Exercise 1. (6 pts) Let u be a weak solution of the scalar conservation law. Show that if $u \in C^1(\Omega)$ for some domain Ω, then it is a classical solution in Ω, that is

$$u_t + f(u)_x = 0 \text{ for } (x,t) \in \Omega, \qquad u(x,0) = u_0 \text{ when } (x,0) \in \Omega. \tag{1}$$

Proof. Without loss of generality, we can assume that $\Omega \cap \{t=0\}$ is not empty.

First, take any $U \Subset \Omega$ such that $U \cap \{t=0\}$ is empty. Take a test function ϕ supported in U. By the definition of weak solution, we have

$$\int_U [u\,\phi_t + f(u)\,\phi_x]\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{2}$$

As $u, f(u) \in C^1$, we can integrate by parts to obtain

$$\int_U \left[u_t + f(u)_x\right]\phi = 0. \tag{3}$$

Here $u_t + f(u)_x$ is a continuous function. We claim that it is identically 0 in U. Assume it is not. Then there is $(x_0, t_0) \in U$ such that $u_t + f(u)_x \neq 0$ at (x_0, t_0). Wlog assume $\left[u_t + f(u)_x\right](x_0, t_0) > 0$. Then there is $\varepsilon > 0$ such that $\left[u_t + f(u)_x\right](x,t) > 0$ in $B_\varepsilon(x_0, t_0)$. Now we can take $\phi \in C_0^1(B_\varepsilon(x_0, t_0))$ with $\phi \geqslant 0$ (for example, a rescaled and translated version of $\exp\left[-\frac{1}{1-r^2}\right]$) and obtain contradiction.

As U is arbitrary, we conclude that $u_t + f(u)_x = 0$ in $\Omega \cap \{t > 0\}$. Now take $\phi \in C_0^1(\Omega)$. Since u is a weak solution, we have

$$\int_{\Omega\cap\{t>0\}} [u\,\phi_t + f(u)\,\phi_x]\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega\cap\{t=0\}} u_0\,\phi = 0. \tag{4}$$

Performing integration by parts on the first integral and using $u_t + f(u)_x = 0$ in $\Omega \cap \{t > 0\}$, we have

$$\int_{\Omega\cap\{t=0\}} \nu \cdot \begin{pmatrix} u \\ f(u) \end{pmatrix} \phi + u_0\,\phi = 0. \tag{5}$$

As $\nu = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, we have

$$\int_{\Omega\cap\{t=0\}} (u - u_0)\,\phi = 0 \tag{6}$$

and $u|_{t=0} = u_0$ follows from the arbitrariness of ϕ. □

Exercise 2. (6 pts) Let $u(x,t)$ be a weak solution of the scalar conservation law with initial value $u_0(x)$. Show that for any $\lambda > 0$, $u(\lambda x, \lambda t)$ is a weak solution for the same equation with initial value $u_0(\lambda x)$.

Proof. Recall that u is a weak solution of the conservation law

$$u_t + f(u)_x = 0, \quad u(x,0) = u_0 \tag{7}$$

if for any $\phi \in C_0^1(\mathbb{R}^2)$, we have

$$\iint_{t>0} u(x,t)\,[\partial_t\phi(x,t)] + f(u(x,t))\,[\partial_x\phi(x,t)]\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} u_0(x)\,\phi(x,0)\,\mathrm{d}x = 0. \tag{8}$$

Therefore we need to prove that

$$\iint_{t>0} u(\lambda x, \lambda t)\,[\partial_t\phi(x,t)] + f(u(\lambda x, \lambda t))\,[\partial_x\phi(x,t)]\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} u_0(\lambda x)\,\phi(x,0)\,\mathrm{d}x = 0. \tag{9}$$

Define $\psi(\lambda x, \lambda t) = \phi(x,t)$. It's clear that $\psi(y,s) \in C_0^1(\mathbb{R}^2)$. The LHS becomes

$$\iint_{t>0} u(\lambda x, \lambda t)\,[\partial_t\psi(\lambda x, \lambda t)] + f(u(\lambda x, \lambda t))\,[\partial_x\phi(\lambda x, \lambda t)]\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} u_0(\lambda x)\,\psi(\lambda x, 0)\,\mathrm{d}x. \tag{10}$$

Make a change of variables $y=\lambda x, s=\lambda t$. We have

$$\begin{aligned}\text{LHS} &= \iint_{t>0} u(y,s)\,\lambda\,\psi_s(y,s)+f(u(y,s))\,\lambda\,\psi_y(y,s)\,\lambda^{-2}\,\mathrm{d}y\,\mathrm{d}s+\int_{\mathbb{R}} u_0(y)\,\psi(y,0)\,\lambda^{-1}\,\mathrm{d}y\\ &= \lambda^{-1}\left[\iint_{t>0} u(y,s)\,\psi_s(y,s)+f(u(y,s))\,\psi_y(y,s)\,\mathrm{d}y\,\mathrm{d}s+\int_{\mathbb{R}} u_0(y)\,\psi(y,0)\,\mathrm{d}y\right]\\ &= 0 \qquad (11)\end{aligned}$$

as u is a weak solution. Thus ends the proof. □

Exercise 3. (6 pts) Consider the Burgers equation with initial data $u_0(x)=\begin{cases}0 & x<0\\ 1 & x>0\end{cases}$. Show that the following two functions

$$u_1(x,t)=\begin{cases}0 & x<t/2\\ 1 & x>t/2\end{cases}, \qquad u_2(x,t)=\begin{cases}0 & x<0\\ x/t & 0<x<t\\ 1 & x>0\end{cases} \qquad (12)$$

are both weak solutions to the problem.

Proof.

– u_1 is a weak solution.

Take any $\phi\in C_0^1(\mathbb{R}^2)$. If the support of ϕ does not intersect with $x=t/2$, then we have

$$\iint_{t>0} u\,\phi_t+\left(\frac{u^2}{2}\right)\phi_x+\int_{\mathbb{R}} u_0\,\phi(x,0)=-\iint_{t>0}\left[u_t+\left(\frac{u^2}{2}\right)_x\right]\phi=0. \qquad (13)$$

If the support of ϕ intersect with $x=t/2$, then denote

$$\Omega_L=\operatorname{supp}\phi\cap\{x<t/2\}\cap\{t>0\}, \qquad \Omega_R=\operatorname{supp}\phi\cap\{x>t/2\}\cap\{t>0\} \qquad (14)$$

and also

$$\Gamma=\operatorname{supp}\phi\cap\{x=t/2\}, \qquad \Gamma_L=\overline{\Omega}_L\cap\{t=0\}, \qquad \Gamma_R=\overline{\Omega}_R\cap\{t=0\}. \qquad (15)$$

We have

$$\begin{aligned}\iint_{t>0} u\,\phi_t+\left(\frac{u^2}{2}\right)\phi_x+\int_{\mathbb{R}} u_0\,\phi(x,0) &= \iint_{\Omega_L} u\,\phi_t+\left(\frac{u^2}{2}\right)\phi_x+\iint_{\Omega_R} u\,\phi_t+\left(\frac{u^2}{2}\right)\phi_x\\ &\quad+\int_{\Gamma_L} u_0\,\phi+\int_{\Gamma_R} u_0\,\phi\\ &= -\iint_{\Omega_L}\left[u_t+\left(\frac{u^2}{2}\right)_x\right]\phi-\int_{\Gamma_L} u_0\,\phi+\int_{\Gamma}\boldsymbol{n}_L\cdot\begin{pmatrix}u\\ \left(\frac{u^2}{2}\right)\end{pmatrix}\\ &\quad-\iint_{\Omega_R}\left[u_t+\left(\frac{u^2}{2}\right)_x\right]\phi-\int_{\Gamma_R} u_0\,\phi+\int_{\Gamma}\boldsymbol{n}_R\cdot\begin{pmatrix}u\\ \left(\frac{u^2}{2}\right)\end{pmatrix}\\ &\quad+\int_{\Gamma_L} u_0\,\phi+\int_{\Gamma_R} u_0\,\phi\\ &= \int_{\Gamma}\boldsymbol{n}\cdot\begin{pmatrix}[u_l-u_r]\\ \frac{1}{2}\left[u_l^2-u_r^2\right]\end{pmatrix}=\int_{x=t/2}\boldsymbol{n}\cdot\begin{pmatrix}-1\\ -1/2\end{pmatrix}=0. \qquad (16)\end{aligned}$$

The last step follows from the fact that u satisfies the jump condition.

– u_2 is a weak solution. Note that u_2 is a classical solution in $\{x<0\}$, $\{0<x<t\}$, $\{x>0\}$ respectively. The proof is done after writing

$$\iint_{t>0} u\,\phi_t + \left(\frac{u^2}{2}\right)\phi_x + \int_{\mathbb{R}} u_0\,\phi(x,0) = \iint_{D_1} + \iint_{D_2} + \iint_{D_3} \tag{17}$$

and then using the same argument as for u_1. Note that as u_2 is continuous across the boundaries between D_1 and D_2 as well as D_2 and D_3, the integrals along them vanish. □

Exercise 4. (12 pts) (Evans 3.5.20)

Compute explicitly the unique entropy solution of

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \quad \text{in } \mathbb{R}\times(0,\infty); \qquad u = g \quad \text{on } \mathbb{R}\times\{t=0\}. \tag{18}$$

for

$$g(x) = \begin{cases} 1 & x < -1 \\ 0 & -1 < x < 0 \\ 2 & 0 < x < 1 \\ 0 & x > 0 \end{cases}. \tag{19}$$

Solution. It is clear that initially we have two shocks and one rarefaction wave. The two shocks are

1. Starting from $(-1,0)$ with slope 2, that is $x = \frac{t}{2} - 1$.
2. Starting from $(1,0)$ with slope 1, that is $x = t + 1$.

Note that after passing $(0,2)$ and $(2,1)$ both shocks are not straight anymore: Both will meet the rarefaction wave between $x=0$ and $x=2t$:

$$u = \frac{x}{t}. \tag{20}$$

Denote these two curvy shocks by $x_1(t)$ and $x_2(t)$. First consider $x_1(t)$.

When $t<2$ we have $x_1(t) = \frac{1}{2}t - 1$. For $t \geqslant 2$ we have

$$\dot{x}_1(t) = \frac{1}{2}\left(\frac{x_1}{t} + 1\right), \qquad x_1(2) = 0. \tag{21}$$

Now let $y(t) = x_1(t) - t$. We have

$$\dot{y}(t) = \dot{x}_1(t) - 1 = \frac{x_1}{2t} - \frac{1}{2} = \frac{y}{2t} \implies y = C\,t^{1/2}. \tag{22}$$

Now as $y(2) = x_1(2) - 2 = -2$, we have $C = -\sqrt{2}$. Thus

$$x_1(t) = \begin{cases} \frac{1}{2}t - 1 & t \leqslant 2 \\ t - \sqrt{2}\,t^{1/2} & t > 2 \end{cases}. \tag{23}$$

For $x_2(t)$ we have

$$\dot{x}_2(t) = \frac{1}{2}\left(\frac{x_2}{t} + 0\right), \qquad x_2(1) = 2. \tag{24}$$

Solving the equation we have

$$\ln x = \frac{1}{2}\ln t + C \implies x = C\,t^{1/2}. \tag{25}$$

Using $x_2(1) = 2$ we have

$$C = 2. \tag{26}$$

Thus the right shock is

$$x_2(t) = \begin{cases} t+1 & t \leqslant 1 \\ 2\,t^{1/2} & t > 1 \end{cases}. \tag{27}$$

Setting $x_1(t) = x_2(t)$ we see that the two shocks meet at the point $\left(4 + 2\sqrt{2}, 6 + 4\sqrt{2}\right)$.

Finally, after $t = 6 + 4\sqrt{2}$, there is only one shock with speed $1/2$:

$$x - \left(4 + 2\sqrt{2}\right) = \frac{t - \left(6 + 4\sqrt{2}\right)}{2} \Longleftrightarrow x = \frac{t}{2} + 1. \tag{28}$$

To the left $u = 1$ and to the right $u = 0$.