Exercise 1. (6 pts) Let u be a weak solution of the scalar conservation law. Show that if $u \in C^1(\Omega)$ for some domain Ω, then it is a classical solution in Ω, that is

$$u_t + f(u)_x = 0 \text{ for } (x, t) \in \Omega, \qquad u(x, 0) = u_0 \text{ when } (x, 0) \in \Omega. \tag{1}$$

Exercise 2. (6 pts) Let $u(x,\ t)$ be a weak solution of the scalar conservation law with initial value $u_0(x)$. Show that for any $\lambda > 0$, $u(\lambda\, x, \lambda\, t)$ is a weak solution for the same equation with initial value $u_0(\lambda\, x)$.

Exercise 3. (6 pts) Consider the Burgers equation with initial data $u_0(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$. Show that the following two functions

$$u_1(x, t) = \begin{cases} 0 & x < t/2 \\ 1 & x > t/2 \end{cases}, \qquad u_2(x, t) = \begin{cases} 0 & x < 0 \\ x/t & 0 < x < t \\ 1 & x > 0 \end{cases} \tag{2}$$

are both weak solutions to the problem.

Exercise 4. (12 pts) (Evans 3.5.20)

Compute explicitly the unique entropy solution of

$$u_t + \left(\frac{u^2}{2} \right)_x = 0 \quad \text{in } \mathbb{R} \times (0, \infty); \qquad u = g \quad \text{on } \mathbb{R} \times \{t = 0\}. \tag{3}$$

for

$$g(x) = \begin{cases} 1 & x < -1 \\ 0 & -1 < x < 0 \\ 2 & 0 < x < 1 \\ 0 & x > 0 \end{cases}. \tag{4}$$