Asymptotics

1. Introduction.

Asymptotics studies the behavior of a function at/near a given point. The simplest asymptotics is the Taylor expansion:

$$f(x) = f(x_0) + f'(x_0)\,(x - x_0) + \cdots \tag{1}$$

Of course, when $f(x)$ can be easily evaluated, for example when f is explicitly given by a simple formula, there is no practical reason to do asymptotics. Therefore, in practice, asymptotics is often performed in the following situations:

1. f is given semi-explicitly by an integral;
2. f is given implicitly by a differential equation.

In many cases, the point x_0 is either 0 or ∞.

Example 1. (Viscous Burgers equation) Consider the Burgers equation with viscosity

$$u_t^\varepsilon + u^\varepsilon\, u_x^\varepsilon - \varepsilon\, u_{xx}^\varepsilon = 0, \qquad u^\varepsilon(x, 0) = g(x). \tag{2}$$

The solution can be semi-explicitly given as an integral

$$u^\varepsilon(x, t) = \frac{\int_{-\infty}^{\infty} \frac{x - y}{t}\, e^{-\frac{K(x,y,t)}{2\varepsilon}}\, \mathrm{d}y}{\int_{-\infty}^{\infty} e^{-\frac{K(x,y,t)}{2\varepsilon}}\, \mathrm{d}y} \tag{3}$$

where

$$K(x, y, t) := \frac{|x - y|^2}{2\,t} + h(y) \tag{4}$$

where h is the antiderivative of g.

The parameter ε is viscosity, and in realistic situations is very small. Thus one is tempted to neglect it and study the Burgers equation

$$u_t + u\, u_x = 0. \tag{5}$$

To justify this, we need to study the behavior of u^ε as $\varepsilon \searrow 0$.

Example 2. (Oscillatory Integrals) Such integrals usually appear in the process of solving wave-related equations using transform methods. For example, when we try to solve the wave equation in a cylinder, the solution can be represented by Bessel functions. Such functions are either given by infinite sums or by integrals. For example, we have

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(n\,t - x \sin t)\, \mathrm{d}t \tag{6}$$

which can be written as

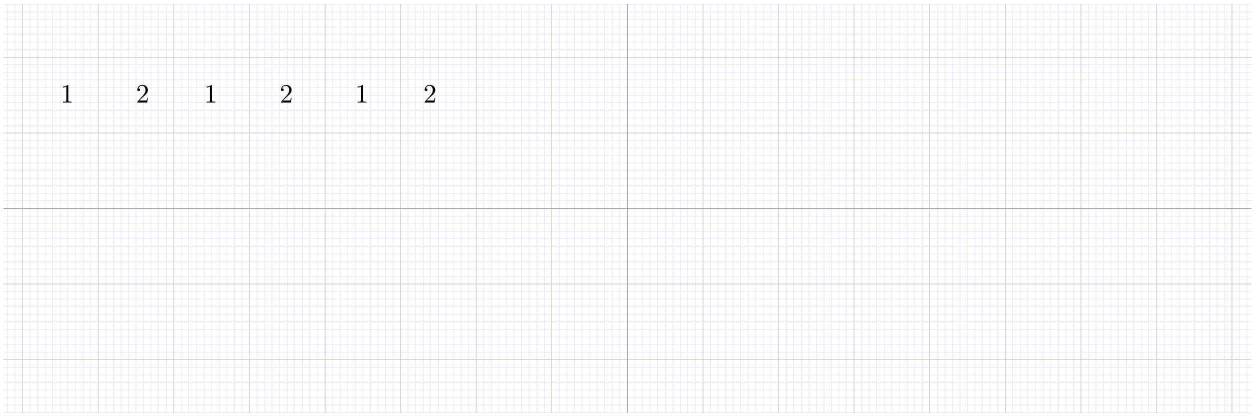
$$\frac{1}{2\,\pi} \sum_{\pm} \int_0^\pi e^{\pm i n t}\, e^{\mp i x \sin t}\, \mathrm{d}t. \tag{7}$$

Suppose we want to understand the behavior of $J_n(x)$ as $x \rightarrow \infty$. Setting $\varepsilon = 1/x$, we are left with an integral of the form

$$\int_a^b f(y)\, e^{i\frac{\phi(y)}{\varepsilon}}\, \mathrm{d}y. \tag{8}$$

And our task is to understand its behavior as $\varepsilon \searrow 0$.

Example 3. (Homogenization) Homogenization is a mathematical theory dealing with problems with multiple spatial scales. Consider a domain filled with two different materials. And let's say they form a "checker board" formation,



and now we would like to study the conductivity of the material. The equation is

$$\nabla \cdot (A(x)\,\nabla u) = 0 \tag{9}$$

where $A(x) = a_1(x)\,I$ for material 1 and $a_2(x)\,I$ for material 2. One way to do this is to solve the equation. However, when the grid size ε is very small, this approach is not efficient or even not practical. Therefore we need to find out what the equation the limit potential satisfies.

2. Evaluation of integrals.

2.1. Laplace's method.

Laplace's method deals with integrals of the form

$$\int_{-\infty}^{\infty} l(y)\, e^{-\frac{k(y)}{\varepsilon}}\, \mathrm{d}y \tag{10}$$

where k, l are continuous functions.

We try to understand the limiting behavior as $\varepsilon \searrow 0$. Now if we assume $k(y)$ has a single minimizer, say at y_0, then clearly $e^{-\frac{k(y)}{\varepsilon}}$ reaches its maximum at y_0. Furthermore, as ε gets smaller, the "peak" at y_0 gets steeper. As a consequence, the integral in a neighborhood of y_0 dominates. Thus we expect, when ε is small,

$$\int_{-\infty}^{\infty} l(y)\, e^{-\frac{k(y)}{\varepsilon}}\, \mathrm{d}y \sim l(y_0) \int_{-\infty}^{\infty} e^{-\frac{k(y)}{\varepsilon}}\, \mathrm{d}y. \tag{11}$$

Lemma 4. *Suppose $k, l: \mathbb{R} \mapsto \mathbb{R}$ are continuous functions, satisfying certain growth conditions at infinity– which will be clear in the proof. Assume also there exists a unique point $y_0 \in \mathbb{R}$ such that*

$$k(y_0) = \min_{y \in \mathbb{R}} k(y) \tag{12}$$

Then

$$\lim_{\varepsilon \searrow 0} \frac{\int_{-\infty}^{\infty} l(y)\, e^{-\frac{k(y)}{\varepsilon}}\, \mathrm{d}y}{\int_{-\infty}^{\infty} e^{-\frac{k(y)}{\varepsilon}}\, \mathrm{d}y} = l(y_0). \tag{13}$$

Remark 5. Note that the above makes sense even if $\int_{-\infty}^{\infty} e^{-\frac{k(y)}{\varepsilon}} \longrightarrow \infty$, for example when $k(y_0) < 0$.

Proof. First notice that, Wlog[1] we can assume $y_0 = 0$. Next by replacing $k(y)$ by $k(y) - k(y_0)$, we can assume $k(0) = 0$.

Let

$$\mu_\varepsilon(x) := \frac{e^{-\frac{k(x)}{\varepsilon}}}{\int_{-\infty}^{\infty} e^{-\frac{k(z)}{\varepsilon}}\, \mathrm{d}z}. \tag{14}$$

1. Without loss of generality.

Then all we need to show is that

$$\lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} \mu_\varepsilon(y)\, l(y)\, \mathrm{d}y = l(y_0). \tag{15}$$

Note that,

$$\mu_\varepsilon \geqslant 0, \qquad \int_{-\infty}^{\infty} \mu_\varepsilon(y)\, \mathrm{d}y = 1. \tag{16}$$

Thus it suffices to show

$$\lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} (l(y) - l(0))\, \mu_\varepsilon(y)\, \mathrm{d}y = 0. \tag{17}$$

For any $\varepsilon > 0$, we find $\delta > 0$ such that

$$|l(y) - l(0)| < \varepsilon \tag{18}$$

when $|y - y_0| < \delta$. Now write

$$\int_{-\infty}^{\infty} (l(y) - l(0))\, \mu_\varepsilon(y)\, \mathrm{d}y = \int_{|y| < \delta} + \int_{|y| > \delta} \tag{19}$$

The first term is clearly bounded by ε.

Now we study the second term. First we show that $\mu_\varepsilon(y) \longrightarrow 0$ uniformly as $\varepsilon \searrow 0$ for all $|y| > \delta$. To see this, let

$$b := \min_{|y| > 0} k(y) > 0. \tag{20}$$

That $b > 0$ is because 0 is the only minimizer and $k(y) \nearrow \infty$ as $|y| \to \infty$. Now as $k(0) = 0$, there is $\delta' > 0$ such that

$$k(y) < b/2 \tag{21}$$

for all $|y| < \delta'$. Note that δ' is independent of ε.

It follows from the above that, for all $|y| > \delta$,

$$\mu_\varepsilon(y) \leqslant \frac{e^{-b/\varepsilon}}{\int_{|z| < \delta'} e^{-b/2\varepsilon}} = \frac{e^{-b/2\varepsilon}}{2\,\delta'} \longrightarrow 0 \text{ as } \varepsilon \searrow 0. \tag{22}$$

Therefore $\mu_\varepsilon(y) \longrightarrow 0$ uniformly for $|y| > \delta$.

From the above analysis we know that

$$\int_{\delta < |y| < R} (l(y) - l(0))\, \mu_\varepsilon(y)\, \mathrm{d}y \longrightarrow 0 \tag{23}$$

for all $R > \delta$, and the proof ends as soon as we have some control at ∞.

We again study

$$\mu_\varepsilon(y) = \frac{e^{-k(y)/\varepsilon}}{\int e^{-k(z)/\varepsilon}}. \tag{24}$$

This time for $|y| > R$ where R can be taken arbitrarily large on condition that it is independent of ε. First notice that the denominator satisfies

$$\int e^{-k(z)/\varepsilon} \geqslant A\, e^{-B/\varepsilon} \tag{25}$$

for some constants $A, B > 0$. To see this, just pick an interval $[-A, A]$ and let $B = \max k(z)$ in that interval. Note that by taking k to grow fast outside this interval, we see that this estimate is actually sharp (meaning: for general k we cannot get better lower bound – the best we can do is find different A, B). Thus we need to study

$$A^{-1} \int_{|y| > R} (l(y) - l(0))\, e^{-(k(y) - B)/\varepsilon}\, \mathrm{d}y. \tag{26}$$

Now assume l and k grows as certain powers of y at infinity. Say $|l(y) - l(0)| \leqslant C\, y^a$, $k(y) - B \geqslant C'\, y^b$. We find out conditions on a, b that guarantee

$$\int_{|y|>R} y^a\, e^{-y^b/\varepsilon}\, \mathrm{d}y \longrightarrow 0. \tag{27}$$

Let $z = y/\varepsilon^{1/b}$. Then the above integral becomes

$$\int_{|z|>R/\varepsilon^{1/b}} \varepsilon^{a/b}\, z^a\, e^{-z^b}\, \varepsilon^{1/b}\, \mathrm{d}z \tag{28}$$

we see that for any $a \geqslant 0$, $b > 0$ the integral goes to 0.

Thus as soon as $k(y) \sim y^b$ at infinity for any $b > 0$, we have

$$\lim_{\varepsilon \searrow 0} \frac{\int_{-\infty}^{\infty} l(y)\, e^{-\frac{k(y)}{\varepsilon}}\, \mathrm{d}y}{\int_{-\infty}^{\infty} e^{-\frac{k(y)}{\varepsilon}}\, \mathrm{d}y} = l(y_0). \tag{29}$$

for any continuous l with polynomial growth at infinity. □

2.2. The method of stationary phase.

Now we study the behavior of the integral

$$\int_a^b e^{i\frac{\phi(y)}{\varepsilon}}\, f(y)\, \mathrm{d}y \tag{30}$$

as $\varepsilon \searrow 0$.

The idea is as follows. Fix at point y_0, we expand $\phi(y)$ by Taylor expansion.

$$\phi(y) \sim \phi(y_0) + \phi'(y_0)\,(y - y_0) + \frac{\phi''(y_0)}{2}\,(y - y_0)^2 + \cdots \tag{31}$$

Thus the contribution of the integral around y_0 is

$$\int_{y_0-\delta}^{y_0+\delta} e^{i\frac{\phi_0}{\varepsilon}}\, e^{i\frac{(y-y_0)}{\varepsilon}\phi'(y_0)} \cdots f(y)\, \mathrm{d}y. \tag{32}$$

Recall the Riemann-Lebesgue lemma:

$$\int_a^b e^{iky}\, f(y)\, \mathrm{d}y \longrightarrow 0 \tag{33}$$

as $k \nearrow \infty$, we see that those points with $\phi'(y_0) \neq 0$ does not contribute as $\varepsilon \searrow 0$.

Now consider those points with $\phi'(y_0) = 0$. Then around such y_0 we have, to the highest order,

$$e^{i\frac{\phi(y_0)}{\varepsilon}} \int_{y_0-\delta}^{y_0+\delta} e^{\frac{i\phi''(y_0)}{2\varepsilon}(y-y_0)^2}\, f(y)\, \mathrm{d}y. \tag{34}$$

Now do a change of variable

$$z = \sqrt{|\phi''(y_0)/2\,\varepsilon|}\,(y - y_0) \tag{35}$$

we reach (using the fact that $f(y) \sim f(y_0)$ in this neighborhood)

$$e^{i\frac{\phi(y_0)}{\varepsilon}}\, f(y_0)\, \sqrt{\frac{2\,\varepsilon}{|\phi''(y_0)|}} \int_{-\sqrt{|\phi''|/\varepsilon}\,\delta}^{\sqrt{|\phi''|/\varepsilon}\,\delta} e^{i\mathrm{sgn}(\phi'')z^2}\, \mathrm{d}z. \tag{36}$$

When $\varepsilon \searrow 0$, the above integral tends to

$$\int_{-\infty}^{\infty} e^{i\mathrm{sgn}(\phi'')z^2}\, \mathrm{d}z = \sqrt{\pi}\, e^{i\frac{\pi}{4}\mathrm{sgn}(\phi'')}. \tag{37}$$

As a consequence, we have

$$\int_a^b e^{i\frac{\phi(y)}{\varepsilon}}\, f(y)\, \mathrm{d}y \sim \sum_{\phi'(y_i)=0} f(y_i)\, e^{i\frac{\phi(y_i)}{\varepsilon}} \sqrt{\frac{2\,\pi\,\varepsilon}{|\phi''(y_0)|}}\, e^{i\frac{\pi}{4}\mathrm{sgn}(\phi'')}. \tag{38}$$

Now we give a rigorous treatment based on the above understanding.

Lemma 6. *Let y_0 be such that $\phi'(y_0) \neq 0$, then there is $\delta > 0$ such that*

$$\int_{y_0-\varepsilon^{1/2}}^{y_0+\varepsilon^{1/2}} e^{i\phi(y)/\varepsilon} f(y)\,\mathrm{d}y = O(\varepsilon) \text{ as } \varepsilon \searrow 0. \tag{39}$$

Proof. Do a change of variable $z = \phi(y)$. Then the integral becomes

$$\int_{z_1}^{z_2} e^{iz/\varepsilon} F(z)\,(\phi'(y))^{-1}\,\mathrm{d}z. \tag{40}$$

Now notice that,

- $|z_2 - z_1| = O\left(\varepsilon^{1/2}\right)$;
- $F(z) - F(z_1) = O\left(\varepsilon^{1/2}\right)$;
- $\phi'(y)^{-1} - \phi'(y_0) = O\left(\varepsilon^{1/2}\right)$;
- $\int e^{iz/\varepsilon}\,\mathrm{d}z = O(\varepsilon)$.

Thus we have

$$\int_{z_1}^{z_2} e^{iz/\varepsilon} F(z)\,(\phi'(y))^{-1}\,\mathrm{d}z = \int_{z_1}^{z_1+O\left(\varepsilon^{1/2}\right)} e^{iz/\varepsilon}\left(F(z_1)\phi'(y_0)^{-1} + O\left(\varepsilon^{1/2}\right)\right) = O(\varepsilon) \tag{41}$$

and the proof ends. □

From this we see that, if $\phi' \neq 0$ over $[a, b]$, then

$$\int_a^b e^{i\phi(y)/\varepsilon} f(y)\,\mathrm{d}y = O\left(\varepsilon^{1/2}\right) \longrightarrow 0. \tag{42}$$

Similarly we can prove

Lemma 7. *Let y_0 be such that $\phi'(y_0) = 0$, $\phi''(y_0) \neq 0$. Then*

$$\int_{y_0-\varepsilon^{1/2}}^{y_0+\varepsilon^{1/2}} e^{i\phi(y)/\varepsilon} f(y)\,\mathrm{d}y = f(y_0)\,e^{i\frac{\phi(y_0)}{\varepsilon}}\sqrt{\frac{2\,\pi\,\varepsilon}{|\phi''(y_0)|}}\,e^{i\frac{\pi}{4}\operatorname{sgn}(\phi''(y_0))} + O\left(\varepsilon^{1/2}\right). \tag{43}$$

Now it is clear that we have

$$\int_a^b e^{i\frac{\phi(y)}{\varepsilon}} f(y)\,\mathrm{d}y = \sum_{\phi'(y_i)=0} f(y_i)\,e^{i\frac{\phi(y_i)}{\varepsilon}}\sqrt{\frac{2\,\pi\,\varepsilon}{|\phi''(y_i)|}}\,e^{i\frac{\pi}{4}\operatorname{sgn}(\phi''(y_i))} + O\left(\varepsilon^{1/2}\right). \tag{44}$$

For multi-dimensional generalization, see Evans pp.210–217.

3. Homogenization.

We discuss the following 1D model problem to get some idea of the homogenization procedure. Consider the 1D problem

$$\left(a\left(\frac{x}{\varepsilon}\right)u'\right)' = 0, \qquad u(0) = 0, \quad u(1) = 1. \tag{45}$$

Here $a(y)$ is assumed to be periodical. The basic approach is to treat $y = \frac{x}{\varepsilon}$ as an independent variable, thus the original derivative becomes

$$\cdot' = \partial_x + \frac{1}{\varepsilon}\,\partial_y. \tag{46}$$

assume

$$u = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon\,u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2\,u_2 + \cdots \tag{47}$$

where each $u_i(x, y)$ is periodic in the variable y.

Substituting this into the equation, we have

$$\big(a(y)\,(u_0+\varepsilon\,u_1+\cdots)'\big)'=0 \tag{48}$$

Using the new variables x, y we reach

$$\big(\partial_x+\varepsilon^{-1}\partial_y\big)\big\{a(y)\big[\varepsilon^{-1}\,\partial_y u_0+(\partial_x u_0+\partial_y u_1)+\varepsilon\,(\partial_x u_1+\partial_y u_2)+\cdots\big]\big\}=0. \tag{49}$$

Expanding, we have

$$\varepsilon^{-2}\,\partial_y(a\,\partial_y u_0)+\varepsilon^{-1}\,[\partial_x\,(a\,\partial_y u_0)+\partial_y(a\,(\partial_x u_0+\partial_y u_1))]+\partial_x(a\,(\partial_x u_0+\partial_y u_1))+\partial_y(a\,(\partial_x u_1+\partial_y u_2))+\cdots=0. \tag{50}$$

Now if our expansion of u is correct, all of $u_1, u_2, \ldots$ should remain bounded as $\varepsilon\searrow 0$. Thus necessarily the quantities at each scale should be 0.

At $O\big(\varepsilon^{-2}\big)$, we have

$$\partial_y(a(y)\,\partial_y u_0(x,y))=0 \tag{51}$$

with periodic boundary condition. This implies $u_0(x,y)$, for any fixed x, is a constant. In other words we have

$$u_0(x,y)=u_0(x). \tag{52}$$

Now move on to the next scale $O\big(\varepsilon^{-1}\big)$. We have

$$\partial_x\,(a(y)\,\partial_y u_0)+\partial_y(a(y)(\partial_x u_0+\partial_y u_1))=0. \tag{53}$$

As $\partial_y u_0=0$ we have

$$\partial_y(a\,\partial_y u_1)=-\,(\partial_y a)\,(\partial_x u_0). \tag{54}$$

If we set $\chi=\chi(y)$ be such that

$$\partial_y(a\,\partial_y\chi)=-\,\partial_y a, \tag{55}$$

then

$$u_1=\chi(y)\,\partial_x u_0+\tilde{u}_1(x). \tag{56}$$

Next consider scale $O(1)$. We have

$$\partial_x(a\,\partial_x u_0)+\partial_x(a\,\partial_y u_1)+\partial_y(a\,\partial_x u_1)+\partial_y(a\,\partial_y u_2)=0. \tag{57}$$

Integrate from 0 to 1 in y, we obtain

$$\partial_x\bigg(\int_0^1 a\,\partial_x u_0\bigg)+\partial_x\bigg(\int_0^1 a\,\partial_y u_1\,\mathrm{d}y\bigg)=0. \tag{58}$$

Recall

$$u_1=\chi(y)\,\partial_x u_0+\tilde{u}_1(x). \tag{59}$$

we have

$$\partial_x\bigg[\bigg(\int_0^1 a(y)(1+\chi'(y))\,\mathrm{d}y\bigg)\partial_x u_0\bigg]=0. \tag{60}$$

This is the equation u_0 satisfies.

In our case (1D), the situation can be further simplified. As

$$(a\,\chi')'=-\,a', \tag{61}$$

we have

$$(a\,(1+\chi'))=A. \tag{62}$$

To find out this constant, we divide both sides by a, and integrate over $(0,1)$:

$$1=\int 1+\chi'=A\int\frac{1}{a}\,\mathrm{d}y. \tag{63}$$

Thus

$$A = \left(\int_0^1 \frac{1}{a(y)} \, \mathrm{d}y \right)^{-1}. \tag{64}$$

As a consequence, the equation satisfied by u_0 is

$$\left(\int_0^1 \frac{1}{a(y)} \, \mathrm{d}y \right)^{-1} u_0'' = 0. \tag{65}$$

References.

- For evaluation of integrals, see e.g. Norman Bleistein, Richard A. Handelsman "Asymptotic Expansions of Integrals", Dover, 1986.
- For asymptotics of differential equations as well as homogenization, see e.g. M. H. Holmes "Introduction to Perturbation Methods", Springer-Verlag, 1994.