MATH 527 A1 HOMEWORK 5 (DUE NOV. 26 IN CLASS)

Exercise 1. (15 pts) (Evans 4.7.7) Consider the viscous conservation law

$$u_t + F(u)_x - a\, u_{xx} = 0 \qquad \text{in } \mathbb{R} \times (0, \infty) \tag{1}$$

where $a > 0$ and F is uniformly convex.

i. **(5 pts)** Show u solves (1) if $u(x, t) = v(x - \sigma t)$ and v is defined implicitly by the formula

$$s = \int_c^{v(s)} \frac{a}{F(z) - \sigma z + b} \, \mathrm{d}z \qquad (s \in \mathbb{R}), \tag{2}$$

where b and c are constants.

ii. **(5 pts)** Demonstrate that we can find a traveling wave satisfying

$$\lim_{s \to -\infty} v(s) = u_l, \qquad \lim_{s \to \infty} v(s) = u_r \tag{3}$$

for $u_l > u_r$, if and only if

$$\sigma = \frac{F(u_l) - F(u_r)}{u_l - u_r}. \tag{4}$$

iii. **(5 pts)** Let u^ε denote the above traveling wave solution of (1) for $a = \varepsilon$, with $u^\varepsilon(0, 0) = \frac{u_l + u_r}{2}$. Compute $\lim_{\varepsilon \to 0} u^\varepsilon$ and explain your answer.

Exercise 2. (5 pts) (5.10.3) Denote by U the open square $\{x \in \mathbb{R}^2 \mid |x_1| < 1, |x_2| < 1\}$. Define

$$u(x) = \begin{cases} 1 - x_1 & x_1 > 0, |x_2| < x_1 \\ 1 + x_1 & x_1 < 0, |x_2| < -x_1 \\ 1 - x_2 & x_2 > 0, |x_1| < x_2 \\ 1 + x_2 & x_2 < 0, |x_1| < -x_2 \end{cases}. \tag{5}$$

For which $1 \leqslant p \leqslant \infty$ does u belong to $W^{1,p}(U)$?

Exercise 3. (10 pts) (5.10.14) Verify that if $n > 1$, the unbounded function $u = \log\log\left(1 + \frac{1}{|x|}\right)$ belongs to $W^{1,n}(U)$, for $U = B^0(0, 1)$.