Nov. 26

The Riemann Problem

The Riemann problem for (scalar or system of) conservation laws is the following

$$u_t + f(u)_x = 0, \qquad u_0(x) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}. \tag{1}$$

1. The Riemann problem for a scalar conservation law.

Our plan is to find out all entropy solutions to this problem. In light of the uniqueness theorem, it suffices to construct one entropy solution for each initial value, that is for each pair of u_l, u_r.

First notice that if $u(x,t)$ is a solution to the scalar conservation law, so is $u(\lambda x, \lambda t)$ for any constant $\lambda > 0$. Thus it is natural to consider solutions of the special form $u(x,t) = U(x/t)$.

It turns out there are three cases.

1. $u_l = u_r$. In this case obviously the constant solution $u \equiv u_0$ is the entropy solution.
2. $f'(u_l) > f'(u_r)$. In this case we know
$$u(x,t) = \begin{cases} u_l & x < s\,t \\ u_r & x > s\,t \end{cases} \tag{2}$$
with $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{f(u_l) - f(u_r)}{u_l - u_r}$ is an entropy solution, and is thus the entropy solution. Such a solution is called a *shock wave*.
3. $f'(u_l) < f'(u_r)$. In this case we try to find a smooth solution of the form $U(x/t)$.

   Set $\xi = x/t$, we have
$$U_t = U'(\xi)\left(-\frac{x}{t^2}\right), \qquad U_x = U'(\xi)\left(\frac{1}{t}\right) \tag{3}$$
and the equation becomes
$$U'(\xi)\left[-\frac{x}{t^2} + f'(U)\,\frac{1}{t}\right] = 0 \tag{4}$$
which reduces to
$$-\xi + f'(U) = 0 \tag{5}$$
if we assume $U'(\xi) \neq 0$ everywhere.

   Now since $f'' > 0$, f' is an increasing function. Therefore for each $\xi \in (f'(u_l), f'(u_r))$ we can find a unique $U \in (u_l, u_r)$ such that $f'(U) = \xi$ and furthermore $U(\xi)$ is differentiable (Implicit Function Theorem).

   Therefore
$$u(x,t) = \begin{cases} u_l & x \leqslant f'(u_l)\,t \\ U(\xi) & x = \xi\,t,\ \xi \in (f'(u_l), f'(u_r)),\ f'(U) = \xi. \\ u_r & x \geqslant f'(u_r)\,. \end{cases} \tag{6}$$

   Such a solution is called a *rarefaction wave*.

Thus we have completely solved the Riemann problem for scalar conservation laws.

2. Shock inequalities for systems of conservation laws.

Before we try to solve systems of conservation laws, we need to gain some intuition of how the solutions should look like (For scalar conservation law, we reach this goal by analyzing rarefaction waves and shock waves).

2.1. System of linear constant coefficient first order hyperbolic equations.

We first consider the linear first order hyperbolic system

$$u_t + A\,u_x = 0 \tag{7}$$

where $u \in \mathbb{R}^n$ and A is an constant $n \times n$ matrix with n distinct real eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ (This means the system is hyperbolic).

To solve this equation, let S be the matrix which diagonalize A, that is $S^{-1}\,A\,S = \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ or equivalently $A = S\,\Lambda\,S^{-1}$. Now setting $v = S^{-1}\,u$, we have

$$v_t + \Lambda\,v_x = 0 \tag{8}$$

If we write $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, we have

$$(v_i)_t + \lambda_i\,(v_i)_x = 0. \tag{9}$$

Thus we can solve each v_i separately.

Keeping in mind that we are aiming at constructing shocks for systems of conservation laws – the relavance can be see by writing

$$u_t + f(u)_x = u_t + (D f(u))\,u_x \tag{10}$$

and note that $D f(u)$ is an $n \times n$ matrix now – we study the constant coefficient problem in the domain $\{(x, t) \colon t > 0, x > s(t)\}$, where $s(t)$ is "modeling" the shock discontinuity. We assume that the initial values for all v_i have been assigned.

Note that each v_i is propagating along the lines $\frac{\mathrm{d}s_i}{\mathrm{d}t} = \lambda_i$. This means for those v_i with $\lambda_i > s(t)$, boundary values need to be set along $x = s(t) + .$[1]

On the other hand, if we are considering $\{(x, t) \colon t > 0, x < s(t)\}$, it's those v_i with $\lambda_i < s(t)$ who need extra boundary conditions along $x = s(t) - .$

2.2. System of linear first order hyperbolic equations.

Now we consider the linear hyperbolic system with variable coefficients.

$$u_t + A(x, t)\,u_x = 0 \tag{11}$$

where the $n \times n$ matrix $A(x, t)$ has n distinct eigenvalues at each (x, t): $\lambda_1(x, t) < \cdots < \lambda_n(x, t)$. Similar to the constant coefficient case, we take $S(x, t)$ such that $S^{-1}\;A\;S = \Lambda(x, t) \equiv \begin{pmatrix} \lambda_1(x,t) & & \\ & \ddots & \\ & & \lambda_n(x,t) \end{pmatrix}$. Setting $v(x, t) = S^{-1}(x, t)\,u(x, t)$, we have

$$v_t + \Lambda(x, t)\,v_x = S_t^{-1}\,u + \Lambda\,S_x^{-1}\,u = \left(S_t^{-1}\,S + \Lambda\,S_x^{-1}\,S\right) v. \tag{12}$$

For each v_i we have

$$(v_i)_t + \lambda_i(x, t)\,(v_i)_x = f_i(S, \Lambda, v, x, t). \tag{13}$$

Although the equation is complicated, each v_i still propagates along the characteristic curve $\frac{\mathrm{d}s_i(t)}{\mathrm{d}t} = \lambda_i(s_i(t), t)$. Therefore we still have the same conclusion about extra boundary conditions.

2.3. Construction of shock waves.

Finally we return to the context of systems of conservation laws. Writing the equation as

$$u_t + (D f(u))\,u_x = 0 \tag{14}$$

we denote the eigenvalues of $D f(u)$ by $\lambda_1(u) < \cdots < \lambda_n(u)$.

Now let $x = s(t)$ be a discontinuity. Denote by u_r the limiting value of u to the right of it, and by u_l the limiting value from the left. Then we have

- If $\lambda_k(u_r) < s < \lambda_{k+1}(u_r)$, then we need $n - k$ boundary conditions along $x = s(t) +$;
- If $\lambda_m(u_l) < s < \lambda_{m+1}(u_l)$, then we need m boundary conditions along $x = s(t) -$.

Thus overall we need $n - k + m$ conditions along $x = s(t)$ to be able to determine the solution on both sides of the discontinuity.

Such conditions can only come from the jump condition (note that s is a scalar, while others are n-vectors)

$$s\,(u_l - u_r) = f(u_l) - f(u_r) \tag{15}$$

1. By writing $x = s(t) +$ (or $-$) we emphasize the fact that the value only needs to be assigned to the "right side" (or "left side") of the curve $x = s(t)$.

which consists of n equation. If we eliminate s from it, we are left with $n-1$ conditions on u_l, u_r.

As a consequence, to make the problem solvable, we have to require

$$n-k+m=n-1 \iff m=k-1. \tag{16}$$

In other words, a "reasonable" discontinuity must satisfy

$$\lambda_k(u_r)<s<\lambda_{k+1}(u_r), \quad \lambda_{k-1}(u_l)<s<\lambda_k(u_l) \tag{17}$$

or equivalently,

$$\lambda_k(u_r)<s<\lambda_k(u_l), \quad \lambda_{k-1}(u_l)<s<\lambda_{k+1}(u_r) \tag{18}$$

for some $k\in\{2, \ldots, n-1\}$. Such a discontinuity is called a k-shock wave, emphasizing the fact that it is a "shock" if we focus on the k-th equation:

$$(v_k)_t+\lambda_k(v)\,(v_k)_x=f_k. \tag{19}$$

Remark 1. In the special case $\lambda_k(u_l)=s=\lambda_k(u_r)$, no shock inequalities are needed. One can check that correct numbers of conditions are present on both sides. This corresponds to the so-called "contact discontinuities".

3. Rarefaction waves.

A rarefaction wave is a smooth solution (in part of space-time, of course) which can be written as

$$u(x,t)=U(x/t). \tag{20}$$

Letting $\xi=x/t$, the system becomes

$$(\xi I-Df)\,U_\xi=0. \tag{21}$$

Therefore ξ is an eigenvalue of Df and U_ξ is the corresponding eigenvector.

The condition

$$(\xi I-Df(U))\,U_\xi=0 \tag{22}$$

can be seen as an ODE system with the unknown U (which is an n-vector) as a function of ξ. In practice, the question is starting from $U=u_l$ at $\xi_l=\lambda(u_l)$, can we solve this equation to obtain $U(\xi)$ so that the solution exists till $\xi_r=\lambda(u_r)$? We will see that in general this cannot be done.

4. Solving systems of conservation laws – the *p*-system.

As a highly non-trivial example illustrating what we have learned so far and at the same time indicating what a general theory should be like, we try to solve the Riemann problem for the p-system:

$$v_t-u_x = 0 \tag{23}$$
$$u_t+p(v)_x = 0 \tag{24}$$

which can be rewritten as

$$U_t+F(U)_x=0 \tag{25}$$

where $U=\begin{pmatrix} v\\ u\end{pmatrix}$, $F(U)=\begin{pmatrix} -u\\ p(v)\end{pmatrix}$, with initial data

$$U=\begin{pmatrix} v\\ u\end{pmatrix}=\begin{cases}\begin{pmatrix} v_l\\ u_l\end{pmatrix} & x<0\\ \begin{pmatrix} v_r\\ u_r\end{pmatrix} & x>0\end{cases}. \tag{26}$$

We further assume $p'<0, p''>0$. An example of such a system is the governing equations for isentropic or polytropic gas dynamics where $p(v)=k\,v^{-\gamma}$ with $\gamma\in(1,3)$.

It is easy to calculate

$$DF(U)=\begin{pmatrix} 0 & -1\\ p'(v) & 0\end{pmatrix} \tag{27}$$

which gives

$$\lambda_1(U) = -\sqrt{-p'(v)} < 0 < \sqrt{-p'(v)} = \lambda_2(U). \tag{28}$$

Our plan is to start from U_l, and try to "connect" to U_r using finitely many shock waves and rarefaction waves. It turns out that, starting from any $\begin{pmatrix} v \\ u \end{pmatrix} \in \mathbb{R}^2$, the states that can be connected to it by either a shock or a rarefaction wave lies on a particular curve. Therefore the Riemann problem is solvable only when U_l and U_r can be connected by finitely many such curves. Thus the first thing is to identify these curves. In other words, we need to identify all possible shocks and rarefaction waves.

4.1. Possible shocks.

From our discussion on shocks, we know there are two possibilities:

- 1-shock:

$$\lambda_1(U_r) < s < \lambda_1(U_l), \;\; s < \lambda_2(U_r). \tag{29}$$

- 2-shock:

$$\lambda_2(U_r) < s < \lambda_2(U_l), \;\; s > \lambda_1(U_l). \tag{30}$$

As $\lambda_1 < 0$ and $\lambda_2 > 0$, we have $s < 0$ for 1-shocks and $s > 0$ for 2-shocks.

A) 1-shocks. We solve all states $\begin{pmatrix} v \\ u \end{pmatrix}$ that can be connected to $\begin{pmatrix} v_l \\ u_l \end{pmatrix}$ by a 1-shock.

The conditions are

$$\begin{aligned} s\,(v - v_l) &= -(u - u_l) && (31)\\ s\,(u - u_l) &= p(v) - p(v_l) && (32)\\ -\sqrt{-p'(v)} < &\; s < -\sqrt{-p'(v_l)} && (33) \end{aligned}$$

Eliminating s from the jump conditions, we obtain

$$u - u_l = \pm\sqrt{-(v - v_l)\,(p(v) - p(v_l))}. \tag{34}$$

Now since

$$-\sqrt{-p'(v)} < -\sqrt{-p'(v_l)} \tag{35}$$

we obtain

$$p'(v) < p'(v_l) \implies v < v_l \tag{36}$$

dueto $p'' > 0$. Now $v - v_l < 0$ and $s < 0$ gives $u - u_l < 0$ which means we need to take the minus sign:

$$u - u_l = \sqrt{-(v - v_l)\,(p(v) - p(v_l))}. \tag{37}$$

Thus any $\begin{pmatrix} v \\ u \end{pmatrix}$ that can be connected to $\begin{pmatrix} v_l \\ u_l \end{pmatrix}$ by a 1-shock must lie on the curve

$$S_1\colon\; u - u_l = \sqrt{-(v - v_l)\,(p(v) - p(v_l))} \equiv s_1(v; U_l). \tag{38}$$

Note that, in this case

$$s = -\sqrt{-p'(\theta)} \tag{39}$$

for some $\theta \in (v, v_l)$ and therefore satisfies the condition for the 1-shock.

B) 2-shocks. Similarly, the conditions are

$$\begin{aligned} s\,(v - v_l) &= -(u - u_l) && (40)\\ s\,(u - u_l) &= p(v) - p(v_l) && (41)\\ \sqrt{-p'(v)} < &\; s < \sqrt{-p'(v_l)}. && (42) \end{aligned}$$

By a similar argument as for the 1-shock, we see that any $\begin{pmatrix} v \\ u \end{pmatrix}$ that can be connected to $\begin{pmatrix} v_l \\ u_l \end{pmatrix}$ via a 2-shock must lie on

$$S_2\colon\; u - u_l = -\sqrt{(v - v_l)\,(p(v) - p(v_l))} \equiv s_2(v; U_l). \tag{43}$$

4.2. Possible rarefaction waves.

The conditions for a rarefaction wave solution is

$$\begin{pmatrix} \lambda & 1 \\ -p'(v) & \lambda \end{pmatrix} \begin{pmatrix} v_\xi \\ u_\xi \end{pmatrix} = 0 \tag{44}$$

We take $\lambda = \lambda_1$ and λ_2 to obtain the 1- (2-) rarefaction waves.

A) 1-rarefaction wave.

Recall that $\lambda_1 = -\sqrt{-p'(v)}$ we have

$$-\sqrt{-p'(v)}\, v_\xi + u_\xi = 0. \tag{45}$$

Note that as λ_1 is an eigenvalue, the other equation is redundant. From this we have

$$\frac{\mathrm{d}u}{\mathrm{d}v} = \sqrt{-p'(v)} \Longrightarrow u - u_l = \int_{v_l}^{v} \sqrt{-p'(y)}\, \mathrm{d}y \tag{46}$$

for all $\begin{pmatrix} v \\ u \end{pmatrix}$ what can be connected to $\begin{pmatrix} v_l \\ u_l \end{pmatrix}$ by a 1-rarefaction wave. Next note that in fact we must have $\xi = \lambda_1$ which means λ_1 is either increasing or decreasing. When λ_1 is decreasing we should use a 1-shock, therefore $\lambda_1(v) > \lambda_1(v_l)$ which means $v > v_l$ due to the assumptions on p. Thus we denote

$$R_1\colon\ u - u_l = \int_{v_l}^{v} \sqrt{-p'(y)}\, \mathrm{d}y \equiv r_1(v, U_l), \quad v > v_l. \tag{47}$$

This is the curve passing all $\begin{pmatrix} v \\ u \end{pmatrix}$ which can be connected to U_l by a 1-rarefaction wave.

B) 2-rarefaction wave.

Similarly, we have

$$R_2\colon\ u - u_l = -\int_{v_l}^{v} \sqrt{-p'(y)}\, \mathrm{d}y \equiv r_2(v; U_l), \quad v_l > v. \tag{48}$$

4.3. Solving the *p*-system.

It turns out that when we vary U_l, the four "half-curves" R_1, R_2, S_1, S_2 cover the whole $\mathbb{R}^2$ for interesting p.[2] This means we can solve the Riemann problem by connecting U_l and U_r using at most two waves, see pp. 317 – 320 of Smoller's book for details.

5. Solving systems of conservation laws – general theory.

We will proceed in a somewhat not-fully-rigorous manner. For details and subtleties see Chapter 17 of Smoller's book.

We denote by $\lambda_1, \ldots, \lambda_k$, $l_1, \ldots, l_k$, $r_1, \ldots, r_k$ the eigenvalues, left eigenvectors and right eigenvectors of Df. The dependence on u should be understood and is made implicit.

5.1. Riemann invariants and rarefaction waves.

Recall that $u(x,t) = U(x/t)$ is a k-rarefaction wave if $U_\xi \parallel \boldsymbol{r}_k$. Now if w is a function of u such that

$$\boldsymbol{r}_k \cdot \nabla_u w = 0 \tag{49}$$

then we have

$$U_\xi \cdot \nabla_u w = 0 \Longleftrightarrow \frac{\mathrm{d}}{\mathrm{d}\xi} w = 0 \tag{50}$$

which means w is invariant along any curve tracing the change of u through a rarefaction wave. Such w is called a *k-Riemann invariant*.

Treating $\boldsymbol{r}_k \cdot \nabla w = 0$ as a 1st order PDE, we see that in general we can find $n-1$ independent solutions in the sense that their gradients are linearly independent at every point. Thus for each k, there are $n-1$ k-Riemann invariants.

2. See pp. 312 – 316 of Smoller's book.

Now consider the $n-1$ level sets of these Riemann invariants. Their intersection is a curve which is tangent to $\boldsymbol{r}_k$ everywhere. If u_l is on one of these curves, all the possible u's that can possibly be connected to u_l has to lie on the same curve.

One should keep in mind that as r_k depends on u, the equation $r_k \cdot \nabla_u w = 0$ is quasi-linear and therefore we can only expect local existence of these curves.

Example 2. Consider the p-system

$$v_t - u_x = 0, \qquad u_t + p(v)_x = 0. \tag{51}$$

We have $f = \begin{pmatrix} -u \\ p(v) \end{pmatrix}$ and $Df = \begin{pmatrix} 0 & -1 \\ p'(v) & 0 \end{pmatrix}$ which leads to

$$\lambda_1 = -\sqrt{-p'(v)}, \ \lambda_2 = \sqrt{-p'(v)} \tag{52}$$

and

$$r_1 = \begin{pmatrix} 1 \\ \sqrt{-p'(v)} \end{pmatrix}, \quad r_2 = \begin{pmatrix} -1 \\ \sqrt{p'(v)} \end{pmatrix}. \tag{53}$$

Thus we can try to solve, say, the 2-Riemann invariants from

$$-\partial_v w + \sqrt{p'(v)}\, \partial_u w = r_2 \cdot \nabla_{\begin{pmatrix} v \\ u \end{pmatrix}} w = 0. \tag{54}$$

The solution is given by

$$w = u + \int^v \sqrt{-p'(y)}\, \mathrm{d}y. \tag{55}$$

From the above discussion we can formulate the following definition.

Definition 3. *Let u be a C^1 solution in a domain D, and suppose that all k-Riemann invariants are constant in D. Then u is called a k-simple wave (or a k-rarefaction wave).*

Remember that for systems of conservations, in general u is not constant along the characteristic curves anymore, and as a consequence, these characteristic curves are not straight lines anymore. However, for simple waves, these curves are still straight!

Theorem 4. *Let u be a k-simple wave in a domain D. Then the characteristics of the kth field*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lambda_k(u) \tag{56}$$

are straight lines and u is constant along them.

Proof. Let w_1, ..., w_{n-1} be the k-Riemann invariants. We know that they remain constants along the characteristic curve, in other words, if s is a parameter along this curve, we have

$$0 = \frac{\mathrm{d}w_j}{\mathrm{d}s} = \frac{\mathrm{d}u}{\mathrm{d}s} \cdot \nabla_u w_j. \tag{57}$$

Furthermore, multiplying the equation by l_k from the left, we have

$$l_k \cdot \frac{\mathrm{d}u}{\mathrm{d}s} = l_k \cdot (u_t + \lambda_k u_x) = 0. \tag{58}$$

Since $l_k \cdot r_k \neq 0$ and $r_k \cdot \nabla w_j = 0$ for all $j = 1, ..., n-1$, we conclude that $l_k, \nabla w_1, ..., \nabla w_{n-1}$ are linearly independent and $\frac{\mathrm{d}u}{\mathrm{d}s} = 0$ immediately follows. $\square$

Now we try to establish the existence of k-simple waves. Recall that $u(x,t) = U(x/t)$ being a k-simple wave if and only if

$$(\xi I - Df(U))\, U_\xi = 0 \tag{59}$$

where $\xi = x/t$. This leads to two conditions

$$U_\xi \parallel r_k(U) \qquad \text{and} \qquad \frac{\mathrm{d}\lambda_k(U)}{\mathrm{d}\xi} = 1. \tag{60}$$

Thus it is natural to try

$$\frac{\mathrm{d}U}{\mathrm{d}\xi} = r_k(U(\xi)), \qquad U(\lambda_k(u_l)) = u_l. \tag{61}$$

As we start from the left side of the wave, necessarily $\xi \geqslant \lambda_k(u_l)$. The solution to the above nonlinear ODE/PDE exists locally.

Now we need to check the second condition, we compute

$$\frac{\mathrm{d}\lambda_k(U(\xi))}{\mathrm{d}\xi} = U_\xi \cdot \nabla_u \lambda_k = r_k \cdot \nabla_u \lambda_k \tag{62}$$

The requirement that it equals 1 becomes the condition

$$r_k \cdot \nabla_u \lambda_k = 1 \tag{63}$$

which is guaranteed only when $r_k \cdot \nabla_u \lambda_k \neq 0$. The kth characteristic family is said to be *genuinely nonlinear* when this is satisfied.

Example 5. Consider the scalar conservation law $u_t + f(u)_x = 0$. In this case $\lambda(u) = f'(u)$ and the genuine nonlinearity thus becomes the condition

$$f''(u) \neq 0 \tag{64}$$

or equivalently $f''(u) > 0$.

Summarizing the above, we obtain the following theorem after some more computation (see pp. 326 – 327 of Smoller's book):

Theorem 6. *Let the kth characteristic field be genuinely nonlinear in a domain N, and normalized so that $r_k \cdot \nabla \lambda_k = 1$. Let u_l be any point in N. There exists a one-parameter family $u = u(\varepsilon)$ with $0 \leqslant \varepsilon < a$, $u(0) = u_l$, which can be connected to u_l on the right by a k-centered simple wave. The parametrization can be chosen so that $\dot{u} = r_k$ and $\ddot{u} = \dot{r}_k$.*

Remark 7. Using the Riemann invariants, we see that u is the implicit function defined by

$$F(u) = \begin{pmatrix} w_1(u) - w_1(u_l) \\ \vdots \\ w_{n-1}(u) - w_{n-1}(u_l) \\ \lambda_k(u) - \lambda_k(u_l) - \varepsilon \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{65}$$

And the genuine nonlinearity is equivalent to the non-singularity of the Jacobian matrix.

Remark 8. One can see that genuine nonlinearity is in fact a condition on the second derivative of f (which is a 3-tensor). Let $A = Df$. We have $A\, r_k = \lambda_k\, r_k$. Taking $r_k \cdot \nabla$ we obtain

$$r_k \cdot \nabla A\, r_k + A\,(r_k \cdot \nabla r_k) = (r_k \cdot \nabla \lambda_k)\, r_k + \lambda_k\,(r_k \cdot \nabla r_k). \tag{66}$$

Multiply by l_k from the left, we have

$$l_k \cdot (r_k \cdot \nabla A) \cdot r_k = (l_k \cdot r_k)\,(r_k \cdot \nabla \lambda_k). \tag{67}$$

As $l_k \cdot r_k \neq 0$, the conclusion follows.

5.2. Shocks.

Any shock must satisfy the jump condition

$$s\,(u - u_l) = f(u) - f(u_l) \tag{68}$$

which after canceling s become $n-1$ equations. It's tempting to use implicit function theorem directly to obtain an one-parameter solution curve, but this turns out to be too good to be true as $u \equiv u_l$ obviously solves the system.

Instead we re-write the jump condition as

$$s\,(u-u_l) = \left[\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} f(u_l + s\,(u-u_l))\,\mathrm{d}s\right](u-u_l) \equiv G(u)\,(u-u_l). \tag{69}$$

Thus u_l, u form a shock with speed s if and only if s is an eigenvalue of G and $u-u_l$ is the corresponding eigenvector.

Now using hyperbolicity, when u is close enough to u_l, we can assume G having n distinct eigenvalues $\mu_1 < \cdots < \mu_n$ with corresponding eigenvectors $L_1, \ldots, L_n$. Thus the condition that $u - u_l$ being the kth eigenvector is the same as

$$L_j \cdot (u-u_l) = 0, \qquad j \neq k. \tag{70}$$

We apply the implicit function theorem to obtain the solution curve $u = u(\varepsilon)$. Such curves are called *k-shocks*.

As $L_j \rightarrow l_j$, it is easy to see that $\dot{u}(0) \parallel r_k$. Furthermore we have

Proposition 9. *Along the k-shock, if the kth characteristic field is genuinely nonlinear, we can choose a parametrization so that $\dot{u}_k(0) = r_k$, and $\ddot{u}_k(0) = \dot{r}_k$ where the RHS are both at u_l. Moreover, with this parametrization $s(0) = \lambda_k(u_l)$ and $\dot{s}(0) = \frac{1}{2}$.*

Proof. This done by taking derivatives of the jump condition as well as the relation $(Df)\,r_k = \lambda_k\,r_k$. See Smoller pp. 330 – 331 for details. $\square$

Remark 10. What's important to notice here, is that the curves for shocks and rarefaction waves has the same first and second derivatives at u_l. Furthermore when u_l and u are close, that is the shock is "weak", the shock speed $s \approx \frac{\lambda_k(u_l) + \lambda_k(u_r)}{2}$.

Remark 11. One can further show by direct computation that the change in a k-Riemann invariant across a k-shock is of third order in ε. See Smoller pp. 332 – 333 for details.

The last thing we need to do for k-shocks is to verify that they are indeed shocks, in the sense that they satisfy the entropy conditions

$$\lambda_{k-1}(u_l) < s(u) < \lambda_{k+1}(u), \quad \lambda_k(u) < s(u) < \lambda_k(u_l). \tag{71}$$

Theorem 12. *The shock inequalities hold along the curve $u = u(\varepsilon)$ if and only if $\varepsilon < 0$.*

Proof. $\lambda_{k-1}(u_l) < s < \lambda_{k+1}(u)$ automatically holds when u and u_l are close enough as $s(0) = \lambda_k(u_l) \in (\lambda_{k-1}(u_l), \lambda_{k+1}(u_l))$. For $s < \lambda_k(u_l)$, we note that $s(0) = \lambda_k(u_l)$ and $\dot{s}(0) = \frac{1}{2}$. Therefore $s(u) < \lambda_k(u_l)$ if and only if $\varepsilon < 0$.

When $\varepsilon < 0$, $\lambda_k(u) < s(u)$ is satisfied because $\dot{s}(0) = \frac{1}{2}$ while $\dot{\lambda_k} = r_k \cdot \nabla \lambda_k = 1$ due to genuine nonlinearity. $\square$

Remark 13. Now we can "connect" the curves for the k-simple waves and the k-shocks to obtain n C^2 curves passing u_l. These curves form a local coordinate system and as a consequence, all states u_r close enough to u_l can be reached through $n-1$ intermediate states.

5.3. Contact discontinuities.

What happens when the k-th characteristic field is not genuinely nonlineair? We assume that $r_k \cdot \nabla \lambda_k = 0$ in a domain N. The characteristic field is said to be linearly degenerate here.

In this case, λ_k is a Riemann invariant and therefore is constant along the curve

$$\frac{\mathrm{d}u}{\mathrm{d}\varepsilon} = r_k, \quad u(0) = u_l. \tag{72}$$

Now for any u on this curve, we can define

$$v(x,t)=\begin{cases} u_l & x<t\,\lambda_k(u_l)\\ u & x>t\,\lambda_k(u)=\lambda_k(u_l)\end{cases}. \tag{73}$$

Such a discontinuous solution is called a "contact discontinuity". To see that it satisfies the jump condition, we differentiate

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}[f(u)-s\,u]=(Df)\,r_k-\lambda_k\,r_k=0 \tag{74}$$

which immediately leads to the jump condition.

Remark 14. Recall how we obtain the shock inequalities by considering boundary conditions along the two sides of a discontinuity. In the case of a contact discontinuity, there is a "jump", but since the characteristic curves are straightlines parallel to this jump, the two sides just do not interfere with each other.

5.4. Solving the general Riemann problem.

Theorem 15. *Let $u_l\in N$ and assume that the system is hyperbolic with each characteristic field either genuinely nonlinear or linearly degenerate in N. Then there is a neighborhood of u_l such that if u_r in this neighborhood, the Riemann problem has a solution. The solution consists of at most $(n+1)$ constant states separated by shocks, centered simple waves, or contact discontinuities. There is precisely one solution of this kind in this neighborhood.*

Proof. See Smoller pp. 335 – 336. □

Further readings.

- J. Smoller, **Shock Waves and Reaction-Diffusion Equations**, §D of Chapter 16, §D of Chapter 15, Chapter 17.

Exercises.

Exercise 1. Let $u(x,t)$ be a weak solution of the scalar conservation law with initial value $u_0(x)$. Show that for any $\lambda>0$, $u(\lambda\,x,\lambda\,t)$ is a weak solution for the same equation with initial value $u_0(\lambda\,x)$.

Exercise 2. Let s be the speed of a k-shock. What is the relation between s and other eigenvalues, that is $\lambda_i(u_l),\lambda_i(u_r)$ with $i\neq k$? Is it possible that a k-shock is at the same time a m-shock for some $m\neq k$?

Exercise 3. Consider the scalar conservation law, write it as $u_t+f'(u)\,u_x=0$. Then use the argument presented in this lecture to show that, if $x=s(t)$ is a discontinuity, then we need the shock condition

$$f'(u_l)>s>f'(u_r) \tag{75}$$

to determine the solution on both sides.