Math 527 A1 Homework 2 (Due Oct. 8 in Class)

Exercise 1. (6 pts) Prove the mean value formula for harmonic functions using Poisson's formula for the ball (see Evans 2.2.4c for the formula).

Proof. The Poisson formula reads (p.41)

$$u(x) = \frac{r^2 - |x|^2}{n\,\alpha(n)\,r} \int_{\partial B_r(0)} \frac{g(y)}{|x-y|^n}\,\mathrm{d}S. \tag{1}$$

Taking $x = 0$ we have $|x - y| = r$ and therefore

$$u(0) = \frac{1}{n\,\alpha(n)\,r^{n-1}} \int_{\partial B_r(0)} g(y)\,\mathrm{d}S = \frac{1}{|\partial B_r|} \int_{\partial B_r(0)} u(y)\,\mathrm{d}S. \tag{2}$$

Thus the mean value formula is proved for $x = 0$. The general case is equivalent to this one due to the translation invariance of Laplace's equation. □

Exercise 2. (7 pts)

a) **(Evans 2.5.3)** Modify the proof of the mean value formulas to show for $n \geqslant 3$ that

$$u(0) = \frac{1}{|\partial B_r|} \int_{\partial B_r} g\,\mathrm{d}S + \frac{1}{n\,(n-2)\,\alpha(n)} \int_{B_r} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f\,\mathrm{d}x. \tag{3}$$

b) **(Optional)** Prove the above using Green's function for the ball B_r.

Proof. Following Evans p.26, we define

$$\phi(r) := \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u(y)\,\mathrm{d}S. \tag{4}$$

Thus all we need to prove is

$$\phi(0) = \phi(r) + \frac{1}{n\,(n-2)\,\alpha(n)} \int_{B_r} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f\,\mathrm{d}x. \tag{5}$$

On Evans p.26 it is derived that

$$\phi'(r) = \frac{r}{n} \frac{1}{|B_r|} \int_{B_r(x)} \triangle u(y)\,\mathrm{d}y = -\frac{1}{\alpha(n)\,n\,r^{n-1}} \int_{B_r} f(y)\,\mathrm{d}y. \tag{6}$$

We compute

$$\begin{aligned} \int_0^r \phi'(t) &= -\frac{1}{\alpha(n)\,n} \int_0^r \frac{1}{t^{n-1}} \int_{B_t} f(y)\,\mathrm{d}y\,\mathrm{d}t \\ &= \frac{1}{\alpha(n)\,n\,(n-2)} \int_0^r \left(\int_{B_t} f(y)\,\mathrm{d}y \right) \mathrm{d}\!\left(\frac{1}{t^{n-2}} \right) \\ &= \frac{1}{\alpha(n)\,n\,(n-2)} \left[\left(\int_{B_t} f(y)\,\mathrm{d}y \right) \left(\frac{1}{t^{n-2}} \right) \Big|_0^r - \int_0^r \frac{1}{t^{n-2}} \int_{\partial B_t} f\,\mathrm{d}S \right] \\ &= \frac{1}{\alpha(n)\,n\,(n-2)} \left[\frac{1}{r^{n-2}} \int_{B_r} f(y)\,\mathrm{d}y - \int_0^r \int_{\partial B_t} \left(\frac{1}{t^{n-2}} \right) f\,\mathrm{d}S\,\mathrm{d}t \right] \\ &= \frac{1}{\alpha(n)\,n\,(n-2)} \int_{B_r} \left(\frac{1}{r^{n-2}} - \frac{1}{|x|^{n-2}} \right) f(x)\,\mathrm{d}x. \end{aligned} \tag{7}$$

and the conclusion immediately follows.

In the above computation we have used the fact that

$$\frac{\mathrm{d}}{\mathrm{d}r} \int_{B_r} f\,\mathrm{d}x = \int_{\partial B_r} f\,\mathrm{d}S. \tag{8}$$

To see this, compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r} \int_{B_r} f\,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}r} \left[\alpha(n)\,r^n \frac{1}{|B_r|} \int_{B_r} f\,\mathrm{d}x \right] \\ &= \frac{\mathrm{d}}{\mathrm{d}r} \left[\alpha(n)\,r^n \frac{1}{|B_1|} \int_{B_1} f(r\,z)\,\mathrm{d}z \right] \\ &= n\,r^{n-1} \int_{B_1} f(r\,z)\,\mathrm{d}z + r^n \int_{B_1} z \cdot Df \\ &= \frac{n}{r} \int_{B_r} f(y)\,\mathrm{d}y + \int_{B_r} \frac{y}{r} \cdot Df \\ &= \frac{1}{r} \int_{B_r} \nabla \cdot (y\,f(y))\,\mathrm{d}y \\ &= \frac{1}{r} \int_{\partial B_r} (\boldsymbol{n} \cdot y)\,f(y)\,\mathrm{d}S \\ &= \int_{\partial B_r} f\,\mathrm{d}S. \end{aligned} \tag{9}$$

□

Exercise 3. (4 pts) (Evans 2.5.12) Suppose u is smooth and solves $u_t - \triangle u = 0$ in $\mathbb{R}^n \times (0, \infty)$.

i. **(1 pt)** Show $u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

ii. **(3 pts)** Use (i) to show $v(x,t) := x \cdot Du(x,t) + 2\, t\, u_t(x,t)$ solves the heat equation as well.

Proof.

i. Direct calculation.

ii. Notice that $v = \frac{\partial u_\lambda}{\partial \lambda}\Big|_{\lambda=1}$. □

Exercise 4. (7 pts) (Evans 2.5.15) Given $g\colon [0, \infty) \mapsto \mathbb{R}$, with $g(0) = 0$, derive the formula

$$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{\frac{-x^2}{4(t-s)}} g(s)\, \mathrm{d}s \tag{10}$$

for a solution of the initial/boundary-value problem

$$u_t - u_{xx} = 0 \text{ in } \mathbb{R}_+ \times (0,\infty); \qquad u = 0 \text{ on } \mathbb{R}_+ \times \{t = 0\}; \qquad u = g \text{ on } \{x = 0\} \times [0, \infty). \tag{11}$$

(Hint: Let $v(x,t) := u(x,t) - g(t)$ and extend v to $\{x < 0\}$ by odd reflections.)

Proof. After the extension v solves

$$v_t - v_{xx} = \begin{cases} -g'(t) & x \geqslant 0 \\ g'(t) & x \leqslant 0 \end{cases}, \qquad v(x,0) = 0. \tag{12}$$

Then the formula gives

$$v(x,t) = \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \left[\int_0^\infty e^{-\frac{(x-y)^2}{4(t-s)}} (-g'(s))\, \mathrm{d}y + \int_{-\infty}^0 e^{-\frac{(x-y)^2}{4(t-s)}} g'(s)\, \mathrm{d}y \right]. \tag{13}$$

Some manipulation yields

$$v(x,t) = \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \left[\int_{-x}^\infty e^{-\frac{z^2}{4(t-s)}} g'(s)\, \mathrm{d}z - \int_x^\infty e^{-\frac{z^2}{4(t-s)}} g'(s)\, \mathrm{d}z \right] \tag{14}$$

This leads to

$$v(x,t) = \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int_{-x}^x e^{-\frac{z^2}{4(t-s)}} g'(s)\, \mathrm{d}z. \tag{15}$$

Now set

$$w = \frac{z}{2\sqrt{t-s}}, \tag{16}$$

and change variable, we have

$$v(x,t) = \frac{1}{\sqrt{4\pi}} \int_0^t g'(s) \left(\int_{-\frac{x}{2\sqrt{t-s}}}^{\frac{x}{2\sqrt{t-s}}} e^{-z^2}\, \mathrm{d}z \right) \mathrm{d}s. \tag{17}$$

Integrating by parts, we get the result. □

Exercise 5. (6 pts) (Evans 2.5.24) Let $u \in C^2(\mathbb{R} \times [0,\infty))$ solve the initial-value problem for the wave equation in one dimension:

$$u_{tt} - u_{xx} = 0 \quad \text{in } \mathbb{R} \times (0,\infty); \qquad u = g,\ u_t = h \quad \text{on } \mathbb{R} \times \{t = 0\}. \tag{18}$$

Suppose g, h have compact support. the *kinetic energy* is $k(t) := \frac{1}{2}\int_{-\infty}^\infty u_t^2(x,t)\, \mathrm{d}x$ and the *potential energy* is $p(t) := \frac{1}{2}\int_{-\infty}^\infty u_x^2(x,t)\, \mathrm{d}x$. Prove

i. **(3 pts)** $k(t) + p(t)$ is constant in t.

ii. **(3 pts)** $k(t) = p(t)$ for all large enough times t.

Proof.

i. Multiply the equation by u_t, integrate.

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}} u_t\,(u_{tt} - u_{xx})\, \mathrm{d}x\, \mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}} \left(\frac{u_t^2}{2} + \frac{u_x^2}{2} \right)_t \mathrm{d}x\, \mathrm{d}t \\ &= \int_{\mathbb{R}} \left(\frac{u_t^2}{2} + \frac{u_x^2}{2} \right)(T) - \left(\frac{u_t^2}{2} + \frac{u_x^2}{2} \right)(0) = k(T) + p(T) - (k(0) + p(0)). \end{aligned} \tag{19}$$

ii. Using d'Alembert's formula we have

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, \mathrm{d}y \tag{20}$$

which gives

$$u_t(x,t) = \frac{1}{2}[g'(x+t) - g'(x-t) + h(x+t) + h(x-t)] \tag{21}$$

$$u_x(x,t) = \frac{1}{2}[g'(x+t) + g'(x-t) + h(x+t) - h(x-t)]. \tag{22}$$

The conclusion follows after noticing that all "cross terms" vanish, e.g.

$$\int_{\mathbb{R}} g'(x+t)\, h(x-t) = 0 \tag{23}$$

for t large enough (say g, h are supported in $(-R, R)$, then when $t > R$, at least one of $x \pm t$ must lie outside the support), and all other terms are independent of t, e.g.

$$\int g'(x+t)\, h(x+t) = \int g'(z)\, h(z)\, \mathrm{d}z. \tag{24}$$

□

Exercise 6. (Optional) (Evans 2.5.9) Let u be the solution of

$$\triangle u = 0 \text{ in } \mathbb{R}^n_+; \qquad u = g \text{ on } \partial\mathbb{R}^n_+ \tag{25}$$

given by Poisson's formula for the half-space. Assume g is bounded and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+$, $|x| \leqslant 1$. Show Du is not bounded near $x = 0$. (Hint: Estimate $\frac{u(\lambda\, e_n) - u(0)}{\lambda}$).

(For those who know the following stuff: This is the unboundedness of Riesz operators on L^∞ in disguise.)