Math 527 B1 Homework 1 (Due Sep. 24 in Class)

Sep. 17, 2010

Exercise 1. (5 pts) (1.5.5) Assume that $f: \mathbb{R}^n \mapsto \mathbb{R}$ is smooth. Prove

$$f(x) = \sum_{|\alpha| \leqslant k} \frac{1}{\alpha!} D^\alpha f(0)\, x^\alpha + O\left(|x|^{k+1}\right) \qquad \text{as } x \to 0$$

for each $k = 1, 2, \ldots$. This is *Taylor's formula* in multiindex notation.

(Hint: Fix $x \in \mathbb{R}^n$ and consider the function of one variable $g(t) := f(t\,x)$.)

Notation: For $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_1, \ldots, \alpha_n \geqslant 0$, $x = (x_1, \ldots, x_n)$,

$$\begin{aligned} |\alpha| &:= \alpha_1 + \cdots + \alpha_n; \\ \alpha! &:= \alpha_1!\, \alpha_2! \cdots \alpha_n! \\ D^\alpha &:= \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}; \\ x^\alpha &:= x_1^{\alpha_1} \cdots x_n^{\alpha_n}; \\ |x| &:= \left(x_1^2 + \cdots + x_n^2\right)^{1/2}. \end{aligned}$$

Proof. From 1D Taylor formula we have

$$g(t) = g(0) + g'(0)\, t + \frac{1}{2} g''(0)\, t^2 + \cdots + \frac{1}{k!} g^{(n)}(0)\, t^k + \frac{1}{(k+1)!} g^{(k+1)}(\xi)\, t^{k+1}. \tag{1}$$

Here ξ lies between 0 and t.

Now as $g(t) = f(t\,x)$, all we need to do is to write the derivatives of g using derivatives of f. We compute

$$\begin{aligned} g^{(k)}(s) &= \left[\frac{\mathrm{d}^k}{\mathrm{d}t^k} f(t\,x)\right]\Big|_{t=s} \\ &= (x_1\, \partial_1 + \cdots + x_n\, \partial_n)^k f(t\,x)\,|_{t=s} \\ &= \sum_{|\alpha| = k} \frac{k!}{\alpha!} x^\alpha D^\alpha f(s\,x). \end{aligned} \tag{2}$$

Here the last equality can be shown in many ways. One of them is mathematical induction. Another is the following counting technique: All we need to do is to pick $\alpha_1, \alpha_2, \ldots, \alpha_n$ from the total of k objects, and assign them to $\partial_1, \ldots, \partial_n$ respectively. To do this, we order the k objects and then assign the first α_1 to ∂_1, the next α_2 to ∂_2 and so on. There are $k!$ different orderings. However, as the α_i ∂_is are identical, we have to devide by $\Pi_i \alpha_i! = \alpha!$.

Now setting $s = 0$ and $t = 1$ we get the desired result. □

Exercise 2. (Well-posedness for ODE) We develop a complete theory of well-posedness for the initial value problem of ODE. Consider an ODE of the form

$$\dot{u} = f(t, u), \qquad u(t_0) = u_0. \tag{3}$$

where f is defined on $D \subseteq \mathbb{R} \times \mathbb{R}^d$ and $(t_0, u_0) \in D$. We say u is a classical solution if $u \in C^1$.

a) **(2 pts)** Existence I: Prove the following theorem.

Theorem. *Assume that f is continuous in t and uniformly Lipschitz in u, then there exists an interval $\left(t^-, t^+\right) \ni t_0$, such that at least one classical solution $u \in C^1\left(t^-, t^+\right)$ exists.*

Remark. The proof still works when $\mathbb{R}^d$ is replaced by any Banach space. As a consequence, it can be applied to many PDEs.

Proof. We consider the Picard iteration (without loss of generality we set $t_0 = 0$)

$$u_{n+1}(t) = u_0 + \int_0^t f(s, u_n(s))\, \mathrm{d}s \tag{4}$$

starting from $u_0(t) \equiv u_0$.

We first show that there is $\left(t^-, t^+\right) \ni t_0$ such that $\{u_n(t)\}$ converges uniformly on this interval. To see this, consider the difference

$$\begin{aligned} |u_{n+1}(t) - u_n(t)| &= \left|\int_0^t f(s, u_n(s)) - f(s, u_{n-1}(s))\, \mathrm{d}s\right| \\ &\leqslant \int_0^t |f(s, u_n(s)) - f(s, u_{n-1}(s))|\, \mathrm{d}s \\ &\leqslant \int_0^t M\, |u_n(s) - u_{n-1}(s)|\, \mathrm{d}s \\ &\quad \text{Here } M \text{ is the Lipschitz constant of } f \\ &\leqslant M\, |t| \max_{s \in (0,t)} |u_n(s) - u_{n-1}(s)|\,. \end{aligned} \tag{5}$$

Now take $|t^-|, |t^+| < \frac{c}{M}$ with $0 < c < 1$, we have

$$|u_{n+1}(t) - u_n(t)| \leqslant M\,|t| \max_{s\in(0,t)} |u_n(s) - u_{n-1}(s)| \leqslant c \max_{s\in(t^-,t^+)} |u_n(s) - u_{n-1}(s)|. \tag{6}$$

Taking $\max_{t\in(t^-,t^+)}$ of the above inequality we obtain

$$\max_{s\in(t^-,t^+)} |u_{n+1}(s) - u_n(s)| \leqslant c \max_{s\in(t^-,t^+)} |u_n(s) - u_{n-1}(s)| \leqslant \cdots \leqslant c^n \max_{s\in(t^-,t^+)} |u_1(s) - u_0(s)|. \tag{7}$$

Therefore $\{u_n(t)\}$ converges uniformly over (t^-, t^+) as $n \nearrow \infty$.

Denote the limit by $u(t)$. We further notice that since

$$|f(s, u_n(s)) - f(s, u_{n-1}(s))| \leqslant M\,|u_n(s) - u_{n-1}(s)|, \tag{8}$$

$\{f(t, u_n(t))\}$ also converges uniformly over (t^-, t^+), and the limit has to be $f(t, u(t))$.

Taking $n \nearrow \infty$ in

$$u_{n+1}(t) = u_0 + \int_0^t f(s, u_n(s))\,\mathrm{d}s \tag{9}$$

we obtain

$$u(t) = u_0 + \int_0^t f(s, u(s))\,\mathrm{d}s. \tag{10}$$

This implies $u \in C^1$. Taking $\frac{\mathrm{d}}{\mathrm{d}t}$ on both sides we see that u is a solution to the ODE. □

b) **(Optional)** Existence II: Prove the following theorem.

Theorem. *The "uniform Lipschitz" condition on f in the above theorem can be replaced by $f \in C(D)$.*

Hint: On any compact subset of D, approximate f uniformly by Lipschitz functions f_n, let u_n be a solution of the corresponding ODE, then use Ascoli-Arzela Theorem (a uniformly bounded, equicontinuous sequence has a subsequence which converges uniformly).

c) Uniqueness:

i. **(2 pts)** Show that the solution obtained in a) is in fact the only solution for the initial value problem.

Proof. We prove by contradiction. Let u, v be two different classical solutions. Then setting $w = u - v$, we have

$$\dot{w} = f(t, u) - f(t, v), \qquad w(0) = 0. \tag{11}$$

which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}(w^2) = 2\,w\,(f(t, u) - f(t, v)) \leqslant 2\,M\,w^2, \qquad w^2(0) = 0. \tag{12}$$

This in turn gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{-2Mt}\,w^2) \leqslant 0, \qquad (e^{-2Mt}\,w^2)\,|_{t=0} = 0. \tag{13}$$

As $e^{-2Mt}\,w^2 \geqslant 0$, the only conclusion is that $e^{-2Mt}\,w^2 = 0$ for all t and therefore $u = v$ for all t. □

ii. **(2 pts)** Construct an example to show that under the condition of the theorem in b), uniqueness may fail.

Solution. Consider the equation $\dot{u} = |u|^{1/3}$, $u(0) = 0$. Both $u \equiv 0$ and $u = \frac{2}{3}\,|t|^{3/2}$ solves the initial value problem.

iii. **(Optional)** Show that uniqueness still holds when the "uniform Lipschitz" condition on f in a) is replaced by the following weaker "Osgood" condition:

$$|(f(t, u) - f(t, v)) \cdot (u - v)| \leqslant g(|u - v|) \tag{14}$$

where the modulus g satisfies

$$\int_0^\delta \frac{1}{g(r)}\,\mathrm{d}r = \infty \tag{15}$$

for any $\delta > 0$.

d) **(2 pts)** Continuous dependence on initial value:

Prove that the unique solution obtained in a) depends continuously on (t_0, u_0). Note that continuous dependence on data automatically fails when the solution is not unique.

Proof. Let u, u' satisfy the equation with data (t_0, u_0) and (t_0', u_0'). Without loss of generality, we assume $t > t_0' > t_0$. Then we have

$$u(t) = u_0 + \int_{t_0}^t f(s, u(s))\,\mathrm{d}s \tag{16}$$

$$u'(t) = u_0' + \int_{t_0'}^t f(s, u'(s))\,\mathrm{d}s. \tag{17}$$

Taking the difference, we have

$$
\begin{aligned}
|u(t)-u'(t)| &= \left|(u_0-u_0')+\int_{t_0}^{t} f(s,u(s))\,\mathrm{d}s-\int_{t_0'}^{t} f(s,u'(s))\,\mathrm{d}s\right| \\
&\leqslant |u_0-u_0'|+\int_{t_0'}^{t}|f(s,u(s))-f(s,u'(s))|\,\mathrm{d}s+\int_{t_0}^{t_0'}|f(s,u(s))|\,\mathrm{d}s \\
&\leqslant |u_0-u_0'|+M\int_{t_0'}^{t}|u(s)-u'(s)|\,\mathrm{d}s+(t_0'-t_0)\,K
\end{aligned} \tag{18}
$$

where M is the Lipschitz constant of f and K is the maximum of $|f|$ over D.

Denote $V(t)=\int_{t_0'}^{t}|u(s)-u'(s)|\,\mathrm{d}s$. We have

$$\dot{V}(t)\leqslant|u_0-u_0'|+(t_0'-t_0)\,K+M\,V(t), \qquad V(t_0')=0. \tag{19}$$

This leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-Mt}V(t)\right)\leqslant e^{-Mt}\left[|u_0-u_0'|+(t_0'-t_0)\,K\right] \tag{20}$$

and consequently

$$V(t)\leqslant e^{Mt}\int_{t_0'}^{t}e^{-Ms}\left[|u_0-u_0'|+(t_0'-t_0)\,K\right]=\left[|u_0-u_0'|+(t_0'-t_0)\,K\right]M^{-1}\left[e^{M(t-t_0')}-1\right]. \tag{21}$$

Now we have

$$|u(t)-u'(t)|\leqslant\left[|u_0-u_0'|+(t_0'-t_0)\,K\right]+M\,V(t)\leqslant e^{M(t-t_0')}\left[|u_0-u_0'|+(t_0'-t_0)\,K\right]. \tag{22}$$

The continuous dependence is obvious. □

e) **(2 pts)** Different definitions of solution, regularity:

One can integrate and obtain the following "weak" formulation

$$u(t)=u_0+\int_{t_0}^{t}f(s,u(s))\,\mathrm{d}s. \tag{23}$$

We say $u\in C(I)$ is a "weak solution" of the ODE if it satisfies this integral formulation. Prove that, $u\in C^m$ if $f\in C^{m-1}$ (as a function of (t,u)) for $m\geqslant 1$.

Proof. We observe that

$$u\in C^k(I)\Longrightarrow f(s,u(s))\in C^k(I)\Longrightarrow\int_{t_0}^{t}f(s,u(s))\,\mathrm{d}s\in C^{k+1}(I)\Longrightarrow u\in C^{k+1}(I) \tag{24}$$

as long as $k\leqslant m-1$. Taking $k=0,1,\dots,m-1$ successively we obtain $u\in C^m$. □

Exercise 3. (10 pts) (2.5.6) Let U be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant C, depending only on U, such that

$$\max_U|u|\leqslant C\left(\max_{\partial U}|g|+\max_U|f|\right) \tag{25}$$

whenever u is a smooth solution of

$$-\triangle u=f\ \text{ in } U; \qquad u=g\ \text{ on } \partial U. \tag{26}$$

(Hint: $-\triangle\left(u+\frac{|x|^2}{2n}\lambda\right)\leqslant 0$ for $\lambda:=\max_{\bar{U}}|f|$)

Proof. Let $\lambda:=\max_{\bar{U}}|f|$. Clearly

$$-\triangle\left(u+\frac{|x|^2}{2n}\lambda\right)\leqslant 0. \tag{27}$$

Thus the weak maximum principle gives

$$u+\frac{|x|^2}{2n}\lambda\leqslant\max_{\partial U}\left[u+\frac{|x|^2}{2n}\lambda\right]. \tag{28}$$

Similarly one has

$$-\triangle\left(-u+\frac{|x|^2}{2n}\lambda\right)\leqslant 0\Longrightarrow -u+\frac{|x|^2}{2n}\lambda\leqslant\max_{\partial U}\left[u+\frac{|x|^2}{2n}\lambda\right]. \tag{29}$$

Combine them we have

$$\pm u\leqslant C\left(\max_{\partial U}|g|+\max_U|f|\right)\Longrightarrow|u|\leqslant C\left(\max_{\partial U}|g|+\max_U|f|\right). \tag{30}$$

Exercise 4. (Optional) Consider the eikonal equation

$$
\begin{aligned}
u_{x_1}^2+\cdots+u_{x_n}^2=1 \qquad & x\in B:=\left\{x_1^2+\cdots+x_n^2<1\right\}, \\
u=0 \qquad & x\in\partial B:=\left\{x_1^2+\cdots+x_n^2=1\right\}.
\end{aligned}
$$

Clearly, the natural class of functions for the solution is $C(\bar{B}) \cap C^1(B)$, that is, functions that are continuously differentiable in B, while continuous up to the boundary. We call such solutions "classical".

a) Show that no classical solution exists. Thus the equation is not well-posed if we consider only classical solutions.

b) One way to define "weak solutions" is through "testing" by smooth functions. For example, suppose we try to define "weak solutions" for the equation $u_{x_1} = f$ in B, $u = 0$ on ∂B, then we can multiply the equation by a smooth function φ with $\varphi = 0$ on ∂B and (formally) integrate by parts and obtain

$$\int u \varphi_{x_1} = -\int f \varphi.$$

and use this integral relation (which we require to hold for all smooth φ) as the definition. We see that as a consequence u need not be in C^1 anymore, in fact u being integrable is enough for the definition to make sense.

Try to define "weak solutions" for the eikonal equation this way. What difficulty do you meet?

c) Another way to relax the regularity requirement is to require $u \in C(\bar{B})$ but not $C^1(B)$, only differentiable almost everywhere. Consider the case $n = 1$. By this definition $u = 1 - |x|$ solves the eikonal equation. Can you establish well-posedness for such kind of "weak solutions" in the $n = 1$ case? If not, why?