MATH 527 A1 HOMEWORK 3 (DUE OCT. 22 IN CLASS)

Exercise 1. (4 pts) (Evans 3.5.5 c) Solve using characteristics:

$$u\,u_{x_1}+u_{x_2}=1, \qquad u(x_1,x_1)=\frac{1}{2}\,x_1. \tag{1}$$

Solution. Using the method of characteristics, we have

$$F(x,z,p)=z\,p_1+p_2-1. \tag{2}$$

$$\dot{p} = -D_xF-D_zF\,p=\begin{pmatrix} p_1^2 \\ p_1\,p_2 \end{pmatrix}, \tag{3}$$

$$\dot{z} = D_pF\cdot p=z\,p_1+p_2 \tag{4}$$

$$\dot{x} = D_pF=\begin{pmatrix} z \\ 1 \end{pmatrix}. \tag{5}$$

Using the equation we have

$$\dot{z}=1 \Longrightarrow z(a,s)=z(a,0)+s. \tag{6}$$

The x equation then gives

$$x_1(a,s)=x_1(a,0)+z(a,0)\,s+\frac{1}{2}\,s^2, \qquad x_2(a,s)=x_2(a,0)+s. \tag{7}$$

The Cauchy data gives

$$x_1(a,0)=x_2(a,0)=a,\ z(a,0)=\frac{1}{2}\,x_1(a,0)=\frac{1}{2}\,a. \tag{8}$$

Thus

$$x_1(a,s)=a+\frac{1}{2}\,a\,s+\frac{1}{2}\,s^2, \qquad x_2(a,s)=a+s, \qquad z(a,s)=\frac{1}{2}\,a+s. \tag{9}$$

From the first two equations we have

$$s=x_2-a \Longrightarrow x_1=a+\frac{1}{2}\,a\,(x_2-a)+\frac{1}{2}\,(x_2-a)^2 \Longrightarrow x_1=a-\frac{1}{2}\,a\,x_2+\frac{1}{2}\,x_2^2. \tag{10}$$

Thus

$$a=\frac{x_1-\frac{1}{2}\,x_2^2}{1-\frac{1}{2}\,x_2} \Longrightarrow u(x)=z(a,s)=x_2-\frac{1}{2}\,a=\frac{2\,x_2-x_1-\frac{1}{2}\,x_2^2}{2-x_2}. \tag{11}$$

Exercise 2. (12 pts) (Evans 3.5.10) Write $L=H^*$, if $H\colon\mathbb{R}^n\mapsto\mathbb{R}$ is convex.

a) **(6 pts)** Let $H(p)=\frac{1}{r}\,|p|^r$, for $1<r<\infty$. Show

$$L(q)=\frac{1}{s}\,|q|^s, \qquad \text{where } \frac{1}{r}+\frac{1}{s}=1. \tag{12}$$

b) **(6 pts)** Let $H(p)=\frac{1}{2}\sum_{i,j=1}^n a_{ij}\,p_i\,p_j+\sum_{i=1}^n b_i\,p_i$, where $A=((a_{ij}))$ is a symmetric, positive definite matrix, $b\in\mathbb{R}^n$. Compute $L(q)$.

Proof.

a) By definition

$$L(q)=\sup_{p\in\mathbb{R}^n}\{p\cdot q-H(p)\}=\sup_{p\in\mathbb{R}^n}\left\{p\cdot q-\frac{1}{r}\,|p|^r\right\}. \tag{13}$$

When $r>1$, it is clear that $p\cdot q-\frac{1}{r}\,|p|^r\searrow-\infty$ as $|p|\nearrow\infty$. As this function is differentiable, we set

$$0=D_p\left\{p\cdot q-\frac{1}{r}\,|p|^r\right\}=q-|p|^{r-1}\,\frac{p}{|p|} \Longrightarrow p=|p|^{2-r}\,q \Longrightarrow |q|=|p|^{r-1}. \tag{14}$$

Since this is the only critical point, it has to be the maximizer. As a consequence,

$$\sup_{p\in\mathbb{R}^n}\left\{p\cdot q-\frac{1}{r}\,|p|^r\right\}=\left(1-\frac{1}{r}\right)|q|^{\frac{r}{r-1}}=\frac{1}{s}\,|q|^s. \tag{15}$$

b) By definition

$$L(q)=\sup_{p\in\mathbb{R}^n}\{p\cdot q-H(p)\}=\sup_{p\in\mathbb{R}^n}\left\{p\cdot q-\frac{1}{2}\,p^T A\,p-p\cdot b\right\} \tag{16}$$

As A is positive definite, the maximum is attained. Set

$$0=D_p\left\{p\cdot q-\frac{1}{2}\,p^T A\,p-p\cdot b\right\}=q-b-A\,p \Longrightarrow p=A^{-1}\,(q-b). \tag{17}$$

Therefore

$$L(q)=A^{-1}\,(q-b)\cdot q-\frac{1}{2}\left[A^{-1}\,(q-b)\right]^T A\left[A^{-1}\,(q-b)\right]-A^{-1}\,(q-b)\cdot b \tag{18}$$

which can be simplified to

$$L(q)=\frac{1}{2}\,(q-b)^T A^{-1}\,(q-b). \tag{19}$$

□

Exercise 3. (4 pts) (Evans 3.5.8) Confirm that the formula $u=g(x-t\,F'(u))$ provides an implicit solution for the conservation law

$$u_t+F(u)_x=0. \tag{20}$$

Proof. We compute

$$u_t=\frac{\partial}{\partial t}g(x-t\,F'(u))=-\,g'(x-t\,F'(u))\,F'(u)-g'(x-t\,F'(u))\,t\,F''(u)\,u_t \tag{21}$$

which leads to

$$u_t=\frac{-\,g'\,F'}{1+g'\,F''\,t}. \tag{22}$$

Similarly, we compute

$$u_x=\frac{\partial}{\partial x}g(x-t\,F'(u))=g'-t\,g'\,F''\,u_x \tag{23}$$

which leads to

$$u_x=\frac{g'}{1+g'\,F''\,t}. \tag{24}$$

Now it is clear that

$$u_t+F'(u)\,u_x=0. \tag{25}$$

Thus ends the proof. □

Exercise 4. (10 pts) (Evans 3.5.14) Let E be a closed subset of $\mathbb{R}^n$. Show that if the Hopf-Lax formula could be applied to the initial-value problem

$$u_t+|D u|^2=0 \quad \text{in } \mathbb{R}^n\times(0,\infty); \qquad u=\begin{cases}0 & x\in E\\ +\infty & x\notin E\end{cases} \quad \text{on } \mathbb{R}^n\times\{t=0\}, \tag{26}$$

it would give the solution

$$u(x,t)=\frac{1}{4\,t}\,\mathrm{dist}\,(x,E)^2. \tag{27}$$

Proof. We have $H(p)=p^2$ which gives $L(q)=\frac{1}{4}\,q^2$.

Now compute

$$\begin{aligned} u(x,t) &= \min_y\left\{t\,L\left(\frac{x-y}{t}\right)+g(y)\right\}\\ &= \min_{y\in E}\left\{t\,L\left(\frac{x-y}{t}\right)\right\}\\ &= \min_{y\in E}\left\{\frac{1}{4\,t}\,|x-y|^2\right\}\\ &= \frac{1}{4\,t}\,\mathrm{dist}(x,E)^2. \end{aligned} \tag{28}$$

□