NAME: ______________________

QUIZ 4

Problem 1. Consider the Burgers equation with initial data $u_0(x) = \begin{cases} 1 & x<0 \\ 0 & x>0 \end{cases}$. Prove using definition that $u(x,t) = \begin{cases} 1 & x<t/2 \\ 0 & x>t/2 \end{cases}$ is a weak solution. Show that it is furthermore an entropy solution.

Proof. First we show that u is a weak solution. Take any $\Omega \subset \mathbb{R}^2 \cap \{t>0\}$. Wlog we assume $\Omega \cap \{x=t/2\}$ is nonempty, and $\partial\Omega \cap \{t=0\}$ is nonempty. Thus we have

$$\int\int_\Omega u\,\phi_t + f(u)\,\phi_x\,\mathrm{d}x\,\mathrm{d}t = \int\int_U \phi_t + \frac{1}{2}\,\phi_x\,\mathrm{d}x\,\mathrm{d}t \tag{1}$$

where

$$U := \Omega \cap \{x < t/2\}. \tag{2}$$

It is easy to see that

$$\partial U = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \tag{3}$$

with $\Gamma_2 = \partial U \cap \{x = t/2\} = \Omega \cap \{x = t/2\}$ and $\Gamma_3 = \partial U \cap \{t = 0, x < 0\}$.

Now using Gauss' theorem (integration by parts) we have

$$\int\int_U \phi_t + \frac{1}{2}\,\phi_x\,\mathrm{d}x\,\mathrm{d}t = \int_{\partial U} \nu \cdot \begin{pmatrix} \phi \\ \phi/2 \end{pmatrix} \mathrm{d}S = \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3}. \tag{4}$$

As $\phi = 0$ along Γ_1, $\int_{\Gamma_1} = 0$; Along Γ_2, we have $\nu \parallel \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$ which makes the integrand 0, thus $\int_{\Gamma_2} = 0$. Along Γ_3, we have $\nu = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and then

$$\int_{\Gamma_3} \nu \cdot \begin{pmatrix} \phi \\ \phi/2 \end{pmatrix} \mathrm{d}S = -\int_{x<0} \phi\,\mathrm{d}x = -\int_{\mathbb{R}} u_0\,\phi\,\mathrm{d}x. \tag{5}$$

Thus u is a weak solution.

Now we show that u is an entropy solution. Fix t. We have

$$\frac{u(x+a,t) - u(x,t)}{a} = \begin{cases} 0 & x+a < t/2, x < t/2 \\ -1 & x < t/2, x+a > t/2 \\ 0 & x > t/2, x+a > t/2 \end{cases} \leqslant \frac{0}{t}. \tag{6}$$

Therefore u is an entropy solution. □