Math 527 A1 Homework 3 (Due Oct. 22 in Class)

Exercise 1. (4 pts) (Evans 3.5.5 c) Solve using characteristics:

$$u\,u_{x_1} + u_{x_2} = 1, \qquad u(x_1, x_1) = \frac{1}{2}\,x_1. \tag{1}$$

Exercise 2. (12 pts) (Evans 3.5.10) Write $L = H^*$, if $H: \mathbb{R}^n \mapsto \mathbb{R}$ is convex.

a) **(6 pts)** Let $H(p) = \frac{1}{r}\,|p|^r$, for $1 < r < \infty$. Show

$$L(q) = \frac{1}{s}\,|q|^s, \qquad \text{where } \frac{1}{r} + \frac{1}{s} = 1. \tag{2}$$

b) **(6 pts)** Let $H(p) = \frac{1}{2}\sum_{i,j=1}^n a_{ij}\,p_i\,p_j + \sum_{i=1}^n b_i\,p_i$, where $A = ((a_{ij}))$ is a symmetric, positive definite matrix, $b \in \mathbb{R}^n$. Compute $L(q)$.

Exercise 3. (4 pts) (Evans 3.5.8) Confirm that the formula $u = g(x - t\,F'(u))$ provides an implicit solution for the conservation law

$$u_t + F(u)_x = 0. \tag{3}$$

Exercise 4. (10 pts) (Evans 3.5.14) Let E be a closed subset of $\mathbb{R}^n$. Show that if the Hopf-Lax formula could be applied to the initial-value problem

$$u_t + |Du|^2 = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty); \qquad u = \begin{cases} 0 & x \in E \\ +\infty & x \notin E \end{cases} \quad \text{on } \mathbb{R}^n \times \{t = 0\}, \tag{4}$$

it would give the solution

$$u(x,t) = \frac{1}{4\,t}\,\mathrm{dist}\,(x, E)^2. \tag{5}$$