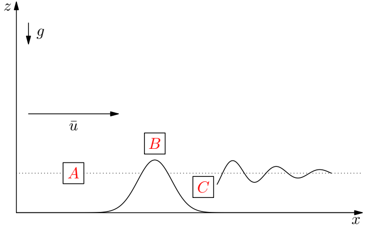

Waves in stratified fluids

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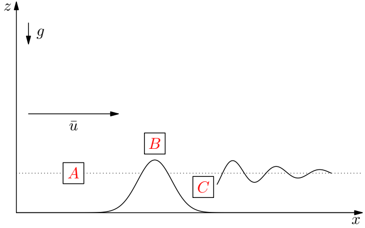
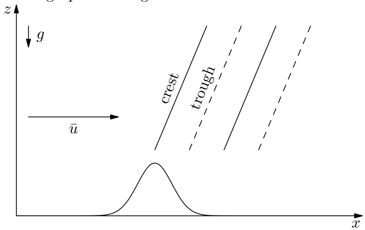
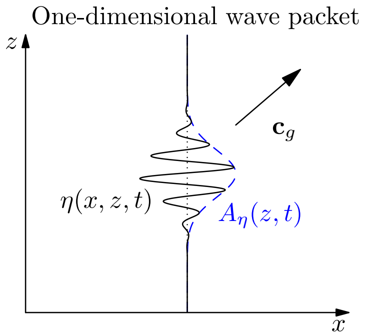
- Buoyancy and gravity alternately cause the fluid parcel to oscillate until it returns to its equilibrium level
- The displaced parcel B, in turn, displaces other fluid parcels, resulting in an *internal gravity wave* (IGW)





1-D wave packets

- Spatially localized groups of waves that travel together with "group speed" $\mathbf{c}_g = (c_{gx}, c_{gz})$



1-D wave packets

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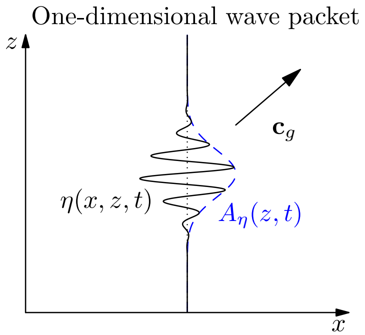
- Spatially localized groups of waves that travel together with "group speed" $\mathbf{c}_g = (c_{gx}, c_{gz})$
- Conveniently expressed as a complex exponential:

$$\eta(\mathbf{x}, t) = \frac{1}{2} A_\eta(\mathbf{x}, t) e^{i(\mathbf{k}_0 \cdot \mathbf{x} - \omega_0 t)} + \text{c.c.}$$

with amplitude "envelope" A_η, wavenumber vector $\mathbf{k}_0 = (k_0, \ell_0, m_0) = 2\pi(\lambda_x^{-1}, \lambda_y^{-1}, \lambda_z^{-1})$, and frequency ω_0, where

$$\omega_0 = \frac{N_0 \sqrt{k_0^2 + \ell_0^2}}{\sqrt{k_0^2 + \ell_0^2 + m_0^2}}$$



$$c_{gx} = \frac{\partial \omega_0}{\partial k_0}, \quad c_{gz} = \frac{\partial \omega_0}{\partial m_0}$$

and $c_{gy} = 0$ if restricted to xz-plane ($\Rightarrow \ell_0 = 0$)

Motivation

Internal gravity waves (IGWs):

- exist on a range of spatial and temporal scales
- parameterized using resolved (grid-scale) variables in operational weather and climate GCMs
- parameterization schemes often rely on theory of linear (i.e., small amplitude) monochromatic waves (Lindzen, J. Geophys. Res., 1981)
  - do not account for weakly or fully nonlinear effects
  - can be better constrained with improved understanding of nonlinear IGW evolution

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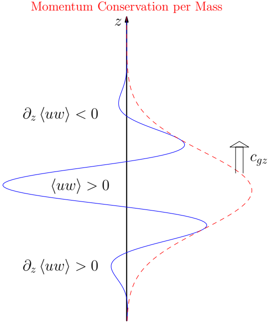
How does the nonlinear interaction between a 3-D wave packet and its induced mean flow affect wave packet transmission and reflection in a retrograde background flow?

Momentum conservation & "divergent-flux" induced flow

- The flux-form of x-momentum eqn: $\partial_t u = -\partial_x(uu) - \partial_z(uw) - \partial_x(p/\rho_0)$
  where at $O(\alpha^1\epsilon^0)$ have
  - $u^{(1)} = \frac{1}{2}\underbrace{(i\omega_0 m_0/k_0)A}_{A_u}\, e^{i(k_0x+m_0z-\omega_0 t)} + \text{c.c.}$,
  - $w^{(1)} = \frac{1}{2}\underbrace{(-i\omega_0)A}_{A_w}\, e^{i(k_0x+m_0z-\omega_0 t)} + \text{c.c.}$.
- Average in x: $\partial_t \langle u\rangle = -\partial_z \langle uw\rangle = -\partial_z\left(-\frac{1}{2}\frac{\omega_0^2 m_0}{k_0}|A|^2\right)$



e.g., Sutherland (2010)

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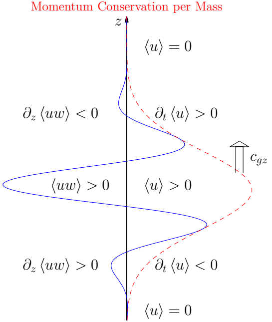
- Average in x: $\partial_t \langle u\rangle = -\partial_z \langle uw\rangle = -\partial_z\left(-\frac{1}{2}\frac{\omega_0^2 m_0}{k_0}|A|^2\right)$
- Suppose $A \equiv A(\epsilon\underbrace{(z-c_{gz}t)}_{Z}, \epsilon^2 t)$ with $\epsilon \equiv (k_0\sigma)^{-1}$.
- Expect $u_{\mathsf{DF}} \equiv \langle U\rangle(\epsilon Z, \epsilon^2 T)$, so at $O(\alpha^2\epsilon)$:

$$\partial_t \langle u\rangle = -\partial_z\langle uw\rangle \;\rightarrow\; -c_{gz}\partial_Z u_{\mathsf{DF}} = -\partial_Z\langle uw\rangle$$

Hence the divergent-flux induced flow of internal waves is

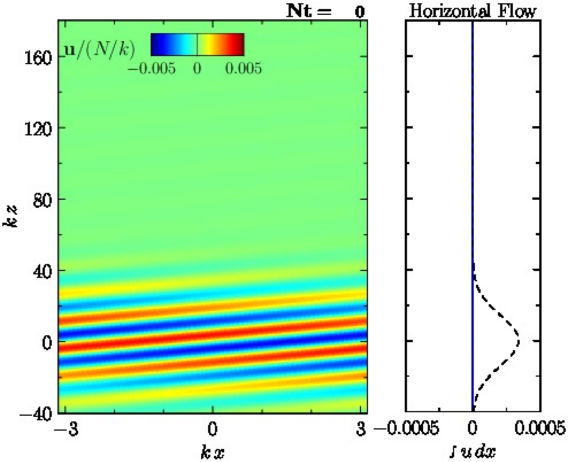
$$u_{\mathsf{DF}} = \frac{1}{2c_{gz}}|A_u A_w^\star| = \frac{1}{2}N_0\|\mathbf{k}_0\||A|^2$$



e.g., Sutherland (2010)

Wave-induced flow of horizontally periodic internal waves

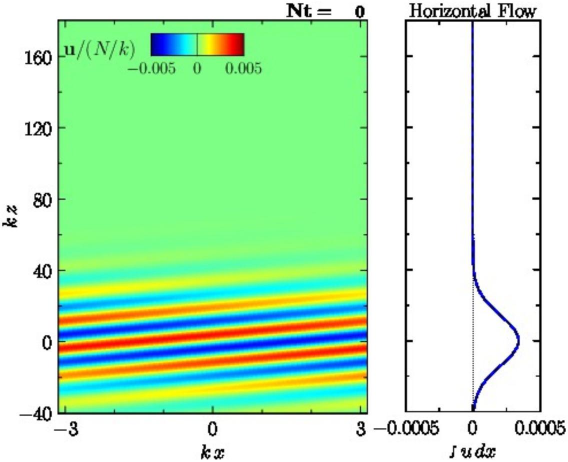
- For a wave packet initialized in an otherwise stationary ambient, a positive jet develops that translates upward with the wave packet while a negative jet develops that remains centred about the initial position of the wave packet



Courtesy of Bruce Sutherland

Wave-induced flow of horizontally periodic internal waves

- For a wave packet initialized with its predicted induced flow superimposed, the induced flow (an existing positive jet) translates upward with the wave packet and no negative flow remains at the initial wave packet location



Courtesy of Bruce Sutherland

Flow induced by 3-D wave packets – "Bretherton flow"

- A 3-D wave packet is localized in all 3 spatial dimensions

van den Bremer & Sutherland, Phys. Fluids. (2018)

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- “Mean” forcing is the contribution from, e.g.,

$$\eta^2 = \left(\frac{1}{2}A_\eta e^{i\varphi} + \frac{1}{2}A_\eta^\star e^{-i\varphi}\right)^2 = \frac{1}{4}A_\eta^2 e^{i2\varphi} + \frac{1}{2}|A_\eta|^2 + \frac{1}{4}A_\eta^{\star 2} e^{-i2\varphi}$$

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- Take curl of 3-D momentum equations, and take mean of the result:

$$\underbrace{\begin{bmatrix} 0 & -\partial_{ttz} & (\partial_{tt} + N_0^2)\partial_y \\ \partial_{ttz} & 0 & -(\partial_{tt} + N_0^2)\partial_x \\ -\partial_{tty} & \partial_{ttx} & 0 \end{bmatrix}}_{\equiv \mathsf{L}} \mathbf{u}_{\mathsf{BF}}$$

$$= \underbrace{\left\langle \nabla \cdot \left\{ -\partial_t(\zeta \otimes \mathbf{u}) - N_0^2\left[(\hat{\mathbf{e}}_y \otimes \partial_x(\xi\mathbf{u})) - (\hat{\mathbf{e}}_x \otimes \partial_y(\xi\mathbf{u}))\right]\right\} + \partial_t(\zeta \cdot \nabla\mathbf{u})\right\rangle}_{\equiv \mathbf{F} = (F_x, F_y, F_z)^\top}$$

van den Bremer & Sutherland, Phys. Fluids. (2018)

Flow induced by 3-D wave packets – "Bretherton flow"

- The induced (Bretherton) flow is forced by the vertical component of vorticity, so take the third row of $\mathbf{L}\mathbf{u}_{\mathrm{BF}} = \mathbf{F}$, i.e., $\partial_{tt}(\nabla \times \mathbf{u}_{\mathrm{BF}}) \cdot \hat{\mathbf{e}}_z = F_z$

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- Suppose

$$A \equiv A(\underbrace{\epsilon_x(x - c_{gx}t)}_{=X}, \underbrace{\epsilon_y y}_{=Y}, \underbrace{\epsilon_z(z - z_0 - c_{gz}t)}_{=Z}, \underbrace{\epsilon^2 t}_{=T})$$

with $\epsilon_{x,y,z} = (k_0\sigma_{x,y,z})^{-1}$, and $\epsilon = \max\{\epsilon_x, \epsilon_y, \epsilon_z\}$, and there is no wave propagation in the y-direction

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  with $\epsilon_{x,y,z} = (k_0\sigma_{x,y,z})^{-1}$, and $\epsilon = \max\{\epsilon_x, \epsilon_y, \epsilon_z\}$, and there is no wave propagation in the y-direction
- At leading non-zero order, $w_{\text{BF}} = 0$ and $F_z = (F_z)_3^{(2)} \sim O(\alpha^2\epsilon^3)$ with $(F_z)_3^{(2)} = \partial_t(\nabla \times \partial_t \mathbf{u}_{\text{DF}}) \cdot \hat{\mathbf{e}}_z + \partial_{tt}(\nabla \times \mathbf{u}_{\text{RF}}) \cdot \hat{\mathbf{e}}_z$

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- At leading non-zero order, $w_{\mathsf{BF}} = 0$ and $F_z = (F_z)_3^{(2)} \sim O(\alpha^2 \epsilon^3)$ with $(F_z)_3^{(2)} = \partial_t(\nabla \times \partial_t \mathbf{u}_{\mathsf{DF}}) \cdot \hat{\mathbf{e}}_z + \partial_{tt}(\nabla \times \mathbf{u}_{\mathsf{RF}}) \cdot \hat{\mathbf{e}}_z$
- Lastly, note the response flow is irrotational: $\partial_{tt}(\nabla \times \mathbf{u}_{\mathsf{RF}}) \cdot \hat{\mathbf{e}}_z \equiv 0$ (Bretherton, J. Fluid. Mech., 1969)

van den Bremer & Sutherland, Phys. Fluids. (2018)

Flow induced by 3-D wave packets – "Bretherton flow"

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- Use incompressibility $\nabla \cdot \mathbf{u}_{\mathsf{BF}} = 0$, and Fourier transforms to solve for u_{BF} and v_{BF}:

$$\begin{bmatrix} u_{\mathsf{BF}} \\ v_{\mathsf{BF}} \end{bmatrix} = \frac{1}{2} N_0 \|\mathbf{k}_0\| \int_{\mathbb{R}^3} \frac{\lambda}{\kappa^2 + \lambda^2} \begin{bmatrix} \lambda \\ -\kappa \end{bmatrix} \widehat{|A|^2} e^{i\boldsymbol{\kappa}\cdot\mathbf{X}} d\kappa \, d\lambda \, d\mu$$

where $\widehat{|A|^2}$ is the Fourier transform of the square of

$$A(\mathbf{x}, 0) = A_0 \exp\left[-\frac{1}{2} \left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{(z - z_0)^2}{\sigma_z^2} \right) \right]$$

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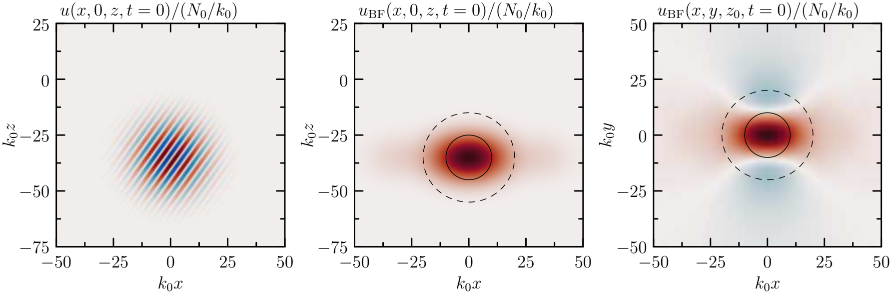
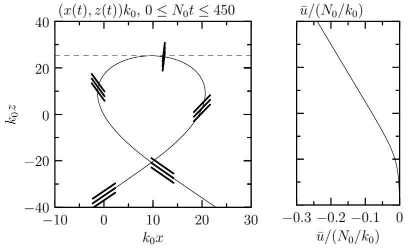


Waves in background shear flow I

- The wave frequency ω_0 of small amplitude waves is Doppler-shifted by the background flow $\bar{u}$:

$$\Omega(z) = \omega_0 - k_0\bar{u}(z)$$



$$\bar{u}(z) = \frac{sL_s}{2}\left[\ln(1 + e^{-2(z-z_s)/L_s}) + 2\frac{z - z_s}{L_s}\right]$$

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$$s < 0$$

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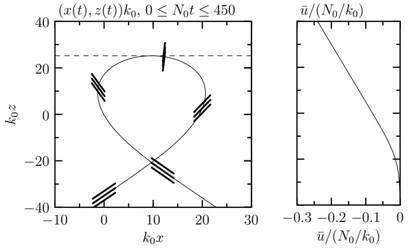

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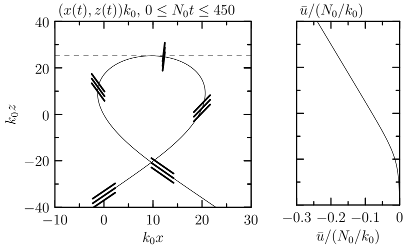
- Wave reflection is predicted at the height z_r where $\Omega(z_r) = N_0 = \sqrt{-\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz}}$
- The path $(x(t), z(t))$ of a wave packet through the fluid is predicted by

$$\frac{dx}{dt} = c_{gx}(m) + \bar{u}(z)$$
$$\frac{dz}{dt} = c_{gz}(m)$$
$$m(z) = \text{sign}(m)k_0\sqrt{N_0^2/\Omega^2(z) - 1}$$



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Waves in background shear flow II

Induced mean flow allows wave packet to *penetrate* partially above the reflection level, provided somewhere $|\partial u_{\mathsf{BF}}/\partial z| > |s|$, where s is strength of background shear

Waves in background shear flow II

Induced mean flow allows wave packet to *penetrate* partially above the reflection level, provided somewhere $|\partial u_{\text{BF}}/\partial z| > |s|$, where s is strength of background shear
Use the Gaussian amplitude envelope

$$A(\mathbf{x},0) = \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{(z-z_0)^2}{\sigma_z^2}\right)\right]$$

in the expression for u_{BF}, then:

Waves in background shear flow II

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Induced mean flow allows wave packet to *penetrate* partially above the reflection level, provided somewhere $|\partial u_{\mathsf{BF}}/\partial z| > |s|$, where s is strength of background shear
Use the Gaussian amplitude envelope

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in the expression for u_{BF}, then:

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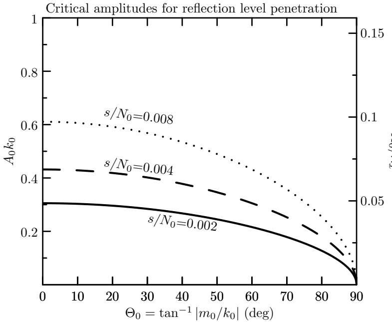
- Substituting this into $|\partial u_{\mathsf{BF}}/\partial z| > |s|$ we predict the critical amplitude for penetration:

$$A_{\mathsf{RP}} = \sqrt{\frac{\sigma_z(1+\sigma_y/\sigma_x)\sqrt{2e}}{\|\mathbf{k}_0\|}\frac{|s|}{N_0}}$$



Conserved quantities: General

To quantify wave transmission above z_r, we must identify a *conserved* quantity, i.e., one whose "action density," $B \sim O(|A|^2)$, satisfies the conservation law

$$\frac{\partial B}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{F}_B,$$

where B is the spatiotemporal action density distribution, and $\mathbf{F}_B \sim O(|A|^2)$ its *flux*

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- the so-called "wave action" quantities are a family of such quantities

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- For small amplitude 3-D wave packets, pseudomomentum density, $\mathcal{P}$, is

$$\mathcal{P}(\mathbf{x}, t) = -\rho_0 \left[\frac{1}{2} \bar{u}'' \xi^2 + \zeta_y \xi \right],$$

where ξ is vertical displacement and $\zeta_y = (\nabla \times \mathbf{u}) \cdot \hat{\mathbf{e}}_y$ is spanwise vorticity (Shaw & Shepherd, J. Fluid Mech., 2008)

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- To quantify transmission $T_{\mathcal{P}}$ above reflection level z_r, we diagnose integrated *upward-propagating pseudomomentum* above z_r:

$$T_{\mathcal{P}}(t) = \frac{1}{\mathscr{P}}\int_{z_r}^{z_{\max}} \underbrace{\left\langle -\rho_0\left[\frac{1}{2}\bar{u}''(H^{\uparrow}\xi)^2 + (H^{\uparrow}\zeta_y)(H^{\uparrow}\xi)\right]\right\rangle}_{\equiv\langle\mathcal{P}^{\uparrow}\rangle} dz,$$

  where $H^{\uparrow}$ is Hilbert filter that extracts upgoing waves, $\langle\cdot\rangle$ is horizontal integral, and $\mathscr{P} = \int_{\mathbb{R}^3}\mathcal{P}(\mathbf{x},0)dxdydz$ is total pseudomomentum

3-D Euler equations to be solved

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- Prognostic equations: Horizontal velocity and perturbation density (recast in terms of vertical displacement, $\xi = -\rho/(d\bar{\rho}/dz)$):

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} - \boldsymbol{\nabla}\cdot(u\mathbf{u}) - \bar{u}\frac{\partial u}{\partial x} - w\frac{d\bar{u}}{dz}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - \boldsymbol{\nabla}\cdot(v\mathbf{u}) - \bar{u}\frac{\partial v}{\partial x}$$

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- Diagnostic equations: Vertical velocity (from $\boldsymbol{\nabla}\cdot\mathbf{u} = 0 \Rightarrow \frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$) and dynamic pressure:

$$\frac{1}{\rho_0}\nabla^2 p = -\left[\frac{\partial^2(u^2)}{\partial x^2} + \frac{\partial^2(v^2)}{\partial y^2} + \frac{\partial^2(w^2)}{\partial z^2}\right] - 2\left[\frac{\partial^2(uv)}{\partial x\partial y} + \frac{\partial^2(uw)}{\partial x\partial z} + \frac{\partial^2(vw)}{\partial y\partial z}\right] - 2\frac{d\bar{u}}{dz}\frac{\partial w}{\partial x} - N_0^2\frac{\partial \xi}{\partial z}$$

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- Gaussian wave packet initial condition:

$$\xi(\mathbf{x},0) = A_0 \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{(z-z_0)^2}{\sigma_z^2}\right)\right]\cos[k_0 x + m_0(z-z_0)]$$

Numerical methods I

Domain and discretization (Gervais et al., Phys. Rev. Fluids, 2021):

- Time and space scales set by $k_0 = 1$ and $N_0 = 1$
- Triply periodic domain $L_x \times L_y \times L_z$
  - $L_x = 2\pi n_{\lambda x}/k_0$
  - $L_z = 2\pi n_{\lambda z}/|m_0|$
  - L_y specified directly
- Discretized so that 16 (32) grid points span one horizontal (vertical) wavelength; y-resolution specified directly

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Problem is well-suited for solving with Fourier transforms:

- It is natural to represent wave packets via their Fourier (wave) spectra
- Direct correspondence between Fourier modes and physical wave properties
- Differentiation (integration) is equivalent to scalar multiplication (division)

Numerical methods II

To damp unphysical growth of power at high wavenumbers ($\kappa, \lambda, \mu \gg 1$) in Fourier spectra of nonlinear terms, code applies an exponential filter after computing RHS of evolution equations (Subich, Lamb, & Stastna, Int. J. Num. Meth. Fluids, 2013):

$$\chi(\kappa) = \begin{cases} 1, & \kappa < \kappa_{\text{cut}} \\ \exp\left[-\alpha \left(\frac{|\kappa| - \kappa_{\text{cut}}}{\kappa_{\text{nyq}} - \kappa_{\text{cut}}} \right)^{\beta} \right], & \kappa \geq \kappa_{\text{cut}} \end{cases}$$

where $\alpha = 20$, $\beta = 2$, and $\kappa_{\text{cut}} = 0.6\kappa_{\text{nyq}}$

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Time stepping: 3rd-order Williamson–Runge–Kutta scheme (Durran, 2010):

$$\dot{\mathbf{u}}_1 \leftarrow F(\mathbf{u}_n, t_n) \qquad \boldsymbol{\phi}_{(1)} \leftarrow \mathbf{u}_n + \frac{\Delta t}{3}\dot{\mathbf{u}}_1$$

$$\dot{\mathbf{u}}_2 \leftarrow F(\boldsymbol{\phi}_{(1)}, t_{n+1/3}) - \frac{5}{9}\dot{\mathbf{u}}_1 \qquad \boldsymbol{\phi}_{(2)} \leftarrow \boldsymbol{\phi}_{(1)} + \frac{15\Delta t}{16}\dot{\mathbf{u}}_2$$

$$\dot{\mathbf{u}}_1 \leftarrow F(\boldsymbol{\phi}_{(2)}, t_{n+5/12}) - \frac{153}{128}\dot{\mathbf{u}}_2 \qquad \mathbf{u}_{n+1} \leftarrow \boldsymbol{\phi}_{(2)} + \frac{8\Delta t}{15}\dot{\mathbf{u}}_1$$

where $\mathbf{u}_n = (u_n, v_n, \xi_n)$, $\dot{\mathbf{u}}_{1,2} = (\dot{u}_n, \dot{v}_n, \dot{\xi}_n) = F(\mathbf{u}_n, t_n) =$ RHS of evolution equations at time $t_n = t_0 + n\Delta t$, and $\boldsymbol{\phi}_{(1)}$, $\boldsymbol{\phi}_{(2)}$ are dummy variables

Subset of simulations performed

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Table: Initial conditions, domain size and resolution, and predicted reflection height and time. . .

ID	A_0k_0	$\frac{A_0}{A_{\rm RP}}$	$\frac{A_0}{A_{\rm SW}}$	$\frac{\lvert s\rvert}{N_0}$	$\lvert\frac{m_0}{k_0}\rvert$	Θ_0	$L_xk_0 \times L_yk_0 \times L_zk_0$
S1	0.01	0.03	0.04	0.002	0.4	22°	$201.1 \times 200 \times 253.1$
S2	0.28	0.95	1.07	0.002	0.4	22°	$402.1 \times 200 \times 502.7$
S3	0.50	1.07	1.21	0.008	1.4	54°	$402.1 \times 200 \times 287.2$

Table: . . . continued

ID	$n_x \times n_y \times n_z$	$(z_r - z_0)k_0$	t_rN_0
S1	$512 \times 256 \times 512$	50.76	248
S2	$1024 \times 256 \times 1024$	50.76	248
S3	$1024 \times 256 \times 2048$	67.35	227

Reflection of a small amplitude wave packet (S1)

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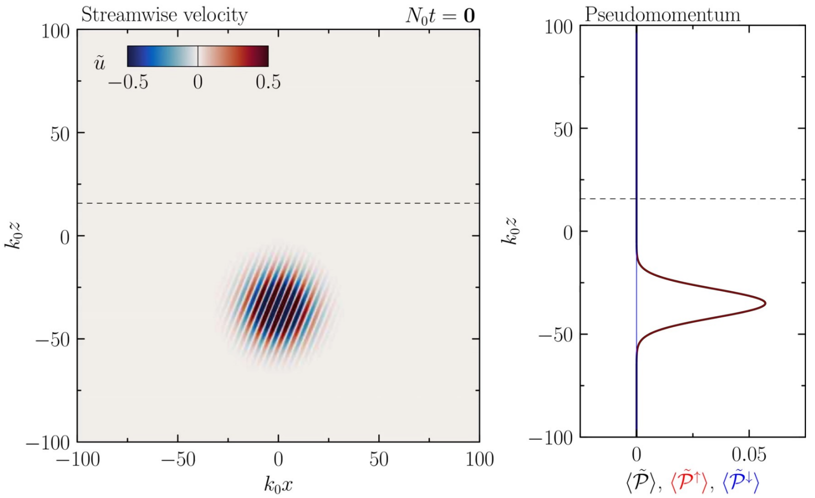
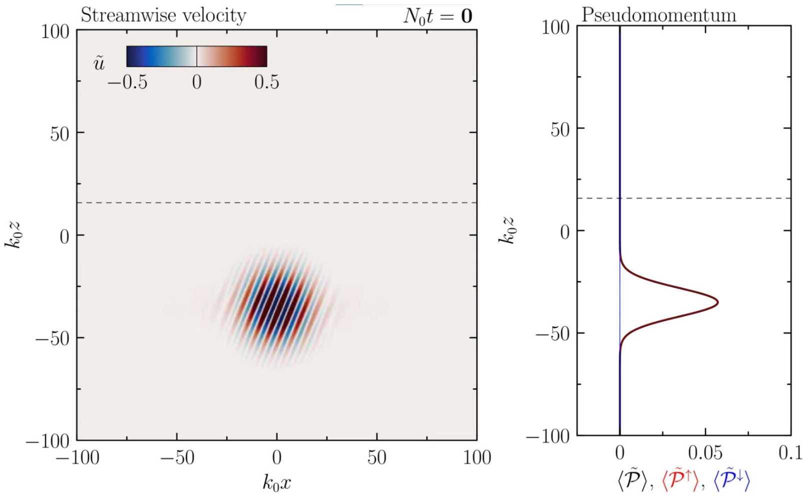
Here, $|s|/N_0 = 0.002$, $m_0/k_0 = -0.4$ ($\Theta_0 \approx 22^\circ$), and $A_0 k_0 = 0.01 \ll A_{\mathrm{RP}} k_0 \approx 0.29$.
Right: Horizontally integrated pseudomomentum density $\langle \mathcal{P} \rangle = -\rho_0 \langle \frac{1}{2} \bar{u}'' \xi^2 + \zeta_y \xi \rangle$



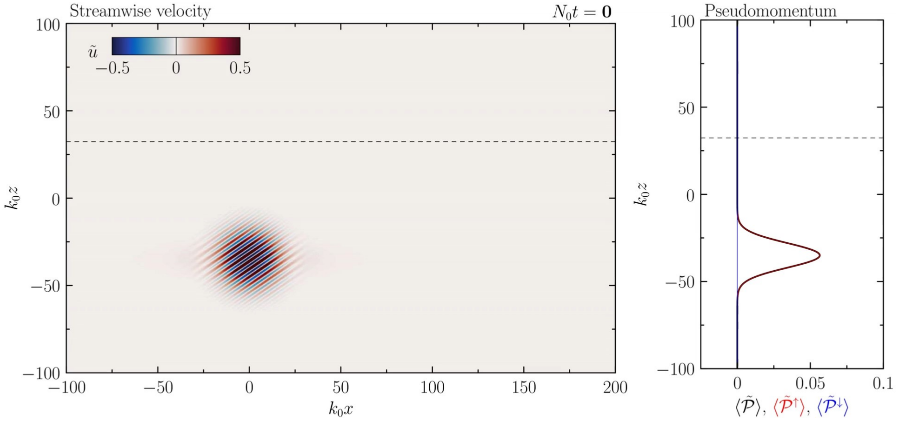
Moderately large amplitude (S2): Transmission & reflection

Here, $|s|/N_0 = 0.002$, $m_0/k_0 = -0.4$ ($\Theta_0 \approx 22^\circ$), and $A_0 k_0 = 0.28 \approx A_{\mathrm{RP}} k_0$ (transmission, reflection, and secondary waves)



Moderately large amplitude (S3): Transmission & reflection

Here, $|s|/N_0 = 0.008$, $m_0/k_0 = -1.4$ ($\Theta_0 \approx 54°$), and $A_0 k_0 = 0.50 \approx 1.07 A_{\mathrm{RP}} k_0$ (transmission, reflection, and secondary waves)



Wave packet transmission

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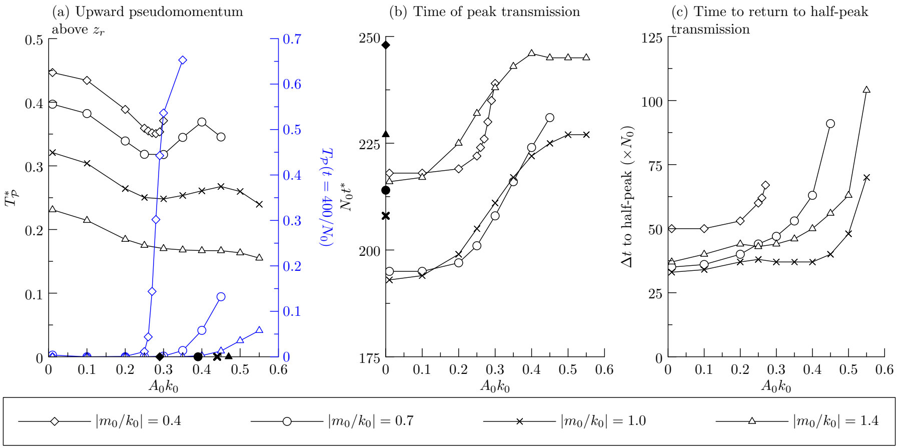
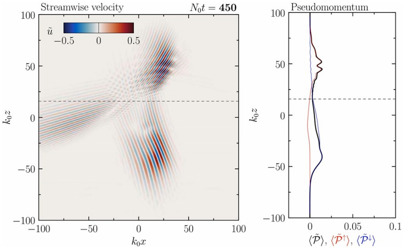


Summary

- Internal gravity waves are disturbances that propagate vertically in stably stratified fluids
- Background flow Doppler-shifts the wave frequency, ultimately leading to reflection of small amplitude waves
- 3-D IGW packets induce a vertically jet-like, horizontally dipole-like mean flow, which can permit the wave packet to transmit partially above the reflection level
- Moderately large amplitude wave packets can partially transmit, reflect, and generate secondary waves
- Current and future work:
    - Manuscript is in final stages of preparation
    - Next: Introduce anelastic effects into theory & numerics



Secondary wave generation

Even if $\bar{u} \equiv 0$, waves can reflect in decreasing stratification

- Can moderately large amplitude waves "self-reflect" in locally decreased stratification due to their own passage?

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$$N_0^2 - N_0^2 \frac{\partial \xi}{\partial z} = \omega_0^2 + 2\omega_0\omega_2|A|^2 + O(|A|^4)$$

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- Assume $\omega_2|A|^2 = k_0 u_{\mathsf{BF}}(0,0,z,0)$
- Minimize $N_0^2(1 - \partial\xi/\partial z)$, solve for critical amplitude: