NAME: ______________________

QUIZ 5

Problem 1. Assume that Ω is bounded and there exists a smooth vector field α such that $\boldsymbol{\alpha} \cdot \boldsymbol{n} \geqslant 1$ along $\partial\Omega$, where $\boldsymbol{n}$ is the outer normal. Assume $1 \leqslant p < \infty$.

Apply the Gauss-Green Theorem (that is "Gauss formula") to $\int_{\partial\Omega} |u|^p \, \boldsymbol{\alpha} \cdot \boldsymbol{n} \, \mathrm{d}S$, to derive a new proof of the trace inequality

$$\int_{\partial\Omega} |u|^p \, \mathrm{d}S \leqslant C \int_U |Du|^p + |u|^p \, \mathrm{d}x. \tag{1}$$

for all $u \in C^1(\overline{\Omega})$. (Hint: You may need Young's inequality $a\, b \leqslant a^p/p + a^{p'}/p'$ for $p, p' \geqslant 1, 1/p + 1/p' = 1$)

Solution. We have

$$\begin{aligned} \int_{\partial\Omega} |u|^p \, \mathrm{d}S &\leqslant \int_{\partial\Omega} |u|^p \, \boldsymbol{\alpha} \cdot \boldsymbol{n} \, \mathrm{d}S & (2) \\ &= \int_\Omega \nabla \cdot (|u|^p \, \boldsymbol{\alpha}) \, \mathrm{d}x & (3) \\ &= \int_\Omega (\nabla \cdot \boldsymbol{\alpha}) \, |u|^p + \boldsymbol{\alpha} \cdot \nabla |u|^p \, \mathrm{d}x & (4) \\ &\leqslant C \int_\Omega |u|^p + |\nabla(|u|^p)| \, \mathrm{d}x. & (5) \end{aligned}$$

Now we compute

$$\nabla(|u|^p) = |u|^{p-1} \, \mathrm{sgn}(u) \, Du. \tag{6}$$

When $p = 1$, the proof ends here.

For $p > 1$, using Young's inequality, we have

$$\int |\nabla(|u|^p)| \, \mathrm{d}x = \int |u|^{p-1} \, |Du| \leqslant \int |u|^p + |Du|^p. \tag{7}$$

Thus ends the proof.